Econometrica Supplementary Material

SUPPLEMENT TO "VIABILITY AND ARBITRAGE UNDER KNIGHTIAN UNCERTAINTY"

(*Econometrica*, Vol. 89, No. 3, May 2021, 1207–1234)

MATTEO BURZONI
Department of Mathematics, Università degli studi di Milano

FRANK RIEDEL
Center for Mathematical Economics, Bielefeld University and School of Economics, College of Business and Economics, University of Johannesburg

H. METE SONER
Department of Operations Research and Financial Engineering, Princeton University

This Online Appendix contains additional technical material.

KEYWORDS: Viability, Knightian uncertainty, no arbitrage, robust finance.

S.1. EXTENSIONS

FOR THE NOTATION AND DEFINITIONS, we refer to the main text. Let $\mathcal{B}(\Omega, \mathcal{F})$ be the set of all $\mathcal{F}$ measurable real-valued functions on Ω. Any Banach space contained in $\mathcal{B}(\Omega, \mathcal{F})$ satisfies the requirements for $\mathcal{H}$. In our examples, we used the spaces $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{M})$, and $\mathcal{B}_b(\Omega, \mathcal{F})$, the set of all bounded functions in $\mathcal{B}(\Omega, \mathcal{F})$, with the supremum norm. In the latter case, the superhedging functional enjoys several properties as discussed in Remark 2.3 in the main text.

Since we require that $\mathcal{I} \subset \mathcal{H}$, in the case of $\mathcal{H} = \mathcal{B}_b(\Omega, \mathcal{F})$ this means that all the trading instruments are bounded. This could be restrictive in some applications and we now provide another example that overcomes this difficulty. To define this set, fix $L^* \in \mathcal{B}(\Omega, \mathcal{F})$ with $L^*(\omega) \geq 1$ for every $\omega \in \Omega$. Consider the linear space

$$\mathcal{B}_\ell := \{X \in \mathcal{B}(\Omega, \mathcal{F}) : \exists \alpha \in \mathbb{R}^+ \text{ such that } |X(\omega)| \leq \alpha L^*(\omega)\ \forall \omega \in \Omega\}$$

equipped with the norm,

$$\|X\|_\ell := \inf\{\alpha \in \mathbb{R}^+ : |X(\omega)| \leq \alpha L^*(\omega)\ \forall \omega \in \Omega\} = \left\| \frac{X}{L^*} \right\|_\infty .$$

We denote the topology induced by this norm by τ_ℓ. Then $\mathcal{B}_\ell(\Omega, \mathcal{F})$ with τ_ℓ is a Banach space and satisfies our assumptions. Note that if $L^* = 1$, then $\mathcal{B}_\ell(\Omega, \mathcal{F}) = \mathcal{B}_b(\Omega, \mathcal{F})$.

Now, suppose that

$$L^*(\omega) := c^* + \hat{\ell}(\omega), \quad \omega \in \Omega, \tag{S.1.1}$$

for some $c^* > 0$, $\hat{\ell} \in \mathcal{I}$. Then one can define the superreplication functional as in Remark A.3 of the main paper.

Matteo Burzoni: matteo.burzoni@unimi.it
Frank Riedel: frank.riedel@uni-bielefeld.de
H. Mete Soner: soner@princeton.edu

Another important extension is to relax the assumption that the consumption sets is equal to the entire space $\mathcal{H}$. This hypothesis is taken in the classical literature as well as in the main body of the paper but it might be restrictive in some applications. We show here that, within our framework, we may accommodate a smaller consumption set, in particular, we are able to restrict to consumption sets bounded from below. Let us fix the lower bound to be 0 for the sake of discussion, which also correspond to the most relevant case of non-negative consumption. Consider a market $(H, \tau, \leq, \mathcal{I}, \mathcal{R})$ where the topology is generated by open intervals with respect to the (strict) order; by definition, the set $O := \{X > 0\}$ is open. Given a preference relation on O, we can extend it to the whole space by treating all elements of $\{X \leq 0\}$ as indifferent and $X \prec Y$ if $X \notin O$ and $Y \in O$. Since the preferences in our definition of $\mathcal{A}$ are only required to be τ-lower semi-continuity, this class satisfies all the required properties. In particular, the class of linear agents constructed in the proofs of Theorems 2.1 and 2.2 in the main text may be modified accordingly: for a given linear continuous functional φ, we may set utility to minus infinity on the complement of O. The induced preference relation is in $\mathcal{A}$. Additionally, agents with power utilities

$$U(X) = \mathbb{E}\left[\frac{(X+c)^{1-\gamma}}{1-\gamma}\right]$$

with a constant c can be included in $\mathcal{A}$. Restricting consumption to be positive may result in the failure of extendability of the pricing functional,[1] therefore the classical theory of Harrison and Kreps (1979), Kreps (1981) does not apply. On the contrary, since we do not insist on a single representative agent, this aspect does not affect the results of Section 2 of the main text.

S.2. NO ARBITRAGE VERSUS NO FREE-LUNCH-WITH-VANISHING-RISK

Let $(\mathcal{H}, \tau, \leq, \mathcal{I}, \mathcal{R})$ be a financial market. An arbitrage opportunity is always a free lunch with vanishing risk. The purpose of this section is to investigate when these two notions are equivalent.

S.2.1. *Attainment*

DEFINITION S.2.1: We say that a financial market has the *attainment property*, if for every $X \in \mathcal{H}$ there exists a minimizer in equation (5.1) of the main text, that is, there exists $\ell_X \in \mathcal{I}$ satisfying,

$$\mathcal{D}(X) + \ell_X \geq X.$$

PROPOSITION S.2.2: *Suppose that a financial market has the attainment property. Then it is strongly free of arbitrage if and only if it has no arbitrages.*

PROOF: Let $R^* \in \mathcal{R}$. By hypothesis, there exist $\ell \in \mathcal{I}^*$ so that $\mathcal{D}(R^*) + \ell^* \geq R^*$. If the market has no arbitrage, then we conclude that $\mathcal{D}(R^*) > 0$. In view of Proposition 5.1 of the main text, this proves that the financial market is also strongly free of arbitrage. Since no arbitrage is weaker condition, they are equivalent. *Q.E.D.*

[1]We thank an anonymous referee for pointing out this aspect.

S.2.2. *Discrete Time Markets With Finite Horizon*

In this subsection and in the next section, we restrict ourselves to arbitrage considerations in finite discrete-time markets.

We start by introducing a discrete filtration $\mathbb{F} := (\mathcal{F}_t)_{t=0}^T$ on subsets of Ω. Let $S = (S_t)_{t=0}^T$ be an adapted stochastic process[2,3] with values in $\mathbb{R}_+^M$ for some M. For every $\ell \in \mathcal{I}$, there exist predictable integrands $H_t \in \mathcal{B}_b(\Omega, \mathcal{F}_{t-1})$ for all $t = 1, \ldots, T$ such that

$$\ell = (H \cdot S)_T := \sum_{t=1}^{T} H_t \cdot \Delta S_t, \quad \text{where } \Delta S_t := (S_t - S_{t-1}).$$

Denote by $\ell_t := (H \cdot S)_t$ for $t \in \mathcal{I}$ and $\ell := \ell_T$.

Set $\hat{\ell} = \sum_{k,i} S_k^i - S_0^i$. Then one can directly show that with an appropriate c^*, we have $L^* := c^* + \hat{\ell} \geq 1$. Define $\mathcal{B}_\ell$ using $\hat{\ell}$, set $\mathcal{H} = \mathcal{B}_\ell$ and denote by $\mathcal{I}_\ell$ the subset of $\mathcal{I}$ with H_t *bounded* for every $t = 1, \ldots, T$.

We next prescribe the equivalence relation and the relevant sets. Our starting point is the set of negligible sets $\mathcal{Z}$ which we assume is given. We also make the following structural assumption.

ASSUMPTION S.2.3: Assume that the trading is allowed only at finite time points labeled through $1, 2, \ldots, T$. Let $\mathcal{I}$ be given as above and let $\mathcal{Z}$ be a lattice which is closed with respect to pointwise convergence.

We also assume that $\mathcal{R} = \mathcal{P}^+$ and the preorder is given by

$$X \preceq Y \quad \Leftrightarrow \quad \exists Z \in \mathcal{Z} \quad \text{such that} \quad X \leq_\Omega Y + Z,$$

where $\leq_\Omega$ denotes the pointwise order of functions. In particular, $X \in \mathcal{P}$ if and only if there exists $Z \in \mathcal{Z}$ such that $Z \leq_\Omega X$.

For an example of the above structure, refer to Example 2.6 of the main text. In that example, $\mathcal{Z}$ consists of the polar sets of a given class $\mathcal{Q}$ of probabilities. Then, in this context all inequalities should be understood as $\mathcal{Q}$ quasi-surely. Also note also that the assumptions on $\mathcal{Z}$ are trivially satisfied when $\mathcal{Z} = \{0\}$. In this latter case, inequalities are pointwise.

Observe that in view of the definition of $\preceq$ and the fact $\mathcal{R} = \mathcal{P}^+$, $\ell \in \mathcal{I}$ is an arbitrage if and only if there is $R^* \in \mathcal{P}^+$ and $Z^* \in \mathcal{Z}$, so that $\ell \geq_\Omega R^* + Z^*$. Hence, $\ell \in \mathcal{I}$ is an arbitrage if and only if $\ell \in \mathcal{P}^+$. We continue by showing the equivalence of the existence of an arbitrage to the existence of a one-step arbitrage.

LEMMA S.2.4: *Suppose that Assumption S.2.3 holds. Then there exists arbitrage if and only if there exists $t \in \{1, \ldots, T\}$, $h \in \mathcal{B}_b(\Omega, \mathcal{F}_{t-1})$ such that $\ell := h \cdot \Delta S_t$ is an arbitrage.*

---

[2]When working with N stocks, a canonical choice for Ω would be

$$\Omega = \{\omega = (\omega_0, \ldots, \omega_T) : \omega_i \in [0, \infty)^N, i = 0, \ldots, T\}.$$

Then one may take $S_t(\omega) = \omega_t$ and $\mathbb{F}$ to be the filtration generated by S.

[3]Note that we do not specify any probability measure.

PROOF: The sufficiency is clear. To prove the necessity, suppose that $\ell \in \mathcal{I}$ is an arbitrage. Then there is a predictable process H so that $\ell = (H \cdot S)_T$. Also $\ell \in \mathcal{P}^+$, hence, $\ell \notin \mathcal{Z}$ and there exists $Z \in \mathcal{Z}$ such that $\ell \geq Z$. Define

$$\hat{t} := \min\{t \in \{1, \dots, T\} : (H \cdot S)_t \in \mathcal{P}^+\} \leq T.$$

First, we study the case where $\ell_{\hat{t}-1} \in \mathcal{Z}$. Define

$$\ell^* := H_{\hat{t}} \cdot \Delta S_{\hat{t}},$$

and observe that $\ell_{\hat{t}} = \ell_{\hat{t}-1} + \ell^*$. Since $\ell_{\hat{t}-1} \in \mathcal{Z}$, we have that $\ell^* \in \mathcal{P}^+$ iff $\ell_{\hat{t}} \in \mathcal{P}^+$, and consequently the lemma is proved.

Suppose now $\ell_{\hat{t}-1} \notin \mathcal{Z}$. If $\ell_{\hat{t}-1} \geq_\Omega 0$, then $\ell_{\hat{t}-1} \in \mathcal{P}$, and thus, also in $\mathcal{P}^+$, which is not possible from the minimality of $\hat{t}$. Hence the set $A := \{\ell_{\hat{t}-1} <_\Omega 0\}$ is nonempty and $\mathcal{F}_{\hat{t}-1}$-measurable. Define $h := H_{\hat{t}} \chi_A$ and $\ell^* := h \cdot \Delta S_{\hat{t}}$. Note that

$$\ell^* = \chi_A(\ell_{\hat{t}} - \ell_{\hat{t}-1}) \geq_\Omega \chi_A \ell_{\hat{t}} \geq_\Omega \chi_A Z \in \mathcal{Z}.$$

This implies $\ell^* \in \mathcal{P}$. Toward a contradiction, suppose that $\ell^* \in \mathcal{Z}$. Then

$$\ell_{\hat{t}-1} \geq_\Omega \chi_A \ell_{\hat{t}-1} \geq \chi_A(Z - \ell^*) \in \mathcal{Z}.$$

Since by assumption, $\ell_{\hat{t}-1} \notin \mathcal{Z}$ we have $\ell_{\hat{t}-1} \in \mathcal{P}^+$ from which $\hat{t}$ is not minimal. *Q.E.D.*

The following is the main result of this section. For the proof, we follow the approach of Kabanov and Stricker (2001) which is also used in Bouchard and Nutz (2015). We consider the financial market $\Theta_* = (\mathcal{B}_\ell, \|\cdot\|_\ell, \leq_\Omega, \mathcal{I}, \mathcal{P}^+)$ described above.

THEOREM S.2.5: *In a finite discrete time financial market satisfying Assumption S.2.3, the following are equivalent*:

1. *The financial market Θ_* has no arbitrages*.
2. *The attainment property holds and Θ_* is free of arbitrage*.
3. *The financial market Θ_* is strongly free of arbitrages*.

PROOF: In view of Proposition S.2.2, we only need to prove the implication $1 \Rightarrow 2$.

For $X \in \mathcal{H}$ such that $\mathcal{D}(X)$ is finite, we have that

$$c_n + \mathcal{D}(H) + \ell_n \geq_\Omega X + Z_n,$$

for some $c_n \downarrow 0$, $\ell_n \in \mathcal{I}$ and $Z_n \in \mathcal{Z}$. Note that since $\mathcal{Z}$ is a lattice we assume, without loss of generality, that $Z_n = (Z_n)^-$ and denote by $\mathcal{Z}^- := \{Z^- \mid Z \in \mathcal{Z}\}$.

We show that $\mathcal{C} := \mathcal{I} - (\mathcal{L}^0_+(\Omega, \mathcal{F}) + \mathcal{Z}^-)$ is closed under pointwise convergence where $\mathcal{L}^0_+(\Omega, \mathcal{F})$ denotes the class of pointwise nonnegative random variables. Once this result is shown, by observing that $X - c_n - \mathcal{D}(X) = W_n \in \mathcal{C}$ converges pointwise to $X - \mathcal{D}(X)$ we obtain the attainment property.

We proceed by induction on the number of time steps. Suppose first $T = 1$. Let

$$W_n = \ell_n - K_n - Z_n \to W, \tag{S.2.1}$$

where $\ell_n \in \mathcal{I}$, $K_n \geq_\Omega 0$ and $Z_n \in \mathcal{Z}^-$. We need to show $W \in \mathcal{C}$. Note that any ℓ_n can be represented as $\ell_n = H^n_1 \cdot \Delta S_1$ with $H^n_1 \in \mathcal{L}^0(\Omega, \mathcal{F}_0)$.

Let $\Omega_1 := \{\omega \in \Omega \mid \liminf |H_1^n| < \infty\}$. From Lemma 2 in Kabanov and Stricker (2001) there exist a sequence $\{\tilde{H}_1^k\}$ such that $\{\tilde{H}_1^k(\omega)\}$ is a convergent subsequence of $\{H_1^k(\omega)\}$ for every $\omega \in \Omega_1$. Let $H_1 := \liminf H_1^n \chi_{\Omega^1}$ and $\ell := H_1 \cdot \Delta S_1$.

Note now that $Z_n \leq_\Omega 0$, hence, if $\liminf |Z_n| = \infty$ we have $\liminf Z_n = -\infty$. We show that we can choose $\tilde{Z}_n \in \mathcal{Z}^-$, $\tilde{K}_n \geq_\Omega 0$ such that $\tilde{W}_n := \ell_n - \tilde{K}_n - \tilde{Z}_n \to W$ and $\liminf \tilde{Z}_n$ is finite on Ω_1. On $\{\ell_n \geq_\Omega W\}$ set $\tilde{Z}_n = 0$ and $\tilde{K}_n = \ell_n - W$. On $\{\ell_n <_\Omega W\}$ set

$$\tilde{Z}_n = Z_n \vee (\ell_n - W), \qquad \tilde{K}_n = K_n \chi_{\{Z_n = \tilde{Z}_n\}}.$$

It is clear that $Z_n \leq_\Omega \tilde{Z}_n \leq_\Omega 0$. From Lemma S.4.1, we have $\tilde{Z}_n \in \mathcal{Z}$. Moreover, it is easily checked that $\tilde{W}_n := \ell_n - \tilde{K}_n - \tilde{Z}_n \to W$. Nevertheless, from the convergence of ℓ_n on Ω_1 and $\tilde{Z}_n \geq_\Omega -(W - \ell_n)^+$, we obtain $\{\omega \in \Omega_1 \mid \liminf \tilde{Z}_n > -\infty\} = \Omega_1$. As a consequence, also $\liminf \tilde{K}_n$ is finite on Ω_1; otherwise, we could not have that $\tilde{W}_n \to W$. Thus, by setting $\tilde{Z} := \liminf \tilde{Z}_n$ and $\tilde{K} := \liminf \tilde{K}_n$, we have $W = \ell - \tilde{K} - \tilde{Z} \in \mathcal{C}$.

On Ω_1^C we may take $G_1^n := H_1^n / |H_1^n|$ and let $G_1 := \liminf G_1^n \chi_{\Omega_1^C}$. Define, $\ell_G := G_1 \cdot \Delta S_1$. We now observe that

$$\{\omega \in \Omega_1^C \mid \ell_G(\omega) \leq 0\} \subseteq \{\omega \in \Omega_1^C \mid \liminf Z_n(\omega) = -\infty\}.$$

Indeed, if $\omega \in \Omega_1^C$ is such that $\liminf Z_n(\omega) > -\infty$, applying again Lemma 2 in Kabanov and Stricker (2001), we have that

$$\liminf_{n\to\infty} \frac{X(\omega) + Z_n(\omega)}{|H_1^n(\omega)|} = 0,$$

implying $\ell_G(\omega)$ is nonnegative. Set now

$$\tilde{Z}_n := Z_n \vee -(\ell_G)^-.$$

From $Z_n \leq_\Omega \tilde{Z}_n \leq_\Omega 0$, again by Lemma S.4.1, $\tilde{Z}_n \in \mathcal{Z}$. By taking the limit for $n \to \infty$, we obtain $(\ell_G)^- \in \mathcal{Z}$, and thus, $\ell_G \in \mathcal{P}$. Since the financial market has no arbitrages $G_1 \cdot \Delta S_1 = Z \in \mathcal{Z}$, and hence one asset is redundant. Consider a partition Ω_2^i of Ω_1^C on which $G_1^i \neq 0$. Since $\mathcal{Z}$ is stable under multiplication (Lemma S.4.2), for any $\ell^* \in \mathcal{I}$, there exists $Z^* \in \mathcal{Z}$ and $H^* \in \mathcal{L}^0(\Omega_2^i, \mathcal{F}_0)$ with $(H^*)^i = 0$, such that $\ell^* = H^* \cdot \Delta S_1 + Z^*$ on Ω_2^i. Therefore, the term ℓ_n in (S.2.1) is composed of trading strategies involving only $d-1$ assets. Iterating the procedure up to d-steps, we have the conclusion.

Assuming now that (S.2.1) holds for markets with $T-1$ periods, with the same argument we show that we can extend to markets with T periods. Set again $\Omega_1 := \{\omega \in \Omega \mid \liminf |H_1^n| < \infty\}$. Since on Ω_1, we have that

$$W_n - H_1^n \cdot \Delta S_1 = \sum_{t=2}^{T} H_t^n \cdot \Delta S_t - K_n - Z_n \to W - H_1 \cdot \Delta S_1.$$

The induction hypothesis allows to conclude that $W - H_1 \cdot S_1 \in \mathcal{C}$ and, therefore, $W \in \mathcal{C}$. On Ω_1^C we may take $G_1^n := H_1^n / |H_1^n|$ and let $G_1 := \liminf G_1^n \chi_{\Omega_1^C}$. Note that $W_n / |H_1^n| \to 0$, and hence

$$\sum_{t=2}^{T} \frac{H_t^n}{|H_1^n|} \cdot \Delta S_t - \frac{K_n}{|H_1^n|} - \frac{Z_n}{|H_1^n|} \to -G_1 \cdot \Delta S_1.$$

Since $\mathcal{Z}$ is stable under multiplication $\frac{Z_n}{|H_1^n|} \in \mathcal{Z}$, and hence, by inductive hypothesis, there exists $\tilde{H}_t$ for $t = 2, \dots, T$ and $\tilde{Z} \in \mathcal{Z}$ such that

$$\tilde{\ell} := G_1 \cdot \Delta S_1 + \sum_{t=2}^{T} \tilde{H}_t \cdot \Delta S_t \geq_\Omega \tilde{Z} \in \mathcal{Z}.$$

The no arbitrage condition implies that $\tilde{\ell} \in \mathcal{Z}$. Once again, this means that one asset is redundant and, by considering a partition Ω_2^i of Ω_1^C on which $G_1^i \neq 0$, we can rewrite the term ℓ_n in (S.2.1) with $d - 1$ assets. Iterating the procedure up to d-steps, we have the conclusion. *Q.E.D.*

The above result is consistent with the fact that in classical "probabilistic" model for finite discrete-time markets only the no-arbitrage condition and not the no-free lunch condition has been utilized.

S.3. COUNTABLY ADDITIVE MEASURES

In this section, we show that in general finite discrete time markets, it is possible to characterize viability through countably additive functionals. Also in this section, $\leq_\Omega$ denotes the pointwise order for functions. We prove this result by combining some results from Burzoni, Frittelli, Hou, Maggis, and Obłój (2019), which we collect in Section S.4.2. We refer to that paper for the precise technical requirements for $(\Omega, \mathbb{F}, S)$, we only point out that, in addition to the previous setting, Ω needs to be a Polish space.

We let $\mathcal{Q}^{\mathrm{ca}}$ be the set of countably additive positive probability measures $\mathbb{Q}$, with finite support, such that S is a $\mathbb{Q}$-martingale and $\mathcal{Z}^- := \{-Z^- \mid Z \in \mathcal{Z}\}$. For $X \in \mathcal{H}$, set

$$\mathcal{Z}(X) := \big\{Z \in \mathcal{Z}^- : \exists \ell \in \mathcal{I} \text{ such that } \mathcal{D}(X) + \ell \geq_\Omega X + Z\big\},$$

which is always nonempty when $\mathcal{D}(X)$, for example, $\forall X \in \mathcal{B}_b$. By the lattice property of $\mathcal{Z}$, if $\mathcal{D}(X) + \ell \geq_\Omega X + Z$ the same is true if we take $Z = Z^-$. From Theorem S.2.5, we know that, under no arbitrage, the attainment property holds and, hence, $\mathcal{Z}(X)$ is nonempty for every $X \in \mathcal{H}$. For $A \in \mathcal{F}$, we define

$$\mathcal{D}_A(X) := \inf\big\{c \in \mathbb{R} : \exists \ell \in \mathcal{I} \text{ such that } c + \ell(\omega) \geq X(\omega), \forall \omega \in A\big\},$$
$$\mathcal{Q}_A^{\mathrm{ca}} := \big\{\mathbb{Q} \in \mathcal{Q}^{\mathrm{ca}} : \mathbb{Q}(A) = 1\big\}.$$

We need the following technical result in the proof of the main theorem.

PROPOSITION S.3.1: *Suppose Assumption S.2.3 holds and the financial market has no arbitrages. Then, for every $X \in \mathcal{H}$ and $Z \in \mathcal{Z}(X)$, there exists $A_{X,Z}$ such that*

$$A_{X,Z} \subset \big\{\omega \in \Omega : Z(\omega) = 0\big\}, \tag{S.3.1}$$

and

$$\mathcal{D}(X) = \mathcal{D}_{A_{X,Z}}(X) = \sup_{\mathbb{Q} \in \mathcal{Q}^{\mathrm{ca}}_{A_{X,Z}}} \mathbb{E}_\mathbb{Q}[X].$$

Before proving this result, we state the main result of this section.

THEOREM S.3.2: *Suppose Assumption S.2.3 holds. Then the financial market has no arbitrages if and only if for every* $(Z,R)\in\mathcal{Z}^-\times\mathcal{P}^+$ *there exists* $\mathbb{Q}_{Z,R}\in\mathcal{Q}^{\mathrm{ca}}$ *satisfying*

$$\mathbb{E}_{\mathbb{Q}_{Z,R}}[R]>0 \quad \textit{and} \quad \mathbb{E}_{\mathbb{Q}_{Z,R}}[Z]=0. \tag{S.3.2}$$

PROOF: Suppose that the financial market has no arbitrages. Fix $(Z,R)\in\mathcal{Z}^-\times\mathcal{P}^+$ and $Z_R\in\mathcal{Z}(R)$. Set $Z^*:=Z_R+Z\in\mathcal{Z}(R)$. By Proposition S.3.1, there exists $A_*:=A_{R,Z^*}$ satisfying the properties listed there. In particular,

$$0<\mathcal{D}(R)=\sup_{\mathbb{Q}\in\mathcal{Q}^{\mathrm{ca}}_{A_*}}\mathbb{E}_{\mathbb{Q}}[R].$$

Hence, there is $\mathbb{Q}^*\in\mathcal{Q}^{\mathrm{ca}}_{A_*}$ so that $\mathbb{E}_{\mathbb{Q}^*}[R]>0$. Moreover, since $Z_R,Z\in\mathcal{Z}^-$,

$$A_*\subset\{Z^*=0\}=\{Z_R=0\}\cap\{Z=0\}.$$

In particular, $\mathbb{E}_{\mathbb{Q}^*}[Z]=0$.

To prove the opposite implication, suppose that there exists $R\in\mathcal{P}^+$, $\ell\in\mathcal{I}$ and $Z\in\mathcal{Z}$ such that $\ell\geq_\Omega R+Z$. Then it is clear that $\ell\geq_\Omega R-Z^-$. Let $\mathbb{Q}^*:=\mathbb{Q}_{-Z^-,R}\in\mathcal{Q}^{\mathrm{ca}}$ satisfying (S.3.2). By integrating both sides against $\mathbb{Q}^*$, we obtain

$$0=\mathbb{E}_{\mathbb{Q}^*}[\ell]\geq\mathbb{E}_{\mathbb{Q}^*}\big[R-Z^-\big]=\mathbb{E}_{\mathbb{Q}^*}[R]>0$$

which is a contradiction. Thus, there are no arbitrages. *Q.E.D.*

We continue with the proof of Proposition S.3.1.

PROOF OF PROPOSITION S.3.1: Since there are no arbitrages, by Theorem S.2.5 we have the attainment property. Hence, for a given $X\in\mathcal{H}$, the set $\mathcal{Z}(X)$ is nonempty.

Step 1. We show that, for any $Z\in\mathcal{Z}(X)$, $\mathcal{D}(X)=\mathcal{D}_{\{Z=0\}}(X)$.

Note that, since $\mathcal{D}(X)+\ell\geq_\Omega X+Z$, for some $\ell\in\mathcal{I}$, the inequality $\mathcal{D}_{\{Z=0\}}(X)\leq\mathcal{D}(X)$ is always true. Toward a contradiction, suppose that the inequality is strict, namely, there exist $c<\mathcal{D}(X)$ and $\tilde{\ell}\in\mathcal{I}$ such that $c+\tilde{\ell}(\omega)\geq X(\omega)$ for any $\omega\in\{Z=0\}$. We show that

$$\tilde{Z}:=(c+\tilde{\ell}-X)^-\chi_{\{Z<0\}}\in\mathcal{Z}.$$

This together with $c+\tilde{\ell}\geq_\Omega X+\tilde{Z}$ yields a contradiction. Recall that $\mathcal{Z}$ is a linear space so that $nZ\in\mathcal{Z}$ for any $n\in\mathbb{N}$. From $nZ\leq_\Omega\tilde{Z}\vee(nZ)\leq_\Omega 0$, we also have $\tilde{Z}_n:=\tilde{Z}\vee(nZ)\in\mathcal{Z}$, by Lemma S.4.1. By noting that $\{\tilde{Z}<0\}\subset\{Z<0\}$, we have that $\tilde{Z}_n(\omega)\to\tilde{Z}(\omega)$ for every $\omega\in\Omega$. From the closure of $\mathcal{Z}$, under pointwise convergence, we conclude that $\tilde{Z}\in\mathcal{Z}$.

Step 2. For a given set $A\in\mathcal{F}_T$, we let $A^*\subset A$ be the set of scenarios visited by martingale measures (see (S.4.2) in the Appendix for more details). We show that, for any $Z\in\mathcal{Z}(X)$, $\mathcal{D}(X)=\mathcal{D}_{\{Z=0\}^*}(X)$.

Suppose that $\{Z=0\}^*$ is a proper subset of $\{Z=0\}$ otherwise, from Step 1, there is nothing to show. From Lemma S.4.6, there is a strategy $\tilde{\ell}\in\mathcal{I}$ such that $\tilde{\ell}\geq 0$ on $\{Z=0\}$.[4]

[4]Note that restricted to $\{Z=0\}$ this strategy yields no risk and possibly positive gains, in other words, this is a good candidate for being an arbitrage.

Lemma S.4.5 (and in particular (S.4.4)) yields a finite number of strategies $\ell_1^t, \dots, \ell_{\beta_t}^t$ with $t = 1, \dots, T$, such that

$$\{\hat{Z} = 0\} = \{Z = 0\}^* \quad \text{where } \hat{Z} := Z - \sum_{t=1}^{T} \sum_{i=1}^{\beta_t} \chi_{\{Z=0\}} (\ell_i^t)^+. \tag{S.3.3}$$

Moreover, for any $\omega \in \{Z = 0\} \setminus \{Z = 0\}^*$, there exists (i, t) such that $\ell_i^t(\omega) > 0$. We are going to show that, under the no arbitrage hypothesis, $\ell_i^t \in \mathcal{Z}$ for any $i = 1, \dots, \beta_t$, $t = 1, \dots, T$. In particular, from the lattice property of the linear space $\mathcal{Z}$, we have $\hat{Z} \in \mathcal{Z}$.

We illustrate the reason for $t = T$, by repeating the same argument up to $t = 1$ we have the thesis. We proceed by induction on i. Start with $i = 1$. From Lemma S.4.5, we have that $\ell_i^T \geq 0$ on $\{Z = 0\}$ and, therefore, $\{\ell_1^T < 0\} \subseteq \{Z < 0\}$. Define $\tilde{Z} := -(\ell_1^T)^- \leq_\Omega 0$. By using the same argument as in Step 1, we observe that $nZ \leq_\Omega \tilde{Z} \vee (nZ) \leq_\Omega 0$ with $nZ \in \mathcal{Z}$ for any $n \in \mathbb{N}$. From $\{\ell_1^T < 0\} \subseteq \{Z < 0\}$ and the closure of $\mathcal{Z}$ under pointwise convergence, we conclude that $\tilde{Z} \in \mathcal{Z}$. From no arbitrage, we must have $\ell_1^T \in \mathcal{Z}$.

Suppose now that $\ell_j^T \in \mathcal{Z}$ for every $1 \leq j \leq i - 1$. From Lemma S.4.5, we have that $\ell_i^T \geq 0$ on $\{Z - \sum_{j=1}^{i-1} \ell_i^T = 0\}$ and, therefore, $\{\ell_i^T < 0\} \subseteq \{Z - \sum_{j=1}^{i-1} \ell_i^T < 0\}$. The argument of Step 1 allows to conclude that $\ell_i^T \in \mathcal{Z}$.

We are now able to show the claim. The inequality $\mathcal{D}_{\{Z=0\}^*}(X) \leq \mathcal{D}_{\{Z=0\}}(X) = \mathcal{D}(X)$ is always true. Toward a contradiction, suppose that the inequality is strict, namely, there exist $c < \mathcal{D}(X)$ and $\tilde{\ell} \in \mathcal{I}$ such that $c + \tilde{\ell}(\omega) \geq X(\omega)$ for any $\omega \in \{Z = 0\}^*$. We show that

$$\tilde{Z} := (c + \tilde{\ell} - X)^- \chi_{\Omega \setminus \{Z=0\}^*} \in \mathcal{Z}.$$

This together with $c + \tilde{\ell} \geq_\Omega X + \tilde{Z}$, yields a contradiction. To see this recall that, from the above argument, $\hat{Z} \in \mathcal{Z}$ with $\hat{Z}$ as in (S.3.3). Moreover, again by (S.3.3), we have $\{\tilde{Z} < 0\} \subset \{\hat{Z} < 0\}$. The argument of Step 1 allows to conclude that $\tilde{Z} \in \mathcal{Z}$.

Step 3. We are now able to conclude the proof. Fix $Z \in \mathcal{Z}(X)$ and set $A_{X,Z} := \{Z = 0\}^*$. Then

$$\mathcal{D}(X) = \mathcal{D}_{\{Z=0\}}(X) = \mathcal{D}_{(A_{X,Z})^*}(X) = \sup_{Q \in \mathcal{Q}^{\mathrm{ca}}_{A_{X,Z}}} \mathbb{E}_Q[X],$$

where the first two equalities follow from Step 1 and Step 2 and the last equality follows from Proposition S.4.7. *Q.E.D.*

S.4. SOME TECHNICAL TOOLS

S.4.1. *Preferences*

We start with a simple but a useful condition for negligibility.

LEMMA S.4.1: *Consider two negligible claims $\hat{Z}, \tilde{Z} \in \mathcal{Z}$. Then any claim $Z \in \mathcal{H}$ satisfying $\hat{Z} \leq Z \leq \tilde{Z}$ is negligible as well.*

PROOF: By definitions, we have

$$X \leq X + \hat{Z} \leq X + Z \leq X + \tilde{Z} \leq X \quad \Rightarrow \quad X \sim X + Z.$$

Thus, $Z \in \mathcal{Z}$. *Q.E.D.*

LEMMA S.4.2: *Suppose that $\mathcal{Z}$ is closed under pointwise convergence. Then $\mathcal{Z}$ is stable under multiplication, that is, $ZH \in \mathcal{Z}$ for any $H \in \mathcal{H}$.*

PROOF: Note first that $Z_n := Z((H \wedge n) \vee -n) \in \mathcal{Z}$. This follows from by Lemma S.4.1 and the fact that $\mathcal{Z}$ is a cone. By taking the limit for $n \to \infty$, the result follows. *Q.E.D.*

We next prove that $\mathcal{E}(Z) = 0$ for every $Z \in \mathcal{Z}$.

LEMMA S.4.3: *Let $\mathcal{E}$ be a sublinear expectation. Then*

$$\begin{aligned}\mathcal{E}\big(c + \lambda[X + Y]\big) &= c + \mathcal{E}\big(\lambda[X + Y]\big) = c + \lambda\mathcal{E}(X + Y) \\ &\leq c + \lambda\big[-\big(-\mathcal{E}(X) - \mathcal{E}(Y)\big)\big],\end{aligned} \tag{S.4.1}$$

for every $c \in$, $\lambda \geq 0$, $X, Y \in \mathcal{H}$. In particular,

$$\mathcal{E}(Z) = 0, \quad \forall Z \in \mathcal{Z}.$$

PROOF: Let $X, Y \in \mathcal{H}$. The subadditivity of $U_{\mathcal{E}}$ implies that

$$U_{\mathcal{E}}\big(X'\big) + U_{\mathcal{E}}\big(Y'\big) \leq U_{\mathcal{E}}\big(X' + Y'\big), \quad \forall X', Y' \in \mathcal{H},$$

even when they take values $\pm\infty$. The definition of $U_{\mathcal{E}}$ now yields

$$\mathcal{E}(X + Y) = -U_{\mathcal{E}}(-X - Y) \leq -\big[U_{\mathcal{E}}(-X) + U_{\mathcal{E}}(-Y)\big] = -\big(-\mathcal{E}(X) - \mathcal{E}(Y)\big).$$

Then (S.4.1) follows directly from the definitions.

Let $Z \in \mathcal{Z}$. Then $-Z, Z \in \mathcal{P}$ and $\mathcal{E}(Z), \mathcal{E}(-Z) \geq 0$. Since $-Z \in \mathcal{P}$, the monotonicity of $\mathcal{E}$ implies that $\mathcal{E}(X - Z) \geq \mathcal{E}(X)$ for any $X \in \mathcal{H}$. Choose $X = Z$ to arrive at

$$0 = \mathcal{E}(0) = \mathcal{E}(Z - Z) \geq \mathcal{E}(Z) \geq 0.$$

Hence, $\mathcal{E}(Z)$ is equal to zero. *Q.E.D.*

S.4.2. *Finite Time Markets*

We here recall some results from Burzoni et al. (2019) (see Section 2 therein for the precise specification of the framework). We are given a filtered space $(\Omega, \mathbb{F}, \mathcal{F})$ with Ω a Polish space and $\mathbb{F}$ containing the filtration generated by a Borel-measurable process S. We denote by $\mathcal{Q}$ the set of martingale measures for the process S, whose support is a finite number of points. For a given set $A \in \mathcal{F}$, $\mathcal{Q}_A = \{Q \in \mathcal{Q} \mid Q(A) = 1\}$. We define the set of scenarios charged by martingale measures as

$$A^* := \big\{\omega \in \Omega \mid \exists Q \in \mathcal{Q}_A \text{ s.t. } Q(\omega) > 0\big\} = \bigcup_{Q \in \mathcal{Q}_A} \operatorname{supp}(Q). \tag{S.4.2}$$

DEFINITION S.4.4: We say that $\ell \in \mathcal{I}$ is a one-step strategy if $\ell = H_t \cdot (S_t - S_{t-1})$ with $H_t \in \mathcal{L}(X, \mathcal{F}_{t-1})$ for some $t \in \{1, \dots, T\}$. We say that $a \in \mathcal{I}$ is a one-point Arbitrage on A iff $a(\omega) \geq 0$ $\forall \omega \in A$ and $a(\omega) > 0$ for some $\omega \in A$.

The following lemma is crucial for the characterization of the set A^* in terms of arbitrage considerations.

LEMMA S.4.5: *Fix any $t \in \{1, \dots, T\}$ and $\Gamma \in \mathcal{F}$. There exist an index $\beta \in \{0, \dots, d\}$, one-step strategies $\ell^1, \dots, \ell^\beta \in \mathcal{I}$ and $B^0, \dots, B^\beta$, a partition of Γ, satisfying:*

1. *if $\beta = 0$ then $B^0 = \Gamma$ and there are No one-point Arbitrages, that is,*

$$\ell(\omega) \geq 0 \quad \forall \omega \in B^0 \quad \Rightarrow \quad \ell(\omega) = 0 \quad \forall \omega \in B^0.$$

2. *if $\beta > 0$ and $i = 1, \dots, \beta$ then:*
   - ▷ $B^i \neq \emptyset$,
   - ▷ $\ell^i(\omega) > 0$ *for all* $\omega \in B^i$,
   - ▷ $\ell^i(\omega) \geq 0$ *for all* $\omega \in \bigcup_{j=i}^{\beta} B^j \cup B^0$.

We are now using the previous result, which is for some fixed t, to identify A^*. Define

$$\begin{aligned} A_T &:= A, \\ A_{t-1} &:= A_t \setminus \bigcup_{i=1}^{\beta_t} B_t^i, \quad t \in \{1, \dots, T\}, \end{aligned} \tag{S.4.3}$$

where $B_t^i := B_t^{i,\Gamma}$, $\beta_t := \beta_t^\Gamma$ are the sets and index constructed in Lemma S.4.5 with $\Gamma = A_t$, for $1 \leq t \leq T$. Note that, for the corresponding strategies ℓ_i^t we have

$$A_0 = \bigcap_{t=1}^{T} \bigcap_{i=1}^{\beta_t} \{\ell_i^t = 0\}. \tag{S.4.4}$$

LEMMA S.4.6: *A_0 as constructed in (S.4.3) satisfies $A_0 = A^*$. Moreover, No one-point Arbitrage on $A \Leftrightarrow A^* = A$.*

PROPOSITION S.4.7: *Let $A \in \mathcal{F}$. We have that for any $\mathcal{F}$-measurable random variable g,*

$$\pi_{A^*}(g) = \sup_{Q \in \mathcal{Q}_A} \mathbb{E}_Q[g] \tag{S.4.5}$$

with $\pi_{A^*}(g) = \inf\{x \in \mathbb{R} \mid \exists a \in \mathcal{I}$ such that $x + a_T(\omega) \geq g(\omega)\ \forall \omega \in A^*\}$. In particular, the left-Fnohand side of (S.4.5) is attained by some strategy $a \in \mathcal{I}$.

REFERENCES

BOUCHARD, B., AND M. NUTZ (2015): "Arbitrage and Duality in Nondominated Discrete-Time Models," *The Annals of Applied Probability*, 25, 823–859. [4]

BURZONI, M., M. FRITTELLI, Z. HOU, M. MAGGIS, AND J. OBŁÓJ (2019): "Pointwise Arbitrage Pricing Theory in Discrete Time," *Mathematics of Operations Research*, 44, 1034–1057. [6,9]

HARRISON, J. M., AND D. M. KREPS (1979): "Martingales and Arbitrage in Multiperiod Securities Markets," *Journal of Economic Theory*, 20, 381–408. [2]

KABANOV, Y., AND C. STRICKER (2001): "A Teacher's Note on No-Arbitrage Criteria," *Séminaire de probabilités de Strasbourg*, 35, 149–152. [4,5]

KREPS, D. M. (1981): "Arbitrage and Equilibrium in Economies With Infinitely Many Commodities," *Journal of Mathematical Economics*, 8, 15–35. [2]

Co-editor Bart Lipman handled this manuscript.

Manuscript received 23 July, 2018; final version accepted 23 November, 2020; available online 3 December, 2020.