Econometrica Supplementary Material

SUPPLEMENT TO "POLICY PERSISTENCE AND DRIFT IN ORGANIZATIONS"

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APPENDIX B: PROOFS (CONTINUOUS TIME LIMIT)

IN THIS SECTION, we assume m is C^2. We first define some useful objects. Say $s(x,t)$ is a policy mapping if $s(x,t+t') \equiv s(s(x,t'),t)$; $s(x,0) \equiv x$; and s is weakly decreasing in t. Given a policy mapping $s(x,t)$, denote $V(\alpha) = U_\alpha(S(m^{-1}(\alpha))) - u_\alpha(m^{-1}(\alpha))$ and $W(x) = U_{m(x)}(S(x)) - u_{m(x)}(x)$, where $S(x) = (s(x,t))_t$. Given a policy path S, denote $V_\alpha(S) = U_\alpha(S) - u_\alpha(m^{-1}(\alpha))$.

REMARK 1: If $s(x,t)$ is C^1 and decreasing in t, there are functions $d(x,y) : [x^*, x^{**})^2 \to \mathbb{R}$, $e(z) : [x^*, x^{**}) \to \mathbb{R}_+$ such that $s(x, d(x,y)) = y$ and $d(x,y) = \int_y^x e(z)\,dz$.

$d(x,y)$ measures the time it takes the policy path to get from x to y, if $x > y$ (if $x < y$, then $d(x,y) = -d(y,x)$). This time can be expressed as an integral of the instantaneous delay $e(z)$ at each policy z.

We first show that, if a CLS exists, it solves Equation (2). We restate it here:

$$e(x) = \frac{1}{r}\left(-\frac{\frac{\partial^2 u}{\partial x^2}}{\frac{\partial u}{\partial x}} - 2\frac{\frac{\partial^2 u}{\partial \alpha \partial x}}{\frac{\partial u}{\partial x}} m'(x) - \frac{\frac{\partial^2 u}{\partial \alpha^2}}{\frac{\partial u}{\partial x}} m'(x)^2 + \frac{\frac{\partial^2 U_{m(x)}\big(S(x)\big)}{\partial \alpha^2}}{\frac{\partial u}{\partial x}} m'(x)^2 + \frac{m''(x)}{m'(x)} \right). \quad (*)$$

LEMMA 5: *Let $x_0 \in (x^*, x^{**})$. If a policy mapping $s(x,t)$ is such that $W(x) = 0$ for all x in a neighborhood of x_0, then $d(x_0, x)$ is differentiable with respect to its second argument at (x_0, x_0), and $e(x) = -\frac{\partial d(x_0,x)}{\partial x}$ is given by Equation* (*).

PROOF: First, assume a C^1 policy mapping $s(x,t)$. Denote $n(\alpha) = m^{-1}(\alpha)$ and $\alpha_0 = m(x_0)$. By the envelope theorem,

$$V'(\alpha_0) = \frac{\partial U_\alpha\big(S(x)\big)}{\partial \alpha}\bigg|_{\alpha_0, x_0} - \frac{du_\alpha\big(n(\alpha)\big)}{d\alpha}\bigg|_{\alpha_0},$$

$$V''(\alpha_0) = \frac{\partial^2 U_\alpha\big(S(x)\big)}{\partial \alpha^2}\bigg|_{\alpha_0, x_0} + n'(\alpha_0)\,\mathrm{re}(x_0)\left[\frac{\partial u_\alpha(x)}{\partial \alpha}\bigg|_{\alpha_0, x_0} - \frac{\partial U_\alpha\big(S(x)\big)}{\partial \alpha}\bigg|_{\alpha_0, x_0}\right]$$
$$- \frac{\partial^2 u}{\partial \alpha^2} - 2n'\frac{\partial^2 u}{\partial \alpha \partial x} - n'^2\frac{\partial^2 u}{\partial^2 x} - n''\frac{\partial u}{\partial x}.$$

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We can use the fact that $V(\alpha) \equiv 0$ in a neighborhood of α_0, and hence $V' \equiv V'' \equiv 0$, to determine $e(x)$:

$$
\begin{aligned}
0 = V'(\alpha_0) &= \frac{\partial U_\alpha\big(S(x)\big)}{\partial\alpha}\bigg|_{\alpha_0,x_0} - \frac{\partial u_\alpha(x)}{\partial\alpha}\bigg|_{\alpha_0,x_0} - n'(\alpha)\frac{\partial u_\alpha(x)}{\partial x}\bigg|_{\alpha_0,x_0} \\
\implies \quad & \mathrm{re}(x)\left[-\frac{1}{m'(x)}\frac{\partial u_{m(x)}(x)}{\partial x}\right] \\
&= \mathrm{re}(x)\left[\frac{\partial u_{m(x)}(x)}{\partial\alpha} - \frac{\partial U_{m(x)}\big(S(x)\big)}{\partial\alpha}\right] \\
&= -m'(x)\frac{\partial^2 U_{m(x)}\big(S(x)\big)}{\partial\alpha^2} + m'(x)\frac{d^2 u_\alpha\big(n(\alpha)\big)}{d\alpha^2}|_{\alpha_0} \\
&= -m'(x)\frac{\partial^2 U_{m(x)}\big(S(x)\big)}{\partial\alpha^2} + \frac{1}{m'(x)}\frac{\partial^2}{\partial x^2}u \\
&\quad + 2\frac{\partial^2}{\partial\alpha\partial x}u + m'(x)\frac{\partial^2}{\partial\alpha^2}u - \frac{m''(x)}{m'(x)^2}\frac{\partial}{\partial x}u.
\end{aligned}
$$

Now we show that d must be differentiable at (x_0, x_0). Let $(z_n)_n$ be a sequence such that $z_n \to x_0$. WLOG assume $z_n < x_0$ for all n. Note that

$$
\begin{aligned}
&\frac{\partial U_{m(x_0)}\big(S(x_0)\big)}{\partial\alpha}\bigg|_{m(x_0)} - \frac{\partial U_{m(x_0)}\big(S(z_n)\big)}{\partial\alpha} \\
&= \int_0^\infty \mathrm{re}^{-rt}\left[\frac{\partial u_{m(x_0)}\big(s(x_0,t)\big)}{\partial\alpha} - \frac{\partial u_{m(x_0)}\big(s(z_n,t)\big)}{\partial\alpha}\right] dt \\
&= \int_0^{d(x_0,z_n)} \mathrm{re}^{-rt}\frac{\partial}{\partial\alpha}u_{m(x_0)}\big(s(x_0,t)\big)\,dt - \big(1-\mathrm{re}^{-rd(x_0,z_n)}\big)\frac{\partial U_{m(x_0)}\big(S(z_n)\big)}{\partial\alpha} \\
&= \big(1-\mathrm{re}^{-rd(x_0,z_n)}\big)\left(\frac{\partial u_{m(x_0)}(\tilde{x})}{\partial\alpha} - \frac{\partial U_{m(x_0)}\big(S(z_n)\big)}{\partial\alpha}\right)
\end{aligned}
$$

for some $\tilde{x} \in (z_n, x_0)$. Then

$$
\begin{aligned}
0 &= V'\big(m(x_0)\big) - V'\big(m(z_n)\big) \\
&= \frac{\partial U_\alpha\big(S(x)\big)}{\partial\alpha}\bigg|_{m(x_0),x_0} - \frac{du_\alpha\big(n(\alpha)\big)}{d\alpha}\bigg|_{m(x_0)} - \left(\frac{\partial U_\alpha\big(S(x)\big)}{\partial\alpha}\bigg|_{m(z_n),z_n} - \frac{du_\alpha\big(n(\alpha)\big)}{d\alpha}\bigg|_{m(z_n)}\right) \\
&= \frac{\partial U_\alpha\big(S(x)\big)}{\partial\alpha}\bigg|_{m(x_0),x_0} - \frac{\partial U_\alpha\big(S(x)\big)}{\partial\alpha}\bigg|_{m(x_0),z_n} + \frac{\partial U_\alpha\big(S(x)\big)}{\partial\alpha}\bigg|_{m(x_0),z_n} - \frac{\partial U_\alpha\big(S(x)\big)}{\partial\alpha}\bigg|_{m(z_n),z_n} \\
&\quad - \big(m(x_0) - m(z_n)\big)\frac{d^2 u_\alpha\big(n(\alpha)\big)}{d\alpha^2}\bigg|_{\tilde{\alpha}}
\end{aligned}
$$

$$= \left(1 - e^{-rd(x_0,z_n)}\right)\left(\frac{\partial u_{m(x_0)}(\tilde{x})}{\partial \alpha} - \frac{\partial U_{m(x_0)}\big(S(z_n)\big)}{\partial \alpha}\right) + \big(m(x_0) - m(z_n)\big)\frac{\partial^2 U_\alpha\big(S(x)\big)}{\partial \alpha^2}\Bigg|_{\hat{\alpha},z_n}$$

$$- \big(m(x_0) - m(z_n)\big)\frac{d^2 u_\alpha\big(n(\alpha)\big)}{d\alpha^2}\Bigg|_{\tilde{\alpha}},$$

$$0 = \frac{\partial^2 U_\alpha\big(S(x)\big)}{\partial \alpha^2}\Bigg|_{\hat{\alpha},z_n} + \frac{1 - e^{-rd(x_0,z_n)}}{m(x_0) - m(z_n)}\left(\frac{\partial u_{m(x_0)}(\tilde{x})}{\partial \alpha} - \frac{\partial U_{m(x_0)}\big(S(z_n)\big)}{\partial \alpha}\right) - \frac{d^2 u_\alpha\big(n(\alpha)\big)}{d\alpha^2}\Bigg|_{\tilde{\alpha}}.$$

This implies that $\lim_{n\to\infty} \frac{1-e^{-rd(x_0,z_n)}}{m(x_0)-m(z_n)} = n'(m(x_0))re(x_0)$, which in turn implies that $\lim_{n\to\infty} \frac{d(x_0,z_n)}{x_0-z_n} = e(x_0)$, as we wanted. *Q.E.D.*

LEMMA 6: *Equation (*) has a unique solution, in the following sense: for any $x_1 > x_0 \geq x^*$ and given a candidate path $S(x_0)$, there is at most one way to choose $e : (x_0, x_1) \to \mathbb{R}_{\geq 0}$ so that Equation (*) holds for all $x \in (x_0, x_1)$.*

PROOF: Let $\tilde{g}(x) = re(x)[-\frac{1}{m'(x)}\frac{\partial u_{m(x)}(x)}{\partial x}]$ and $g(x) = \max(\tilde{g}(x), 0)$. The issue is that Equation (*) is an integral equation, since $\frac{\partial^2 U_{m(x)}(S(x))}{\partial\alpha\partial x}$ is an integral that depends on $S(x)$, which depends on $g(x')$ for $x' < x$. We prove the result for the case $x_0 = x^* < x_1$, but other cases are analogous.

Given $x_1 \in (x^*, x^{**})$, let $\mathcal{C}_{x_1} = \{h : [x^*, x_1] \to \mathbb{R}_{\geq 0} \text{ continuous}\}$ with the norm $\|h\|_\infty$, and define $T_{x_1} : \mathcal{C}_{x_1} \to \mathcal{C}_{x_1}$ as follows:

$$T_{x_1}(g)(x) = \max\left(-m'(x)\frac{\partial^2 U_{m(x)}\big(S(x)\big)}{\partial \alpha^2} + m'(x)\frac{d^2 u_{m(x)}(x)}{dm(x)^2}, 0\right)$$

for $x \in [x^*, x_1]$. Let $g_1, g_2 \in \mathcal{C}_{x_1}$ with $\|g_1 - g_2\| \leq K$, and, for each $x_2 \in (x^*, x_1)$, define g_{x_2} by: $g_{x_2}(x) = g_1(x)$ if $x \leq x_2$ and $g_{x_2}(x) = g_2(x)$ otherwise. Then

$$\begin{aligned}
&\big|T_{x_1}(g_1)(x) - T_{x_1}(g_2)(x)\big| \\
&\quad\leq m'(x)\left|-\frac{\partial^2 U_{m(x)}\big(S_{g_1}(x)\big)}{\partial \alpha^2} + \frac{\partial^2 U_{m(x)}\big(S_{g_2}(x)\big)}{\partial \alpha^2}\right| \\
&\quad= m'(x)\left|\int_{x^*}^{x} \frac{\partial}{\partial x_2}\left(\frac{\partial^2 U_{m(x)}\big(S_{g_{x_2}}(x)\big)}{\partial \alpha^2}\right) dx_2\right| \\
&\quad= m'(x)\left|\int_{x^*}^{x} \mathrm{r}e^{-rd(x,x_2)}\,\frac{m'(x_2)}{r\frac{\partial u_{m(x)}(x)}{\partial x}}\big(g_1(x_2) - g_2(x_2)\big)\right. \\
&\qquad\left.\times\left[\frac{\partial^2 U_{m(x)}(x_2)}{\partial \alpha^2} - \frac{\partial^2 U_{m(x)}\big(S_{g_{x_2}}(x)\big)}{\partial \alpha^2}\right] dx_2\right| \\
&\quad\leq \int_{x^*}^{x} \frac{\overline{m}'^2 KL\big(x_2 - x^*\big)}{M'\big(x_2 - m(x_2)\big)}\,dx_2 \leq \int_{x^*}^{x} \frac{\overline{m}'^2 KL\big(x_2 - x^*\big)}{M'\big(x_2 - x^*\big)\big(1 - m'\big(\tilde{x}(x_2)\big)\big)}\,dx_2 \leq \mathrm{KC}\big(x - x^*\big)
\end{aligned}$$

for some constant $C > 0$.[1] If x_1 is close enough to x^*, $C(x - x^*) < 1$ and hence T_{x_1} is a contraction. Thus g (and hence e) is uniquely determined in a neighborhood of x^*. By repeating the same argument, we can extend the solution uniquely on any interval (x^*, x) where $e(x') > 0$ for all $x \in (x^*, x)$. *Q.E.D.*

PROOF OF PROPOSITION 6: For (i), the uniqueness is proven by Lemmas 5 and 6; that s is C^1 follows from the fact that the RHS of Equation (*) is continuous. Part (ii) will be proven as part of Proposition 8.

For (iii), suppose not. Then there is a sequence $(\delta_n)_n$ with $\delta_n \to 1$ and a sequence of Q1Es s_n for each δ_n, such that s_n is not a 1E for all n. By part (ii), we know that $s_n(x, t) \to s(x, t)$ for all x, t. Suppose that, for each n, there is x^n_k for which $S_n(x^n_{k+1})$ is not a Condorcet winner in $I(x^n_k)$ because a strict majority strictly prefers $S_n(y_n)$, and assume $x^n_k \to x$ and $y_n \to y$. Note that $y_n \leq x^n_{k+1}$, as otherwise all agents to the left of $m(x^n_k)$ and some to the right would strictly prefer $S_n(x^n_k)$ over $S_n(y_n)$; and thus $y \leq x$. If $x \in (x^*, x^{**})$ and $y < x$, this leads to a contradiction as $U_\alpha(S_n(y_n)) \to U_\alpha(S(y))$ and $U_\alpha(S_n(x^n_k)) \to U_\alpha(S(x))$ for all α, and $U_\alpha(S(x)) > U_\alpha(S(y))$ for all $\alpha \in (m(x) - \epsilon, x + d^+_x)$ for some $\epsilon > 0$. If $y = x$, suppose $y_n \in (x^n_{k+l_n}, x^n_{k+l_n-1})$, where $l_n \geq 2$. It is clear that $U_\alpha(S_n(x^n_{k+1})) > U_\alpha(S_n(x^n_{k+l_n-1})) > U_\alpha(S_n(y_n))$ for all $\alpha \in (x^n_{k+l_n-1}, x^n_k + d^+_{x^n_k})$ and $U_\alpha(S_n(x^n_{k+1})) > U_\alpha(S_n(x^n_{k+l_n})) > U_\alpha(S_n(y_n))$ for all $\alpha \in (m(x), x^n_{k+l_n})$, so it must be that some $\alpha_n \in (x^n_{k+l_n-1}, x^n_{k+l_n})$ prefers $S_n(y_n)$ to $S_n(x^n_{k+1})$. But then

$$
\begin{aligned}
0 &\geq \frac{1-\delta}{1-\delta^{l_n-1}}\big[U_{\alpha_n}\big(S_n\big(x^n_{k+1}\big)\big) - U_{\alpha_n}\big(S_n(y_n)\big)\big] = \\
&= \frac{1-\delta}{1-\delta^{l_n-1}}\left[\sum_{t=0}^{l_n-1}\delta^t u_{\alpha_n}\big(x^n_{k+1+t}\big) - u_{\alpha_n}(y_n) - \big(1-\delta^{l_n-1}\big)\sum_{t=1}^{\infty}\delta^t u_{\alpha_n}\big(x^n_{k+l_n+t}\big)\right] = \\
&= \frac{1-\delta}{1-\delta^{l_n-1}}\left[u_{\alpha_n}\big(x^n_{k+1}\big) - u_{\alpha_n}(y_n) + \sum_{t=1}^{l_n-1}\delta^t u_{\alpha_n}\big(x^n_{k+1+t}\big) - \big(1-\delta^{l_n-1}\big)\delta U_{\alpha_n}\big(S\big(x^n_{k+l_n+1}\big)\big)\right] \\
&\xrightarrow[n\to\infty]{} 0 + \delta\big[u_x(x) - (1-\delta)U_x\big(S(x)\big)\big] > 0,
\end{aligned}
$$

a contradiction.

An analogous proof can be written if $x = x^*$ after a normalization argument. Briefly, if $x = x^*$, assume WLOG that $x^* = 0$ to simplify notation, and denote $T_n(y) = x^n_k y$ and $U^n_\alpha(y) = U_{\alpha x^n_k}(\alpha x^n_k) - \frac{1}{(x^n_k)^2}(U_{\alpha x^n_k}(\alpha x^n_k) - U_{\alpha x^n_k}(y x^n_k))$. In the normalized version of the problem, x^n_k maps to $y^n_k = 1 > 0$ and we can apply the above arguments. The case $x = x^{**}$ is similar. *Q.E.D.*

PROOF OF PROPOSITION 7: WLOG assume $r = 1$. Suppose that there is a CLS with $e(x) \geq A$ for all $x \leq x_0$. Take $D > 0$ fixed, and let $L > 0$ be such that, for all $\alpha, x, x' \in [-1, 1]$, $|\frac{\partial u^2_\alpha(x)}{\partial \alpha^2} - \frac{\partial u^2_\alpha(x')}{\partial \alpha^2}|, |\frac{\partial u^2_\alpha(x)}{\partial \alpha \partial x} - \frac{\partial u^2_\alpha(x')}{\partial \alpha \partial x}|, |\frac{\partial u^2_\alpha(x)}{\partial x^2} - \frac{\partial u^2_\alpha(x')}{\partial x^2}| \leq L|x - x'| + D$. (For any D, such L exists because u is C^2.) Note then that $\frac{\partial^2 u_y(y)}{\partial x^2} \equiv -\frac{\partial^2 u_y(y)}{\partial \alpha \partial x} \in [M', M]$; $|\frac{\partial^2 u_{m(y)}(y)}{\partial \alpha \partial x} - \frac{\partial^2 u_{m(y)}(m(y))}{\partial \alpha \partial x}|, |\frac{\partial^2 u_{m(y)}(y)}{\partial x^2} - \frac{\partial^2 u_{m(y)}(m(y))}{\partial x^2}| \leq L(y - m(y)) + D$; and $|\frac{\partial u_{m(y)}(y)}{\partial x}| \in [M'(y -$

[1] L is a Lipschitz constant for $\frac{\partial^2 u}{\partial \alpha^2}$, and $\overline{m}' = \sup_x m'(x)$. The argument still goes through if we only require $\frac{\partial^2 u}{\partial \alpha^2}$ to be Hölder continuous for some positive exponent.

$m(y)), M(y-m(y))]$. In addition, $|x-s(x,t)| \leq \frac{t}{A}$ for all t, so $|\frac{\partial^2 \max(u_{m(x)}(s(x,t)),0)}{\partial\alpha^2} - \frac{\partial^2 u_{m(x)}(x)}{\partial\alpha^2}| \leq \frac{\tilde{L}t}{A} + D$, where $\tilde{L} = \max(L, \frac{\max|\frac{\partial^2 u}{\partial\alpha^2}|}{\min d_{\overline{y}}})$. In turn, this means that

$$\left|\frac{\partial^2 U_{m(x)}\big(S(x)\big)}{\partial\alpha^2} - \frac{\partial^2 u_{m(x)}(x)}{\partial\alpha^2}\right| \leq \int_0^\infty e^{-t}\left(\frac{\tilde{L}t}{A} + D\right)dt = \frac{\tilde{L}}{A} + D.$$

Putting this all together, by Equation (*),

$$e(x_0) \geq \frac{\big(2m'(x_0)-1\big)M' - \big(2m'(x_0)+1\big)(LB+D)}{MB} - \frac{\left(1+B'\right)^2\tilde{L}}{AMB} - \frac{\left(1+B'\right)^2 D}{MB} - \frac{B''}{1-B'} \overset{?}{\geq} A.$$

We now choose $A = \sqrt{\frac{(1+B')^2\tilde{L}}{MB}}$. Then it is enough if

$$\frac{\big(1-2B'\big)M' - \big(3+2B'\big)(LB+D)}{MB} - \frac{\left(1+B'\right)^2 D}{MB} - \frac{B''}{1-B'} \geq 2\sqrt{\frac{\left(1+B'\right)^2\tilde{L}}{MB}}.$$

Choose any $B' < \frac{1}{2}$ and any B''. Choose D such that $(1-2B')M' > (3+2B')D + (1+B')^2 D$. Then this condition holds for B small enough. We can show with a similar argument that, under these parameter conditions, $e(x) \underset{x\to x^*}{\longrightarrow} \infty$, so the unique solution to Equation (*) must satisfy $e(x) \geq A\ \forall x$ by an argument similar to Lemma 6. *Q.E.D.*

We now define a (not necessarily continuous) limit solution (LS) as a policy mapping s such that

(i) $x = \arg\max_{y\in[-1,1]} U_{m(x)}(S(y))\ \forall x \in [x^*, x^{**})$.
(ii) If there is $c > 0$ s.t. $W(x_0 - \epsilon) = 0\ \forall \epsilon \in [0, c]$, then $d(x_0^+, x_0) = 0$.
(iii) If $W(x_0) = 0$ and $W(x') > 0$ for all x' in a left-neighborhood of x_0, then $d(x_0^+, x_0^-)$ satisfies[2]

$$e^{\frac{rd(x_0^+, x_0^-)}{2}} = 1 + m'(x_0)\frac{\frac{\partial V_{m(x_0)}\big(S\big(x_0^-\big)\big)}{\partial\alpha}}{\frac{\partial u_{m(x_0)}(x_0)}{\partial x}}.$$

This definition is backward-engineered so that the transition path generated by an LS will be the limit of Q1E transition paths as $\delta \to 1$. Property (i) requires that an LS has to be as if agents could choose their preferred continuation. Property (ii) says that, if in a left-neighborhood of x_0 pivotal agents are indifferent between the LS transition path and a constant path, then the policy path cannot stop at x_0 for a positive length of time. The significance of (iii) is that non-CLS transition paths will have intervals in which $W(x) > 0$ and, as a result, the policy moves quickly (in the limit, instantaneously) through such intervals. At a point x_0 where W hits 0 again, the transition path has to slow down dramatically (in the limit, stop for some time $d(x_0^+, x_0^-)$) in order to increase the average policy of the

[2]We denote $f(x^-) = \lim_{t\nearrow x} f(x)$ and $f(x^+) = \lim_{t\searrow x} f(x)$.

path so that W' can have a kink and $W'(x_0^+)$ can be nonnegative. But Q1Es have "inertia": when they slow down, they do so for long enough that the average policy increases enough that $W'(x_0^+) > 0$. Property (iii) requires the correct value of $d(x_0^+, x_0^-)$ to match the behavior of Q1E transition paths around such points.

The following properties, defined jointly for the parameters u, m and a policy mapping s, will help us to ascertain the properties of an LS:

B2.1 $m'(x^*) > \frac{1}{2}$. (This implies that $e(x) > 0$ for x in a neighborhood of x^*.)[3]

B2.1′ $V_{m(x)}(S(x)) \equiv 0$ and there is $K > 0$ s.t. $e(x) \geq K$ for all $x \in (x^*, x^{**})$.

B2.2 u, m are C^3, and there is no point $x \in [x^*, x^{**})$ for which $V_{m(x)}(S(x)) = \frac{\partial V_{m(x)}(S(x))}{\partial \alpha} = \frac{\partial^2 V_{m(x)}(S(x))}{\partial \alpha^2} = \frac{\partial^3 V_{m(x)}(S(x))}{\partial \alpha^3} = 0$.

B2.3 For all $x \in (x^*, x^{**})$ such that $W(x) = 0$ and $W(x') > 0$ for $x' < x$ arbitrarily close to x, $W'(x^-) < 0$. We refer to such points x as *vertex points*.

Conditions B2.1 and B2.1′ are not generic, but hold in an open set. Conditions B2.2 and B2.3 are generic conditions under the assumption that u, m are C^3; this is shown in Appendix C. On an intuitive level, B2.3 requires that, at points where an interval of fast policy change ends (that is, where W hits 0), the derivative of W does not happen to also equal zero; Condition B2.2 is a similar but weaker condition involving higher-order derivatives of W.

We will now build toward a characterization of LS.

LEMMA 7: *If any LS $s(x, t)$ satisfying Conditions* B2.2 *and* B2.3 *is such that $W(x) > 0$ for some $x \in (x^*, x^{**})$, then there are sequences $(y_l)_{l \in \mathbb{N}_{\geq 0}}$, $(e_l)_{l \in \mathbb{N}_{\geq 1}}$ such that $(y_l)_l$ is strictly increasing in l; $W(y_l) = 0$ for all l; $W(y) > 0$ for all $y \in (y_l, y_{l+1})$ for any l; $d(y_{l+1}^-, y_l^+) = 0$; $d(y_l^+, y_l^-) = e_l$; $W'(y_l^+) > 0 > W'(y_l^-)$ for all $l \geq 1$;*

$$e^{\frac{re_l}{2}} = 1 + m'(y_l) \frac{\dfrac{\partial V_{m(y_l)}(S(y_l^-))}{\partial \alpha}}{\dfrac{\partial u_{m(y_l)}(y_l)}{\partial x}}$$

for all l; $y_l \to x^{**}$ as $l \to \infty$; and $W(z) = 0$ for $z < y_0$.

PROOF: Let (a, b) be the largest interval containing x such that $W(y) > 0$ for all $y \in (a, b)$, and denote $a = y_0$, $b = y_1$. That $d(x, x') = 0$ for all $x' < x \in (a, b)$ follows from the following argument. Take $x \in (a, b)$. Since $W(x) > 0$, there is $\tilde{x} \in (m(x), x)$ such that $u_{m(x)}(\tilde{x}) = U_{m(x)}(S(x)) > u_{m(x)}(x)$. Then $d(x, \tilde{x}) = 0$, as otherwise we would have $U_{m(x)}(S(\tilde{x})) > U_{m(x)}(S(x))$, contradicting the definition of a LS. Now suppose $d(x, x') > 0$ for some $x' < x \in (a, b)$. Construct a decreasing sequence $x = \tilde{x}_0 > \tilde{x}_1 > \tilde{x}_2 > \cdots$ such that, for all n, $u_{m(\tilde{x}_n)}(\tilde{x}_{n+1}) = U_{m(\tilde{x}_n)}(S(\tilde{x}_n))$ and $d(\tilde{x}_n, \tilde{x}_{n+1}) = 0$ per the above argument. Let $\tilde{x}_\infty = \lim \tilde{x}_n$. If $\tilde{x}_\infty < x'$, we have a contradiction and the proof is done. If not, it follows by continuity that $u_{m(\tilde{x}_\infty)}(\tilde{x}_\infty) = U_{m(\tilde{x}_\infty)}(S(\tilde{x}_\infty))$, that is, $W(\tilde{x}_\infty) = 0$, a contradiction.

Let $d(b^+, b^-) = e_1$. That e_1 is as required follows from the definition of LS. Note that Condition B2.3 implies that $e_1 > 0$. In addition, $W'(b^+) > 0$. To see this, in general let

[3]As seen in Lemma 5, $\frac{1}{m'(x^*)} \frac{\partial^2}{\partial x^2} u|_{x^*,x^*} + 2 \frac{\partial^2}{\partial \alpha \partial x} u|_{x^*,x^*} > 0$ is enough to guarantee that $e(x) > 0$ for x close to x^*—in fact, the condition guarantees $e(x) \geq \frac{C}{x - x^*}$ for some $C > 0$. In addition, Assumptions A1, A4 imply that $\frac{\partial^2}{\partial x^2} u|_{x^*,x^*} = -\frac{\partial^2}{\partial \alpha \partial x} u|_{x^*,x^*}$.

$\epsilon_l = \frac{\partial}{\partial\alpha} V_{m_l}(S(y_l^-))$ and $\epsilon_l' = \frac{\partial}{\partial\alpha} V_{m_l}(S(y_l^+))$, and suppose $\epsilon_l < 0$ as per Condition B2.3. Then

$$\epsilon_l' = \frac{\partial V_{m_l}(S(y_l^+))}{\partial\alpha} = e^{-\mathrm{re}_l}\left(\frac{\partial V_{m_l}(S(y_l^-))}{\partial\alpha}\right) + \left(1 - e^{-\mathrm{re}_l}\right)\left(-\frac{1}{m'(x)}\frac{\partial u_{m(x)}(x)}{\partial x}\right)$$

$$= \frac{\dfrac{\partial V_{m_l}(S(y_l^-))}{\partial\alpha}\dfrac{1}{m'(x)}\dfrac{\partial u_{m(x)}(x)}{\partial x}}{-\dfrac{\partial V_{m_l}(S(y_l^-))}{\partial\alpha} - \dfrac{1}{m'(x)}\dfrac{\partial u_{m(x)}(x)}{\partial x}} = -\frac{\partial V_{m_l}(S(y_l^-))}{\partial\alpha}\frac{1}{1 + \dfrac{\dfrac{\partial V_{m_l}(S(y_l^-))}{\partial\alpha}}{\dfrac{1}{m'(x)}\dfrac{\partial u_{m(x)}(x)}{\partial x}}}$$

$$= -\epsilon_l \frac{1}{1 + \dfrac{\epsilon_l}{\dfrac{1}{m'(x)}\dfrac{\partial u_{m(x)}(x)}{\partial x}}} > 0.$$

This implies that there is $y_2 > y_1$ such that $W(y) > 0$ for $y \in (y_1, y_2)$, with $W(y_2) = 0$ and $W'(y_2^-) < 0$, and so on.

Next we argue that $y_l \to x^{**}$ as $l \to +\infty$. Suppose instead that $y_l \to y^* < x^{**}$, and let $m_l = m(y_l)$, $m^* = m(y^*)$. Since V is continuous, $V_{m(y^*)}(S(y^*)) = 0$. In addition, $\frac{\partial}{\partial\alpha} V_{m^*}(S(y^{*-}))$ must equal zero.[4]

Suppose, then, that $\frac{\partial^2}{\partial\alpha^2} V_{m^*}(S(y^{*-})) \neq 0$. If this is positive, we have $V_{m(x)}(S(x)) > 0$ for all $x < y^*$ in a neighborhood of y^*, a contradiction.

If it is negative, we will obtain a contradiction by showing that $(\epsilon_l)_l$ cannot go fast enough to 0 for $(y_l)_l$ to converge. Note that $\epsilon_l < 0 < \epsilon_l'$ and $\epsilon_l + \epsilon_l' \in \mathcal{O}(\epsilon_l^2)$ since $\epsilon_l' = -\epsilon_l \frac{1}{1 + \frac{-\epsilon_l}{-\frac{1}{m'(x)}\frac{\partial u_{m(x)}(x)}{\partial x}}}$, as shown above, and $-\frac{1}{m'(x)}\frac{\partial u_{m(x)}(x)}{\partial x}$ is bounded away from 0 in a neighborhood of y^*. Next, we argue that $\epsilon_{l+1} = \epsilon_l + \mathcal{O}(\epsilon_l^2)$.

Let $N(\alpha) = \frac{\partial V_\alpha(S(m^{-1}(\alpha)))}{\partial\alpha}$ and $M(\alpha) = \frac{\partial^2 V_\alpha(S(m^{-1}(\alpha)))}{\partial\alpha^2}$. We claim that M is left-continuous at m^*—indeed, for this to not be the case, we would require $\sum_l e_l = +\infty$, which implies M is not bounded in a neighborhood of y^*, a contradiction. Thus, since $M(m^*) < 0$, $M(\alpha) < 0$ for all $\alpha < m^*$ in a neighborhood of m^*.

Let $\overline{M}_l = \max_{\alpha \in (\alpha_l, \alpha_{l+1})} -M(\alpha)$, $\underline{M}_l = \min_{\alpha \in (\alpha_l, \alpha_{l+1})} -M(\alpha)$. Note that $\overline{M}_l - \underline{M}_l \leq L(m_{l+1} - m_l)$ for some fixed constant L, that is, $\overline{M}_l - \underline{M}_l \in \mathcal{O}(m_{l+1} - m_l)$.[5] Since $V_{m_l} = V_{m_{l+1}} = 0$,

$$0 = \int_{m_l}^{m_{l+1}} N(\alpha) = N(m_l^+)(m_{l+1} - m_l) + \int_{m_l}^{m_{l+1}} M(\alpha)(m_{l+1} - \alpha),$$

[4]Indeed, if this derivative is negative, it follows that $V_{m(x)}(S(x)) \geq V_{m(x)}(S(y^{*-})) > 0$ for all $x < y^*$ in a neighborhood of y^*, contradicting that $y_l \to y^*$. If it is positive, then $V_{m(x)}S(y^{*-}) \leq -c(y^* - x)$ for x in such a neighborhood and some $c > 0$. From the fact that $V_{m(x)}S(x) \geq 0$ and $0 = V_{m^*}S(y^*) \geq V_{m(x)}S(y^*)$, it then follows that $E(S(y^*)) - E(S(x)) \geq c' > 0$ for all $x < y^*$, which is impossible.

[5]This follows from the assumption that u is C^3.

where $N(m_l^+) = \epsilon_l'$. This implies

$$\underline{M}_l \frac{(m_{l+1}-m_l)^2}{2} \leq \epsilon_l'(m_{l+1}-m_l) \leq \overline{M}_l \frac{(m_{l+1}-m_l)^2}{2},$$
$$\underline{M}_l \frac{m_{l+1}-m_l}{2} \leq \epsilon_l' \leq \overline{M}_l \frac{m_{l+1}-m_l}{2}.$$

Now $\epsilon_{l+1} = \epsilon_l' + \int_{m_l}^{m_{l+1}} M(\alpha) = \epsilon_l' - (m_{l+1}-m_l)\tilde{M}$, for some $\tilde{M} \in (\underline{M}_l, \overline{M}_l)$. From the above, $\epsilon_l' = \tilde{M}_l \frac{m_{l+1}-m_l}{2} + \mathcal{O}((m_{l+1}-m_l)^2)$. Then $\epsilon_{l+1} = -\epsilon_l' + \mathcal{O}((m_{l+1}-m_l)^2)$. In addition, it follows that $\mathcal{O}(\epsilon_l') = \mathcal{O}(m_{l+1}-m_l)$. Since $\epsilon_l' = -\epsilon_l + \mathcal{O}(\epsilon_l^2)$, we have that $\epsilon_{l+1} = \epsilon_l + \mathcal{O}(\epsilon_l^2)$, that is, $(\epsilon_l)_l$ at most decays (or grows) at the rate of a harmonic series, whence $\sum_l \epsilon_l = \infty$. Since $\epsilon_l \in \mathcal{O}(m_{l+1}-m_l)$, we have $\sum_l (m_{l+1}-m_l) = \infty$ as well, which contradicts $y_l \to y^*$.

Finally, suppose that $P = \frac{\partial^3}{\partial\alpha^3} V_{m^*}(S(y^{*-})) \neq 0$. If it is negative, we again have $V(\alpha) > 0$ for α in a left-neighborhood of m^*, a contradiction, so it must be positive; and, as before, $P(\alpha) = \frac{\partial^3}{\partial\alpha^3} V_\alpha(S(m^{-1}(\alpha)))$ must be left-continuous at m^*, that is, it must be close to P for α close to m^*. Note that

$$0 = \int_{m_l}^{m_{l+1}} N(\alpha) = \epsilon_l'(m_{l+1}-m_l) + \int_{m_l}^{m_{l+1}} M(\alpha)(m_{l+1}-\alpha)$$
$$= \epsilon_l'(m_{l+1}-m_l) + \frac{(m_{l+1}-m_l)^2}{2} M(\tilde{\alpha}_l)$$

for some $\tilde{\alpha}_l \in (m_l, m_{l+1})$. This implies

$$\epsilon_l' = -M(\tilde{\alpha}_l)\frac{m_{l+1}-m_l}{2}, \qquad \epsilon_{l+1}' = -M(\tilde{\alpha}_{l+1})\frac{m_{l+2}-m_{l+1}}{2},$$
$$\epsilon_{l+1} = \epsilon_l' + \int_{m_l}^{m_{l+1}} M(\alpha) = \epsilon_l' + (m_{l+1}-m_l)M(\tilde{\tilde{\alpha}}_l)$$
$$\implies \quad \epsilon_{l+1} = (m_{l+1}-m_l)\frac{2M(\tilde{\tilde{\alpha}}_l) - M(\tilde{\alpha}_l)}{2},$$

where $\tilde{\alpha}_l$, $\tilde{\tilde{\alpha}}_l \in (m_l, m_{l+1})$. To finish the proof, we will need to be more specific about the positions of these values in the interval (m_l, m_{l+1}). Due to the left-continuity of $P(\alpha)$, $M(\alpha)$ is roughly linear in each interval (m_l, m_{l+1}). This, coupled with the above, implies that $\tilde{\alpha}_l = \frac{2m_l+m_{l+1}}{3} + o(m_{l+1}-m_l)$ and $\tilde{\tilde{\alpha}}_l = \frac{m_l+m_{l+1}}{2} + o(m_{l+1}-m_l)$. In addition, $M(m_l^+) - M(m_l^-) \in \mathcal{O}(\epsilon_l) \in o(m_l - m_{l-1})$. Then

$$\begin{aligned}
&M(\tilde{\alpha}_{l+1}) - 2M(\tilde{\tilde{\alpha}}_l) + M(\tilde{\alpha}_l) \\
&= M(\tilde{\alpha}_{l+1}) - M(\tilde{\tilde{\alpha}}_l) + M(\tilde{\alpha}_l) - M(\tilde{\tilde{\alpha}}_l) \\
&= \mathcal{O}(\epsilon_{l+1}) + \big(P + o(l)\big)\left(\frac{2m_{l+1}+m_{l+2}}{3} - \frac{m_l+m_{l+1}}{2} + o(m_{l+2}-m_l)\right) \\
&\quad + \big(P + o(l)\big)\left(\frac{2m_l+m_{l+1}}{3} - \frac{m_l+m_{l+1}}{2} + o(m_{l+2}-m_l)\right) \\
&= P\left(\frac{m_{l+2}}{3} - \frac{m_l}{3}\right) + o(m_{l+2}-m_l) > 0,
\end{aligned}$$

so that

$$\frac{-\epsilon_{l+1}}{-M(\tilde{\alpha}_{l+1})} = \frac{m_{l+1}-m_l}{2}\frac{-2M(\tilde{\tilde{\alpha}}_l)+M(\tilde{\alpha}_l)}{-M(\tilde{\alpha}_{l+1})} =$$

$$= \frac{m_{l+1}-m_l}{2}\left(1+\frac{M(\tilde{\alpha}_{l+1})-2M(\tilde{\tilde{\alpha}}_l)+M(\tilde{\alpha}_l)}{-M(\tilde{\alpha}_{l+1})}\right) \geq \frac{m_{l+1}-m_l}{2}$$

$$\implies \quad \frac{m_{l+2}-m_{l+1}}{2} = \frac{\epsilon'_{l+1}}{-M(\tilde{\alpha}_{l+1})} = \frac{-\epsilon_{l+1}+\mathcal{O}(\epsilon^2_{l+1})}{-M(\tilde{\alpha}_{l+1})}$$

$$= \frac{-\epsilon_{l+1}}{-M(\tilde{\alpha}_{l+1})}\big(1+\mathcal{O}(m_{l+1}-m_l)\big)$$

$$\geq \frac{m_{l+1}-m_l}{2} + \mathcal{O}\big((m_{l+1}-m_l)^2\big),$$

which, as before, implies $m_l \to \infty$, a contradiction. As a result, we can conclude that if $y_l \to y^* < x^{**}$, then m violates Condition B2.2 at y^*, which is what we wanted.

As for what happens to the left of a, if $W'(a^+) = 0$, we are done. If $W'(a^+) > 0$ instead, then by the same arguments as before, $W'(a^-) < 0$ and there must be $y_{-1} < y_0$ s.t. $W(y) > 0$ for $y \in (y_{-1}, y_0)$ and $W(y_{-1}) = 0$, etc. If this sequence of intervals to the left of a is finite, re-index the sequence $(y_l)_l$ appropriately and we are done. If it is infinite, we would have an infinite decreasing sequence $y_0 > y_{-1} > \cdots$ such that $y_l \to y_*$ as $l \to -\infty$. If $y_* > x^*$, we obtain a contradiction analogously to our previous arguments. If $y_* = x^*$, we still obtain a contradiction by a slightly different argument—near x^* and x^{**}, it is not true that $\epsilon'_l = -\epsilon_l + \mathcal{O}(\epsilon_l^2)$, as $\frac{\partial u_{m(x)}(x)}{\partial x}$ approaches zero, but it is still true that $\epsilon'_l \leq -\epsilon_l$, so it is possible that ϵ_l is shrinking fast enough for $(y_l)_l$ to converge as $l \to +\infty$, but not as $l \to -\infty$.

Finally, the fact that $W(x) = 0$ for $x < y_0$ follows from the fact that, if this were false, there would be a sequence $(\tilde{y}_l)_l$ with $\tilde{y}_0 < x < y_0$ and $\tilde{y}_l \to x^{**}$ as $l \to +\infty$, which contradicts $W'(y_0^+) = 0$. *Q.E.D.*

We can now construct a *canonical LS*, s_*, as follows. Under Condition B2.1, construct a smooth LS based on Lemma 5 for a maximal interval (x^*, x_0) where this is possible—either (x^*, x^{**}) if $e(x) > 0$ everywhere, or else up to a point x_0 where $e(x_0) = 0$. In the latter case, to the right of x_0, Condition B2.2 guarantees that $W'''(x_0^+) > 0$, so $W(x) > 0$ in a right-neighborhood of x_0. We can then construct the solution based on sequences $(y_l)_l$, $(e_l)_l$ as described above, with Condition B2.3 guaranteeing that $e_l > 0$ and $y_{l+1} > y_l$ for all l.

LEMMA 8: *If s_* satisfies Conditions* B2.1, B2.2, *and* B2.3, *then it is the unique LS*.

PROOF: Let $\hat{s}$ be another LS. Suppose that $W(x) > 0$ for some x, so s_* features a sequence $(y_l)_l$ as in Lemma 7. Let $\hat{y}_0 = \inf\{y \in (x^*, x^{**}) : \hat{W}(y) > 0\}$. Note that s_* and $\hat{s}$ must be identical for x between x^* and $\min(y_0, \hat{y}_0)$ by Lemma 6.

If $y_0 < \hat{y}_0$, it follows that $\hat{V}_{m(y)}(\hat{S}(y)) = 0$ for y in a right-neighborhood of y_0, but at the same time $\hat{V}_{m(y)}(\hat{S}(y)) \geq \hat{V}_{m(y)}(\hat{S}(y_0)) = V_{m(y)}(S(y_0)) > 0$, a contradiction.

If $\hat{y}_0 < y_0$, there are two cases. First, suppose that $\hat{W}(y) > 0$ for all y in a right-neighborhood of $\hat{y}_0$. Then we can apply the previous argument at $\hat{y}_0$. Second, suppose $W(y) > 0$ and $W(y) = 0$ are both obtained for $y > \hat{y}_0$ arbitrarily close to $\hat{y}_0$. Then there

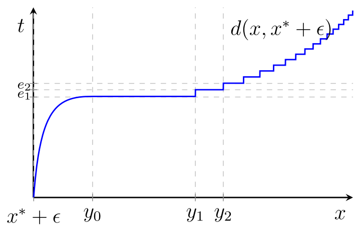


FIGURE 4.—Discontinuous limit solution.

must be an infinite collection of intervals $(a_n, b_n)_{n\in\mathbb{Z}_{\le 0}}$ such that $b_n > a_n \ge b_{n-1}$ for all n; $a_n \xrightarrow[n\to-\infty]{} \hat{y}_0$; $W(y) > 0$ for all $y \in (a_n, b_n)$ and $W(y) = 0$ for all $y = a_n$ or $= b_n$. This case leads to a contradiction by arguments developed in Lemmas 5 and 7. Briefly, for $y > \hat{y}_0$ close enough to $\hat{y}_0$, $S(y)$ and $\hat{S}(y)$ are similar; $V''(m(y)) = 0$; and $e(y) \ge C > 0$, so $\hat{V}''(\alpha) \le \tilde{C} < 0$ for any α such that $m(\alpha) \in (a_n, b_n)$. This implies that $\hat{s}$ in fact satisfies Conditions B2.2 and B2.3, which contradicts $a_n \xrightarrow[n\to-\infty]{} \hat{y}_0$ by Lemma 7.

Hence $y_0 = \hat{y}_0$. Then s_* and $\tilde{s}$ must be identical for $x > y_0$ because their behavior is uniquely pinned down by Lemma 7. Finally, note that, in the case where $e(x) > 0\ \forall x \in (x^*, x^{**})$, the same proof goes through and we do not need to assume that s_* satisfies Condition B2.2 (in particular, we do not need to assume u or m are C^3). *Q.E.D.*

The following proposition summarizes our results and extends Proposition 6 to the case without a CLS.

PROPOSITION 8: *Let $\tilde{e}(x)$ be the solution to Equation (*). Then, if $\tilde{e}(x) \ge 0$ for all $x \in [x^*, x^{**}]$, there is a CLS s_* given by $e \equiv \tilde{e}$. Moreover, this is the unique LS.*

Otherwise, assume that the canonical LS s_* satisfies Conditions B2.1, B2.2, *and* B2.3. *Then it is the unique LS, and it is given by $e(x) = \tilde{e}(x)$ for x up to some $\tilde{x}$, and by two sequences $(y_l)_{l\in\mathbb{N}_{\ge 0}}$, $(e_l)_{l\in\mathbb{N}_{\ge 1}}$ such that: $(y_l)_l$ is increasing, $y_0 = \tilde{x}$ and $y_l \xrightarrow[l\to\infty]{} x^{**}$; $d(y_l^-, y_{l+1}^+) = 0$ and $d(y_l^+, y_l^-) = e_l$ for all $l \ge 1$; and*

$$U_{m(y_{l+1})}\big(S(y_l^+)\big) = u_{m(y_{l+1})}(y_{l+1}),$$

$$e^{\frac{re_l}{2}} = 1 + m'(y_l)\frac{\dfrac{\partial V_{m(y_l)}\big(S(y_l^-)\big)}{\partial \alpha}}{\dfrac{\partial u_{m(y_l)}(y_l)}{\partial x}}.$$

For any sequence $(s_j)_j$, where s_j is a Q1E of the j-refined game, and for any fixed x, $s_j(x, t) \xrightarrow[j\to\infty]{} s_*(x, t)$ a.s. (more precisely, $s_j(x, t) \xrightarrow[j\to\infty]{} s_*(x, t)\ \forall t$ where $s_*(x, t)$ is continuous in t).

In addition, if $m(y_l) < y_{l-1}$ for all l or $\tilde{e}(x) \ge 0\,\forall x$, there is $\overline{\delta} < 1$ such that all Q1Es for discount factor δ are 1Es within $[x^*, m^{-1}(x^* + d^+_{x^*}))$.

PROOF OF PROPOSITION 8: The last claim is a corollary of Proposition 5, and the characterization of s_* follows from Lemmas 7 and 8. It remains to show that all sequences of

Q1Es of the j-refined games $(s_j)_j$ converge to s_* a.e., that is, $s_j(x,t) \underset{j\to\infty}{\longrightarrow} s_*(x,t)\ \forall x,t$ where $s_*(x,t)$ is continuous. Take a fixed x_0 and let $p(x) = d(x,x_0)$. Then we have to show $p_j(x) \underset{j\to\infty}{\longrightarrow} p(x)\ \forall x$ where p is continuous.

Suppose not, so there is a sequence $(s_j)_j$ and an x_1 for which p is continuous at x_1 but $p_j(x_1) \nrightarrow p(x_1)$. Take a subsequence $(s_l)_l$ such that p_l converges pointwise to some $\hat{p}$, and label the associated policy mapping $\hat{s}$.[6] $\hat{p} \neq p$ as, in particular, $\hat{p}(x_1) \neq p(x_1)$.

We will now prove the result simply by proving that $\hat{s}$ is a LS.

(i) Let $x \in (x^*, x^{**})$. For each j and x', $U_{m(x)}(S_j(s_j(x))) \geq U_{m(x)}(S_j(x'))$. If $s_j(x) \underset{j\to\infty}{\longrightarrow} x$, we obtain $U_{m(x)}(\hat{S}(x))) \geq U_{m(x)}(\hat{S}(x'))$ by taking the limit. If not, and $s_{j'}(x) \underset{j'\to\infty}{\longrightarrow} \tilde{x} < x$ for some subsequence, then $U_{m(x)}(\hat{S}(\tilde{x}))) \geq U_{m(x)}(\hat{S}(x'))$. But this also implies $\hat{p}(x) - \hat{p}(\tilde{x}) = 0$, so $U_{m(x)}(\hat{S}(x)) = U_{m(x)}(\hat{S}(\tilde{x}))) \geq U_{m(x)}(\hat{S}(x'))$.

In turn, the fact that $\hat{s}$ satisfies (i) means that Lemmas 5 and 6 apply to it.

(ii) Suppose $\hat{s}$ violates this condition at some $a \in (x^*, x^{**})$, that is, $\hat{p}(a^+) - \hat{p}(a^-) = e^* > 0$. By an argument similar to Lemma 5, we have $\hat{W}'(a^+) > 0$ and hence $\hat{p}$ is constant on some interval (a,b).

Take $\epsilon > 0$ small, and let $(x_{jn})_n$ be the recognized sequence of s_j for each j. By construction, $(x_{jn})_n$ must have $je^* + j\hat{d}(a^-, a-\epsilon) + o(j)$ elements in $(a-\epsilon, a+\epsilon)$, and $j\hat{d}(a-\epsilon, a-2\epsilon) + o(j)$ elements in $(a-2\epsilon, a-\epsilon)$. In particular, given $\eta > 0$, for high enough j there must be an element $x_{jt} \in (a-2\epsilon, a-\epsilon)$ such that $x_{jt} - x_{j(t+1)} \geq \frac{1}{j\overline{e}(1+\eta)}$ for $\overline{e} = \max_{x\in(a-2\epsilon,a-\epsilon)} e(x)$. Let $x_{jt'}$ be the right-most element of $(x_{jn})_n$ contained in $(a-\epsilon, a+\epsilon)$. The above implies $t' - t \geq je^* + j\hat{d}(a^-, a-\epsilon) + o(j)$.

Now, denoting $x_{jn} = x_n$, $m(x_{jn}) = m_n$, and $S_j(x_{jn}) = S(x_n)$, and exploiting the indifference conditions $U_{m_{n-1}}(S(x_n)) = U_{m_{n-1}}(S(x_{n+1})) = u_{m_{n-1}}(x_n)$ and $U_{m_n}(S(x_{n+1})) = U_{m_n}(S(x_{n+2})) = u_{m_n}(x_{n+1})$,

$$\begin{aligned}
&V_{m_{n-1}}\big(S(x_{n+1})\big) - V_{m_n}\big(S(x_{n+1})\big) \\
&\quad = U_{m_{n-1}}\big(S(x_{n+1})\big) - U_{m_n}\big(S(x_{n+1})\big) - u_{m_{n-1}}(x_{n-1}) + u_{m_n}(m_n) \\
&\quad = -\big(u_{m_{n-1}}(x_{n-1}) - u_{m_{n-1}}(x_n)\big) + \big(u_{m_n}(m_n) - u_{m_n}(x_{n+1})\big), \\
&m'(\hat{x}_n)(x_{n-1} - x_n)\frac{\partial}{\partial\alpha}V_{\tilde{\alpha}_n}\big(S(x_{n+1})\big) \\
&\quad = -(x_{n-1} - x_n)\frac{\partial}{\partial x}u_{m_{n-1}}(\tilde{x}_n) + (x_n - x_{n+1})\frac{\partial}{\partial x}u_{m_n}(\tilde{x}_{n+1}), \\
&(x_{n-1} - x_n) = (x_n - x_{n+1})\frac{-\dfrac{\partial}{\partial x}u_{m_n}(\tilde{x}_{n+1})}{-m'(\hat{x}_n)\dfrac{\partial}{\partial\alpha}V_{\tilde{\alpha}_n}\big(S(x_{n+1})\big) - \dfrac{\partial}{\partial x}u_{m_{n-1}}(\tilde{x}_n)},
\end{aligned}$$

[6] Use a diagonal argument to find a subsequence $(s_{l'})_{l'}$ such that $(p_{l'})_{l'}$ converges at all rational points. This guarantees convergence at all points except points of discontinuity of $\limsup_{l'\to\infty} p_{l'}$, which are countable because the function in question is increasing. Use another diagonal argument to get $(s_l)_l$ such that p_l also converges at all discontinuities of $\limsup_{l'\to\infty} p_{l'}$.

for some $\tilde{\alpha}_n \in (m_n, m_{n-1})$, $\tilde{x}_n, \hat{x}_n \in (x_n, x_{n-1})$, $\tilde{x}_{n+1} \in (x_{n+1}, x_n)$. In addition,

$$\begin{aligned}&\frac{\partial}{\partial \alpha} V_\alpha\big(S(x_n)\big)\\&= e^{-\frac{r}{j}} \frac{\partial}{\partial \alpha} V_\alpha\big(S(x_{n+1})\big) + \big(1 - e^{-\frac{r}{j}}\big) \left(\frac{\partial u_\alpha(x_n)}{\partial \alpha} - \frac{\partial u_\alpha\big(m^{-1}(\alpha)\big)}{\partial \alpha} - \frac{\frac{\partial u_\alpha\big(m^{-1}(\alpha)\big)}{\partial x}}{m'\big(m^{-1}(\alpha)\big)} \right),\\&\frac{\partial}{\partial \alpha} V_\alpha\big(S(x_n)\big)\\&= e^{-\frac{rk}{j}} \frac{\partial}{\partial \alpha} V_\alpha\big(S(x_{n+k})\big) + \big(1 - e^{-\frac{rk}{j}}\big) \left(\mathcal{O}(\epsilon) - \frac{\frac{\partial}{\partial x} u_\alpha\big(m^{-1}(\alpha)\big)}{m'\big(m^{-1}(\alpha)\big)} \right)\end{aligned}$$

for $\alpha \in (m(a - 2\epsilon), m(a + \epsilon))$, $x_n, x_{n+k} \in (a - 2\epsilon, a + \epsilon)$.

Then, for $n \in \{t, \dots, t'\}$,

$$\begin{aligned}&x_{j(n-1)} - x_{jn}\\&\geq (x_{jn} - x_{j(n+1)}) \frac{-\frac{\partial}{\partial x} u_{m(a)}(a) - K\epsilon}{-\tilde{m}' \frac{\partial}{\partial \alpha} V_{\tilde{\alpha}_n}\big(S_j(x_{j(n+1)})\big) - \frac{\partial}{\partial x} u_{m(a)}(a) + K'\epsilon}\\&\geq (x_{jn} - x_{j(n+1)}) \frac{-\frac{\partial}{\partial x} u_{m(a)}(a) - K\epsilon}{-e^{-\frac{r(t-n)}{j}} \tilde{m}' \frac{\partial}{\partial \alpha} V_{\tilde{\alpha}_n}\big(S_j(x_{j(t+1)})\big) - e^{-\frac{r(t-n)}{j}} \frac{\partial}{\partial x} u_{m(a)}(a) + K''\epsilon}\\&\geq (x_{jn} - x_{j(n+1)}) \frac{-\frac{\partial}{\partial x} u_{m(a)}(a) - K\epsilon}{G(j) - e^{-\frac{r(t-n)}{j}} \frac{\partial}{\partial x} u_{m(a)}(a) + K'''\epsilon}\\&= (x_{jn} - x_{j(n+1)}) \frac{1 - K''''\epsilon}{e^{-\frac{r(t-n)}{j}} + \tilde{G}(j) + K'''''\epsilon}\end{aligned}$$

for some function $G(j)$ such that $G(j) \xrightarrow[j\to\infty]{} 0$, as

$$\left| \frac{\partial}{\partial \alpha} V_{\tilde{\alpha}_n}\big(S_j(x_{j(t+1)})\big) \right| \xrightarrow[j\to\infty]{} \left| \frac{\partial}{\partial \alpha} V_{\alpha^*}\big(\hat{S}(x^*)\big) \right| \leq \left| \frac{\partial}{\partial \alpha} V_{m(x^*)}\big(\hat{S}(x^*)\big) \right| + C\epsilon = C\epsilon$$

for some $\alpha^* \in [m(a-2\epsilon), m(a+\epsilon)]$, $x^* \in [a-2\epsilon, a-\epsilon]$ and $C > 0$.[7] Then

$$x_{j(n-1)} - x_{jn} \geq (x_{jt} - x_{j(t+1)}) \prod_{k=0}^{t-n} \frac{1-K'\epsilon}{e^{-\frac{rk}{j}} + \tilde{G}(j) + K''''\epsilon}$$

$$\implies \quad x_{jt'} - x_{j(t'+1)} \geq \frac{1}{j\overline{e^*}(1+\eta)} \prod_{k=0}^{je^*-1} \frac{1-K'\epsilon}{e^{-\frac{rk}{j}} + \tilde{G}(j) + K'''''\epsilon}.$$

If we take ϵ small enough that $1-K'\epsilon > 0$, the right-hand side grows to infinity as $j \to \infty$. In particular, for j high enough, $x_{jt'} - x_{j(t'+1)} > 3\epsilon$, a contradiction.

(iii) This follows from a calculation analogous to the one used for (ii). Briefly, if (iii) is violated at x_0 and $\hat{p}(x_0^+) - \hat{p}(x_0^-)$ is higher than the value required by (iii), then $x_{jt'} - x_{j(t'+1)} \xrightarrow[j\to\infty]{} \infty$, a contradiction. If $\hat{p}(x_0^+) - \hat{p}(x_0^-)$ is lower than the value required by (iii), then it can be shown that $j(x_{jn} - x_{j(n+1)}) \xrightarrow[j\to\infty]{} 0$ for all n such that $x_{j(n+1)} \geq a$, which implies that the number of elements of $(x_{jn})_j$ in $(a-\epsilon, a+\epsilon)$ grows faster than j, a contradiction. *Q.E.D.*

APPENDIX C: GENERICITY OF CONDITIONS ON m

In this section, we show that Conditions B1, B2.2, and B2.3 imposed on the function m are "generic."[8] We employ two different notions of genericity. On the one hand, we show that these conditions hold on an open and dense set (or, at least, a residual set) within the function space with a natural metric. In addition, we show that some of these conditions hold on a *prevalent* set, a notion introduced in Hunt, Sauer, and Yorke (1992) that generalizes the measure-theoretic notion of "almost everywhere" to infinite-dimensional spaces where an analog of the Lebesgue measure is not available.

CLAIM 1: *Consider the set of functions*

$$X_1 = \{m : [-1, 1] \longrightarrow [-1, 1] : m \in C^1, m \text{ weakly increasing}\}$$

with the norm $\|m\| = \max(\|m\|_\infty, \|m'\|_\infty)$. *The subset* $Y_1 \subseteq X_1$ *satisfying Condition* B1 *is open and dense,*[9] *and also prevalent in the sense of Hunt, Sauer, and Yorke* (1992).

PROOF: We first show Y_1 is open. Let $m_0 \in Y_1$; $x_1^* < \cdots < x_N^*$ be the fixed points of m_0; $\alpha_i = m_0'(x_i^*)$ for $i = 1, \dots, N$; $\epsilon > 0$ and $\nu > 0$ such that $|m_0'(y) - 1| \geq \nu$ for $y \in I_i = (x_i^* - \epsilon, x_i^* + \epsilon)$ for any i; $\upsilon > 0$ such that $|m(y) - y| \geq \upsilon$ for $y \notin I_i$ for any i; $\eta = \min(\epsilon, \upsilon, \nu)$; and $m_1 \in B(m_0, \eta)$. Then $m_1(y) = y$ implies $|m_0(y) - y| < \eta \leq \upsilon$, so $y \in I_i$ for some i, so $|m_1'(y) - 1| \geq |m_0'(y) - 1| - |m_1'(y) - m_0'(y)| > \nu - \nu = 0$. This shows that $m'(y) \neq 1$ at any fixed point y of m_1. Moreover, by construction, either $m_0'(y) > 1$ for all $y \in I_i$ and $m_1'(y) > 1$ for all $y \in I_i$ as well, or the reverse inequalities hold, whence m_1 can have at most one fixed point in I_i for each i, and the set of fixed points is finite.

Next, we show Y_1 is dense. Let $m_0 \in X_1$ and $\epsilon > 0$. We want to show that there is $m_1 \in B(m_0, \epsilon) \cap Y_1$. Since m_0' is continuous in $[-1, 1]$, it is uniformly continuous, so we can take

[7] If necessary, take a convergent subsequence so that $(\tilde{\alpha}_{n_j})_j$ and $x_{j(t_j+1)}$ converge for this argument.

[8] We say m satisfies Conditions B2.1, B2.2, and B2.3 if there is an LS for this m that satisfies them—equivalently, if the canonical LS for this m satisfies them.

[9] The statement is also true within the space of C^3 functions, taken with the appropriate norm.

$\nu > 0$ such that if $|y - y'| < \nu$, then $|m_0'(y) - m_0'(y')| < \frac{\epsilon}{4}$. Partition $[-1, 1]$ into intervals $I_1, I_2, \ldots, I_J$ as follows: $I_j = [y_{j-1}, y_j)$, where $y_j = -1 + j\nu$, for $j < J$, and $I_J = [y_{J-1}, 1]$. For each j, if $m_0'(y_{j-1}) \geq 1$, let $m_2'(y) = m_0'(y) + \frac{\epsilon}{4}$ for all $y \in I_j$ (which implies $m_2'(y) > 1$ for $y \in I_j$); otherwise let $m_2'(y) = m_0'(y) - \frac{\epsilon}{4}$ for all $y \in I_j$ (so $m_2'(y) < 1$ for $y \in I_j$), and then define m_2 by integrating m_2', with $m_2(-1) = m_0(-1)$. By construction, m_2 has at most one fixed point in each interval I_j and $m_2' \neq 1$ at such points. Moreover, $\|m_2' - m_0'\| \leq \frac{\epsilon}{4}$ and $\|m_2 - m_0\| \leq 2\frac{\epsilon}{4} = \frac{\epsilon}{2}$. If $m_2(y_j) \neq y_j$ for all y_j, we can construct a "smoothed-out" version of m_2', which we will call m_1', that is in $B(m_0, \epsilon) \cap Y_1$. If $m_2(y_j) = y_j$ for some j, and $m_2'(y) > 1$ for $y \in I_j \cup I_{j+1}$ or $m_2'(y) < 1$ for $y \in I_j \cup I_{j+1}$, this is not a problem. If $m_2(y_j) = y_j$ for some j and $m_2'(y) > 1$ for $y \in I_j$, $m_2'(y) < 1$ for $y \in I_{j+1}$, we can construct a smooth m_1' such that $m_1'(y_j) = 1$, $m_1'(y) > 1$ to the left of y_j and < 1 to the right, and $m_1(y_j) < m_2(y_j) = y_j$. The remaining case is analogous.

For the last claim, note that, if a C^1 function m defined on a compact interval has $m' \neq 1$ at all its fixed points, it automatically has a finite number of them. Consider the translation $X_1 - v$, where v is the identity function. Then $m \in Y_1$ iff $m - v$ has no points where $(m - v)(y) = (m - v)'(y) = 0$. Finally, the fact that $Y_1 - v$ is prevalent in $X_1 - v$ follows from Proposition 3 in Hunt, Sauer, and Yorke (1992). *Q.E.D.*

CLAIM 2: *Condition* B2.1 *holds in an open set* Y_2 *within*

$$X_2 = \{m : [x^*, x^{**}] \longrightarrow [x^*, x^{**}] : m \in C^2, m \text{ weakly increasing}, \\ m(x^*) = x^*, m(x^{**}) = x^{**}, m(x) < x \,\forall x \in (x^*, x^{**})\}$$

taken with the norm $\|m\| = \max(\|m\|_\infty, \|m'\|_\infty, \|m''\|_\infty)$.[10]

PROOF: Trivial. *Q.E.D.*

CLAIM 3: *Condition* B2.1′ *holds in an open set* Y_4 *within* X_2.

PROOF: This amounts to showing that e is continuous in m'', and it follows from an argument similar to the proof of the uniqueness of e from Lemma 6. *Q.E.D.*

CLAIM 4: *Assume* u *is* C^3. *Let*

$$X_3 = \{m \in Y_2 : m \in C^3, m(x^*) = x^*, m(x^{**}) = x^{**}, m(x) < x \;\forall x \in (x^*, x^{**}), \\ m \text{ strictly increasing}, m \text{ satisfies Condition B2.1}\}$$

taken with the norm $\|m\| = \max(\|m\|_\infty, \|m'\|_\infty, \|m''\|_\infty, \|m'''\|_\infty)$. *For each* $y \in (x^*, x^{**})$, *the set* $Y_3(y) \subseteq X_3$ *of functions* m *for which Conditions* B2.2 *and* B2.3 *hold in* $[x^*, y]$ *is open and dense*.

PROOF: We proceed in two steps. First, we show that the set $Y_5(y) \subseteq X_3$ for which Condition B2.2 holds in $[x^*, y]$ is open and dense. Second, we show that the set $Y_3(y)$ is open and dense within $Y_5(y)$.

To show that $Y_5(y)$ is open, take $m \in Y_5(y)$ and suppose there is a sequence $(m_n)_n$ such that $m_n \notin Y_5(y)$ for all n but $m_n \to m$. For each m_n we can construct a LS s_n (possibly

[10]Again, this is also true within the space of C^3 functions.

not unique) by finding a convergent sequence of discrete-time equilibria $(s_{nj})_j$ for $\delta = e^{-\frac{r}{j}}$ with $j \to \infty$, as in Proposition 8. Using a diagonal argument, we can find a convergent subsequence of $(s_n)_n$, which by continuity must converge to a LS for m, $\hat{s}$. WLOG assume $(s_n)_n \to \hat{s}$. We will need the following lemma:

LEMMA 9: *If s_* satisfies Conditions* B2.1 *and* B2.2, *then it is the unique LS. Moreover,* s_* *has a finite number of vertex points in* $[x^*, y]$ *for any* $y < x^{**}$.

PROOF: Briefly, if s_* has an infinite number of vertex points in $[x^*, y]$, they must accumulate at some $y^* \in (x^*, y]$, which must satisfy $V_{m(y^*)}(S(y^*)) = \frac{\partial V_{m(y^*)}(S(y^*))}{\partial \alpha} = 0$. If $\frac{\partial^2 V_{m(y^*)}(S(y^*))}{\partial \alpha^2} > 0$, we obtain $V > 0$ in a neighborhood of y^*, a contradiction. If $\frac{\partial^2 V_{m(y^*)}(S(y^*))}{\partial \alpha^2} < 0$, this guarantees Condition B2.3 in a neighborhood of y^*, which means the vertex points near y^* must be part of a single sequence, contradicting Lemma 7.

Suppose that there are infinitely many vertex points on a left-neighborhood of y^* (the other case is analogous). Similar arguments apply if $\frac{\partial^3 V_{m(y^*)}(S(y^*))}{\partial \alpha^3} < 0$ or $\frac{\partial^3 V_{m(y^*)}(S(y^*))}{\partial \alpha^3} > 0$, respectively.

As for the uniqueness of s_*, the proof in Lemma 8 can be extended to this case. *Q.E.D.*

From this we conclude that $\hat{s} = s_*$. Letting W_n be the value function for s_n, we then have $W_n \to W$. It can be shown in addition that, at every y that is not a vertex point of s_*, $W_n'(y) \to W'(y)$, $W_n''(y) \to W''(y)$, and $W_n'''(y) \to W'''(y)$, by using Lemmas 6 and 9.

Next, we show that $Y_5(y)$ is dense. Take $m \in X_3$ and $\epsilon > 0$. Consider $\hat{m}$ given by: $\hat{m(x^*)} = x^*$, $\hat{m}'(x^*) = m'(x^*)$, $\hat{m}''(x^*) = m''(x^*)$, and $\hat{m}'''(x) = m'''(x) + \eta(x)$, where $|\eta(x)| \leq \epsilon$ will be defined as 0 except where we specify otherwise. We will argue that, by picking η correctly, we can find a $\hat{m} \in Y_5(y)$ that is close to m.

Apply the following algorithm. Take $\nu > 0$ small and $N > 0$ large. Let $\eta_0 \equiv 0$ and $m_0 \equiv m$. Let

$$x_0 = \inf\left\{x \in (x^*, y] : \max\left(\left|V_{m_0(x)}\big(S_0(x)\big)\right|, \left|\frac{\partial}{\partial \alpha} V_{m_0(x)}\big(S_0(x)\big)\right|, \left|\frac{\partial^2}{\partial \alpha^2} V_{m_0(x)}\big(S_0(x)\big)\right|, \left|\frac{\partial^3}{\partial \alpha^3} V_{m_0(x)}\big(S_0(x)\big)\right|\right) \leq \frac{\epsilon}{N}\right\},$$

where $S_0(x)$ is a policy path starting at x for a LS given median voter function m_0.[11] Let $\alpha_0 = m(x_0)$. Define $\eta_1(x) = -\epsilon$ for $x \in [x_0, x_0')$ and $\eta_1(x) = 0$ for all other x, with x_0' taken so that $m_1(x_0') = m_1(x_0) + \nu$. Next, let x_1 be the infimum of $x \in (x_0 + \nu, y]$ for which $|V_{m_1(x)}(S_1(x))|$, $|\frac{\partial}{\partial \alpha} V_{m_1(x)}(S_1(x))|$, $|\frac{\partial^2}{\partial \alpha^2} V_{m_1(x)}(S_1(x))|$, and $|\frac{\partial^3}{\partial \alpha^3} V_{m_1(x)}(S_1(x))| \leq \frac{\epsilon}{N}$, and define $\alpha_1 = m(x_1)$ and $\eta_2(x) = -\epsilon$ for $x \in [x_1, x_1')$ and $\eta_2(x) = \eta_1(x)$ for all other x, with x_1' taken so that $m_2(x_1') = m_2(x_1) + \nu$. Define x_k, α_k, η_{k+1}, m_{k+1} for $k = 2, 3, \ldots$ in the same fashion until $x_K = +\infty$ for some K.[12] Let $\tilde{m} = m_K$.

We argue that, if ν and N are taken to be small and large enough, respectively, $\tilde{m}$ satisfies Condition B2.2. To explain why, we will need the following:

[11]Note that, for ϵ small enough, $x_0 > x^*$ since $e(x) \geq C > 0$ in a neighborhood of x^*, which implies $\frac{\partial^2}{\partial \alpha^2} V_{m_0(x)}(S_0(x)) \geq C' > 0$.

[12]This must happen for a finite K, as $\alpha_k - \alpha_{k-1} \geq \nu > 0$ for all k.

REMARK 2: A function $f:[a,b]\to\mathbb{R}$ is uniformly continuous iff there is an increasing function $h:[0,+\infty)\to[0,+\infty)$ such that $h(0)=0$, h is continuous at 0, and $|f(x)-f(y)|\le h(|x-y|)$ for all $x,y\in[a,b]$. We say a function h satisfying these properties is a *bounding function*.

Now note that, for any k and any $x<x'$ such that $m(x)=a$, $m(x')=a'$ satisfy $a,a'\in[\alpha_k,\alpha_k+\nu]$, we have

$$\begin{aligned}
&\left|\frac{\partial^3}{\partial\alpha^3}V_a\big(\tilde{S}(x)\big)-\frac{\partial^3}{\partial\alpha^3}V_{a'}\big(\tilde{S}(x')\big)\right|\\
&\quad\le\left|\frac{\partial^3}{\partial\alpha^3}U_a\big(\tilde{S}(x)\big)-\frac{\partial^3}{\partial\alpha^3}U_a\big(\tilde{S}(x')\big)\right|\\
&\qquad+\left|\frac{\partial^3}{\partial\alpha^3}U_a\big(\tilde{S}(x')\big)-\frac{\partial^3}{\partial\alpha^3}U_{a'}\big(\tilde{S}(x')\big)\right|+\left|-\frac{d^3}{d\alpha^3}u_a\big(\tilde{m}^{-1}(a)\big)+\frac{d^3}{d\alpha^3}u_{a'}\big(\tilde{m}^{-1}(a')\big)\right|\\
&\quad\le\left|\frac{\partial^3}{\partial\alpha^3}U_a\big(\tilde{S}(x)\big)-\frac{\partial^3}{\partial\alpha^3}U_a\big(\tilde{S}(x')\big)\right|+h_1\big(a'-a\big)+h_2\big(a'-a\big)\\
&\quad\le K\frac{\epsilon}{N}+K'\big(a'-a\big)+h_1\big(a'-a\big)+h_2\big(a'-a\big)\le K\frac{\epsilon}{N}+h_3\big(a'-a\big),
\end{aligned}$$

where h_1, h_2, h_3 are bounding functions, $K,K'>0$, and K, h_1, h_2, h_3 are independent of ν and N.

The bound $|\frac{d^3}{d\alpha^3}u_a(\tilde{m}^{-1}(a))+\frac{d^3}{d\alpha^3}u_{a'}(\tilde{m}^{-1}(a'))|\le h_2(|a-a'|)$ uses the uniform continuity of $\frac{d^3}{d\alpha^3}u_{\tilde{a}}(\tilde{m}^{-1}(\tilde{a}))$, which follows from the fact that u is C^3 and $\tilde{m}$ is C^3 on $[a,a']\subseteq[\alpha_k,\alpha_k+\nu]$. The bound $|\frac{\partial^3}{\partial\alpha^3}U_a(\tilde{S}(x'))-\frac{\partial^3}{\partial\alpha^3}U_{a'}(\tilde{S}(x'))|\le h_1(|a-a'|)$ uses the uniform continuity of $\frac{\partial^3}{\partial\alpha^3}U$, and the fact that the mapping $x\mapsto\max(x,0)$ is Lipschitz. The first bound is the trickiest, and is based on the idea that, if $\frac{\partial V}{\partial\alpha}$ and $\frac{\partial^2 V}{\partial\alpha^2}$ are low, then $\tilde{s}(x,t)$ changes relatively quickly as a function of t, so $\tilde{S}(x)$ and $\tilde{S}(x')$ are similar. Formally,

$$\begin{aligned}
&\left|\frac{\partial^3}{\partial\alpha^3}U_a\big(\tilde{S}(x)\big)-\frac{\partial^3}{\partial\alpha^3}U_a\big(\tilde{S}(x')\big)\right|\\
&\quad\le\big(1-e^{-r\tilde{d}(x',x)}\big)2\max_{a,y}\frac{\partial^3}{\partial\alpha^3}u_\alpha(y)1-e^{-r\tilde{d}(x',x)}\le r\tilde{d}\big(x',x\big)\\
&\quad=r\Bigg(\sum_{\substack{\tilde{y}_l\in(x,x')\\ \text{vertex point}}}e_l+\int_x^{x'}e(y)dy\Bigg)\le r\frac{2}{r}\sum_{\tilde{y}_l\in(x,x')\text{v.p.}}\big(e^{\frac{\mathrm{re}_l}{2}}-1\big)+r\overline{e}\big(x'-x\big)\\
&\quad\le 2\sum_{\tilde{y}_l\in(x,x')\text{v.p.}}\frac{\left|\max\tilde{m}'\right|}{\min\limits_{x\in[x_0,y]}\left|\frac{\partial u_{m(x)}(x)}{\partial x}\right|}\left|\frac{\partial}{\partial\alpha}V_{a_l}\big(\tilde{S}(y_l^-)\big)\right|+r\overline{e}\big(x'-x\big)\\
&\quad\le K''\left(\left|\frac{\partial}{\partial\alpha}V_a\big(\tilde{S}(x)\big)\right|+\int_a^{a'}\left|\frac{\partial^2}{\partial\alpha^2}V_{\tilde{a}}\big(\tilde{S}\big(n(\tilde{a})\big)\big)\right|d\tilde{a}\right)+r\overline{e}\big(x'-x\big)\\
&\quad\le K''\frac{\epsilon}{N}+K'''\big(a'-a\big).
\end{aligned}$$

Pick ν and N so that $K\frac{\epsilon}{N}+h_3(\nu)\leq\frac{\epsilon}{2}$ and $N\geq 4$. Now $\tilde{m}$ satisfies Condition B2.2 because, for $a\in[\alpha_k,\alpha_k+\nu)$,

$$\left|\frac{\partial^3}{\partial\alpha^3}V_a\big(\tilde{S}(x)\big)\right|\geq\left|\frac{\partial^3}{\partial\alpha^3}V_{\alpha_k}\big(\tilde{S}(x_k)\big)\right|-\left|\frac{\partial^3}{\partial\alpha^3}V_{\alpha_k}\big(\tilde{S}(x_k)\big)-\frac{\partial^3}{\partial\alpha^3}V_a\big(\tilde{S}(x)\big)\right|\geq\epsilon-\frac{\epsilon}{N}-\frac{\epsilon}{2}\geq\frac{\epsilon}{4}>0.$$

On the other hand, if $a\notin[\alpha_k,\alpha_k+\nu)$ for any k, then $|\frac{\partial^i}{\partial\alpha^i}V|>\frac{\epsilon}{N}$ for some $i=0,1,2,3$ by construction.

The only remaining issue is that $\tilde{m}$ is not C^3 because η_K is not continuous at α_k and $\alpha_k+\nu$ for $k=0,1,\dots,K-1$. However, it is easy to construct a continuous η close to η_K that fixes this problem.[13]

Next, we argue that $Y_3(y)$ is open. As shown in Proposition 8, Conditions B2.1, B2.2, and B2.3 taken together imply that the equilibrium path $s(x,t)$ will be given by either a smooth path with $e(x)>0$ for all $x\in(x^*,y]$ or a smooth path up to some y_0 followed by a finite sequence of jumps and stops with stops at $y_1,y_2,\dots,y_l$. It is enough to show that e' is continuous in m''', which follows from the arguments in Proposition 8, and that y_i is continuous in m''' for $i=1,2,\dots,l$, which is elementary (in fact, y_i is continuous in m').

Finally, we argue that $Y_3(y)$ is dense. Take $m\in X_3$ and $\epsilon>0$. Because $Y_5(y)$ is dense, there is $\hat{m}\in B(m,\epsilon)\cap Y_5(y)$. Because $Y_5(y)$ is open, there is $\epsilon'>0$ such that $B(\hat{m},\epsilon')\subseteq B(m,\epsilon)\cap Y_5(y)$. Next, we claim that there is $\hat{\hat{m}}\in B(\hat{m},\epsilon')\cap Y_3(y)$, which completes the proof. This can be shown by construction. If $\hat{e}(x)>0$ for all $x\in[x^*,y]$, we are done. If not, $\hat{s}$ induces a policy path that is continuous up to some y_0 and then features a sequence of jumps and stops with stops at $y_1,y_2,\dots,y_l$. (By the arguments in Proposition 8, this sequence cannot be infinite.) If Condition B2.3 holds at $y_1,\dots,y_l$, we are done. If not, suppose WLOG that it first fails at y_l. $\hat{m}$ can be perturbed near y_l to obtain $\hat{m}_2\in B(\hat{m},\epsilon')$ that satisfies Condition B2.3 at $y_1,\dots,y_l$. Similarly, if $\hat{m}_2$ first fails Condition B2.3 at some $y_{l'}>y_l$, we can construct $\hat{m}_3\in B(\hat{m},\epsilon')$, a perturbation of $\hat{m}_2$ near $y_{l'}$, that satisfies Condition B2.3 up to $y_{l'}$. If this process stops in a finite number of steps, we are done. If not, let $\hat{m}_\infty$ be the pointwise limit of $(\hat{m}_k)_k$. $\hat{m}_\infty$ must feature an infinite sequence of vertex points $y_1<y_2<\cdots$ with $y_l\xrightarrow[l\to+\infty]{}y^*\leq y$, but, as $\hat{m}_\infty\in Y_5(y)$, $\hat{m}_\infty$ satisfies Condition B2.2, leading to a contradiction. *Q.E.D.*

COROLLARY 3: *The set $Y_3\subseteq X_3$ of functions for which Conditions* B2.2 *and* B2.3 *hold in* $[x^*,x^{**}]$ *is a residual set.*

CLAIM 5: *In the case of quadratic utility, the set of functions m for which Condition* B2.2 *holds for $x\in(x^*,x^*+d)$ is prevalent.*

PROOF: The result follows from Theorem 3 in Hunt, Sauer, and Yorke (1992). Following their notation, take $M=\{y\in\mathbb{R}^4:\frac{2y_3-1}{y_1-y_2}+\frac{y_4}{y_3}=0\}$ and $Z=\{y\in\mathbb{R}^5:(y_1,y_2,y_3,y_4)\in M$ and $\frac{2y_4(y_1-y_2)-2y_3(1-y_3)}{(y_1-y_2)^2}+\frac{y_5y_3-y_4^2}{y_3^2}=0\}$. We need to check that M is a manifold of codimension 1, and that the projection $\pi:M\longrightarrow\mathbb{R}$ given by $y\longmapsto y_1$ is a submersion; both

[13]WLOG, take $a=\alpha_k$. If $V_a(S(x^-))>0$, it is easy to perturb η_K to make it continuous at a without violating Condition B2.2. If not, but $V_{\tilde{a}}(S(\tilde{x}^-))>0$ for $\tilde{a}<a$ arbitrarily close to a, we can perturb η_K at one such $\tilde{a}$ instead. If $V_{\tilde{a}}(S(\tilde{x}^-))=0$ for all $\tilde{a}<a$ close to a, but $\frac{\partial^2}{\partial\alpha^2}V$ is nonzero close to a, we can do the same argument. If $\frac{\partial^2}{\partial\alpha^2}V$ is also zero in an interval to the left of a, then Condition B2.2 would be violated for $\tilde{a}<a$, a contradiction.

follow from the implicit function theorem. Finally, we need to check that Z is a zero set in $M \times \mathbb{R}$, which can also be shown using the implicit function theorem. Theorem 3 from Hunt, Sauer, and Yorke (1992) then implies that the set of functions m for which there is an x such that

$$e(x) = \frac{2m'(x) - 1}{x - m(x)} + \frac{m''(x)}{m'(x)} = 0$$
$$= \frac{2m''(x)\big(x - m(x)\big) - 2m'(x)\big(1 - m'(x)\big)}{\big(x - m(x)\big)^2} + \frac{m'''(x)m'(x) - m''(x)^2}{m'(x)^2} = e'(x)$$

is shy, that is, its complement is prevalent. *Q.E.D.*

I conjecture that Conditions B2.2 and B2.3 hold in a prevalent set even for general utility functions, but this is hard to prove.

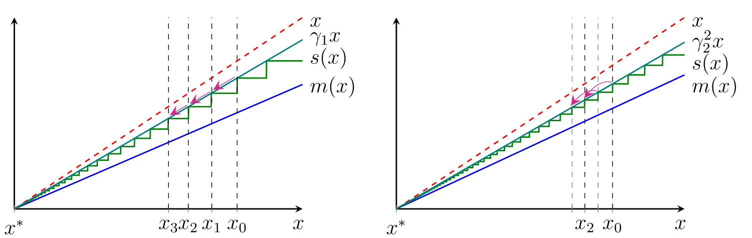
APPENDIX D: OTHER EQUILIBRIA

The discrete-time model in Section 2 may admit MVEs other than 1-equilibria. We discuss two possible types here: k-equilibria (kE), which are composed of k interleaved sequences, and continuous equilibria. For brevity, we present our analysis for the model in Section 5.

DEFINITION 4: Let s be a MVE on $[x^*, x^{**}]$. s is a k-equilibrium if there is a sequence $(x_n)_{n\in\mathbb{Z}}$ such that $x_{n+1} < x_n$ for all n, $x_n \xrightarrow[n\to-\infty]{} x^{**}$, $x_n \xrightarrow[n\to\infty]{} x^*$, and $s(x) = x_{n+k}$ if $x \in [x_n, x_{n-1})$. s is a continuous equilibrium if it is an MVE and continuous.

Figure 5 shows a 2E (5(b)) compared to a 1E (5(a)). Although kEs and continuous equilibria do not exhaust the set of possible equilibria, studying them sheds light on the general behavior of non-1Es. Our main conclusion will be that the existence of these equilibria is not robust in any sense analogous to what is shown in Propositions 5 and 6 for 1Es. This is why the paper does not focus on them.

We first note that, when m is linear[14] and u is quadratic, we can explicitly find kEs for all k, as well as a continuous equilibrium.



(a) 1-equilibrium (b) 2-equilibrium

FIGURE 5.—Equilibria for $u_\alpha(x) = C - (\alpha - x)^2$, $m(x) = 0.7x$, $\delta = 0.7$.

[14] We can construct densities f such that $m(x) = ax$ for $x \in [-d, d]$. For example, for a continuous f symmetric around $x = 0$, take $f(y) = 1 - \frac{1-a}{d}y$ for $y \in [0, d]$ and $f(y) = a + (1-a)(2a^2+1) - \frac{(1-a)(2a^2+1)}{d}y$ thereafter.

PROPOSITION 9: *Let* $u_\alpha(x) = C - (\alpha - x)^2$ *and* $m(x) = ax$ *for* $x \in [-d, d]$. *Assume* $\delta \geq \frac{2}{3}$ *and* $a \in (\frac{1}{2}, 1)$. *Then, for each* k *and* $\underline{x} < d$, *there is a* k*E* s_k *restricted to* $[-d, d]$ *such that* $x_0 = \underline{x}$, *given by* $x_n = \gamma_k^n \underline{x}$, *where* $\gamma_k \in (0, 1)$. *There is also a continuous equilibrium* s_∞ *given by* $s_\infty(x) = \gamma_\infty x$. γ_k^k *is decreasing in* k *and* $\gamma_k^k \to \gamma_\infty$.

PROOF OF PROPOSITION 9: Given $k \geq 1$, assume a kE of the form $s(x_n) = \gamma_k^k x_n$. Since $s(x_n) = x_{n+k}$ but $s(x_n - \epsilon) = x_{n+k+1}$, $m(x_n)$ must be indifferent between x_{n+k} and x_{n+k+1}. This implies

$$-\sum \delta^t (ax_n - x_{n+(t+1)k})^2 = -\sum \delta^t (ax_n - x_{n+(t+1)k+1})^2,$$
$$\sum \delta^t \big(a - \gamma^{(t+1)k}\big)^2 = \sum \delta^t \big(a - \gamma^{(t+1)k+1}\big)^2,$$
$$\frac{\gamma^k(1+\gamma)}{1 - \delta\gamma^{2k}} = \frac{2a}{1 - \delta\gamma^k}.$$

We now argue that the expression $\frac{\gamma^k - \delta\gamma^{2k}}{1-\delta\gamma^{2k}}$ is increasing in γ. It is equivalent to show that $\frac{x-\delta x^2}{1-\delta x^2}$ is increasing in x for $x \in [0, 1]$. This is true because $x - \delta x^2 \leq 1 - \delta x^2$ but $1 - 2\delta x > -2\delta x$, so the log-derivative of $\frac{x-\delta x^2}{1-\delta x^2}$ is positive. Hence, $\frac{(\gamma^k - \delta\gamma^{2k})(1+\gamma)}{1-\delta\gamma^{2k}}$ is also increasing in γ, and equals $2a$ for a unique value of γ which we denote $\gamma_k \in (0, 1)$.

Denote $A_k(x) = \frac{(x-\delta x^2)(1+x^{\frac{1}{k}})}{1-\delta x^2}$. Then $A_k(\gamma_k^k) = 2a$ for all k, and $A_k(x)$ is increasing in x and k. It follows that γ_k^k is decreasing in k.

We now show that the constructed s_k supports an MVE. By the same argument given in Proposition 5 for 1Es, since $m(x_n)$ is indifferent between $S(x_{n+k})$ and $S(x_{n+k+1})$, all $\alpha > m(x_n)$ strictly prefer $S(x_{n+k})$ to $S(x_{n+k+1})$, and $\alpha < m(x_n)$ strictly prefer $S(x_{n+k+1})$. Hence, $m(x_n)$ prefers $S(x_{n+k})$ to $S(x_r)$ for all $r \neq n + k$.

Next, we show that $m(x_n)$ prefers x_{n+k} to other policies $x \notin (x_n)_n$. We do this in two steps. First, we argue that $\gamma^{k+1} > a$, which implies $x_{n+k+1} > m(x_n)$. Second, we note that this yields our result by the same argument as in Proposition 5. For the first part, note that $\gamma^{k+1} > a$ iff

$$\begin{aligned} &\big(\gamma^k + \gamma^{k+1}\big)\big(1 - \delta\gamma^k\big) < 2\gamma^{k+1}\big(1 - \delta\gamma^{2k}\big) \\ &\iff \quad (1-\gamma) < \delta\big(\gamma^k\big(1 - \gamma^{k+1}\big) + \gamma^{k+1}\big(1 - \gamma^k\big)\big) \\ &\iff \quad 1 < \delta\big(\gamma^k + 2\gamma^{k+1} + \cdots + 2\gamma^{2k}\big). \end{aligned}$$

Note that A_k is decreasing in δ, so γ is increasing in δ and a. Hence the worst case is $\delta = \frac{2}{3}$, $a = \frac{1}{2}$. Now suppose $k = 1$. Then the required inequality is $1 < \frac{2}{3}(\gamma + 2\gamma^2)$, which holds if $\gamma \geq \frac{2}{3}$, so it is enough to verify $1 > A_1(\frac{2}{3}) = \frac{50}{57}$. If $k \geq 2$, then it is enough to satisfy $1 < \frac{2}{3}(\gamma^k + 4\gamma^{2k})$, which is true if $\gamma^k \geq \frac{1}{2}$. We then check that $1 > A_k(\frac{1}{2})$. Because A_k is increasing in k, it is enough to check $1 > \lim_{k\to\infty} A_k(\frac{1}{2}) = \frac{4}{5}$.

Finally, we construct a continuous equilibrium. In general, s must solve

$$s(x) = \arg\max_y \sum_{t=0}^{\infty} \delta^t \big(C - \big(m(x) - s^t(y)\big)^2\big)$$
$$\implies \quad 0 = \sum_{t=0}^{\infty} \delta^t \left(-2\big(m(x) - s^t(y)\big) \prod_{i=0}^{t-1} s'\big(s^i(y)\big) \right)$$

if s is smooth. Since $m(x) = ax$, we look for a solution of the form $s_\infty(x) = \gamma x$:

$$\sum_{t=0}^{\infty} \delta^t \left(\big(a - \gamma^{t+1}\big) \prod_{i=0}^{t-1} \gamma \right) = \sum_{t=0}^{\infty} \delta^t \big(\big(a - \gamma^{t+1}\big)\gamma^t\big) = 0,$$

whence $\frac{a}{1-\delta\gamma} = \frac{\gamma}{1-\delta\gamma^2}$. By similar arguments as before, this equation has a unique solution $\gamma_\infty \in (0,1)$ and $\gamma_k^k \to \gamma_\infty$ because $A_k(x) \xrightarrow[k\to\infty]{} \frac{x-\delta x^2}{1-\delta x^2}$. Finally, $\frac{\partial U_{m(x)}(S(y))}{\partial y}|_{y=y_0} > 0$ ($<$) for $y_0 < s(x)$ ($>$) follows from combining Assumption A2 with the fact $\frac{\partial U_{m(x)}(S(y))}{\partial y}|_{y=s(x)} = 0$ $\forall x$. Hence $y = s(x)$ maximizes $U_{m(x)}(S(y))$. *Q.E.D.*

In the general case, however, kEs for $k > 1$ and continuous equilibria may not exist. The issue is the following. Suppose that a kE s_k exists in a right-neighborhood of a stable steady state, $[x^*, x^* + \epsilon)$ (even this is not guaranteed in general). s_k can then be extended at least to $[x^*, x^* + d^+_{x^*}]$ (Lemma 10), but its extension may fail to be a kE. Similarly, the unique extension of a continuous equilibrium may have discontinuities; whether this happens depends on arbitrarily small details of m.

Here is an intuition. Assume $u_\alpha(x) = C - (\alpha - x)^2$, $\delta > \frac{2}{3}$, $a > 0.5$, and $\tilde{m}(x) = ax + \frac{a}{4}\max(c - |x - x'|, 0)$, where $c > 0$ is small. We are in the linear case, except $\tilde{m}$ has a small "bump" around x'. Let s be a 2E for $m(x) = ax$ such that $x_0 = x'$, and let $\tilde{s}$ be a 2E for $\tilde{m}$ such that $\tilde{x}_n = x_n$ for $n > 0$. As $m(x_0)$ and $\tilde{m}(\tilde{x}_0)$ must both be indifferent between $S(x_2)$ and $S(x_3)$, $m(x_0) = \tilde{m}(\tilde{x}_0)$, but $\tilde{m}(x_0) > m(x_0)$, so $\tilde{x}_0 < x_0$. Meanwhile, $\tilde{x}_{-1} = x_{-1}$. But $m(\tilde{x}_{-2})$, being indifferent between $\tilde{S}(\tilde{x}_0)$ and $S(x_1)$, must be *lower* than $m(x_{-2})$ because $\tilde{x}_0$ being lower makes the former path more attractive than $S(x_0)$, so $\tilde{x}_{-2}$ is lower. On the other hand, $\tilde{x}_{-3} > x_{-3}$ because it is defined by indifference between $S(x_{-1})$ and $\tilde{S}(\tilde{x}_0)$ (more attractive than $S(x_0)$). Continuing in this fashion, the subsequence $(\tilde{x}_0, \tilde{x}_{-2}, \tilde{x}_{-4}, \ldots)$ is lower than $(x_0, x_{-2}, \ldots)$, and the opposite is true for the odd elements. Eventually, $\tilde{x}_{2l} < \tilde{x}_{2l+1}$ for some l, that is, the even subsequence becomes so attractive that a voter $m(\tilde{x}_{2l+1})$, though indifferent between $\tilde{S}(\tilde{x}_{2l+3})$ and $\tilde{S}(\tilde{x}_{2l+4})$, instead prefers $\tilde{S}(\tilde{x}_{2l+2})$ to both, so no one votes for x_{2l+3}.

kEs for $k > 1$ are unstable for this reason. We now give a local argument to this effect. Let $E_n = (1-\delta)\sum_{t=0}^{\infty} \delta^t x_{n+tk}$, $W_n = (1-\delta)\sum_{t=0}^{\infty} \delta^t x^2_{n+tk}$, and characterize a kE recursively as follows, using the indifference of $m(x_n)$ between S_{n+k} and S_{n+k+1}, that is, $-(m(x_n) - E_{n+k})^2 - W_{n+k} = -(m(x_n) - E_{n+k+1})^2 - W_{n+k+1}$:

$$E_n = (1-\delta)x_n + \delta E_{n+k} = (1-\delta)m^{-1}\left(\frac{1}{2}\frac{W_{n+k} - W_{n+k+1}}{E_{n+k} - E_{n+k+1}}\right) + \delta E_{n+k},$$
$$W_n = (1-\delta)x_n^2 + \delta W_{n+k} = (E_n - \delta E_{n+k})x_n + \delta W_{n+k}.$$

Taking $Y_n = (E_n, \dots, E_{n+k+1}, W_{n+1}, \dots, W_{n+k+1})$ as the state variable of the recursion, its linearization around an equilibrium is given by $Y_n = M_n Y_{n+1}$, where

$$M_n = \begin{pmatrix} 0 & \dots & 0 & A & B & 0 & \dots & 0 & C & D \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 2x & 0 & \dots & 0 & -2\delta x & 0 & 0 & \dots & 0 & \delta \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{matrix} \}1 \\ \\ \\ \}k \\ \\ \}1 \\ \\ \\ \}k-1 \\ \\ \end{matrix}$$

where $x = x_{n+1}$; $B = \frac{\partial E_n}{\partial E_{n+k+1}}$, $D = \frac{\partial E_n}{\partial W_{n+k+1}}$, and so on. Note that

$$\det(M_n) = -\delta B - 2\delta x_{n+1} D = \delta(1-\delta)\frac{x_{n+1} - m(x_n)}{m'(x_n)(E_{n+k} - E_{n+k+1})},$$

$$\begin{aligned} \det(M_n \dots M_{n-k+1}) &\geq \delta^k (1-\delta)^k \left[\min_{0\leq l\leq k-1}\left(\frac{x_{n-l+1} - m(x_{n-l})}{m'(x_{n-l})}\right)\right]^k \frac{1}{\prod_{l=0}^{k-1}(E_{n-l+k} - E_{n-l+k+1})} \\ &\geq \delta^k (1-\delta)^k \left[\min_{0\leq l\leq k-1}\left(\frac{x_{n-l+1} - m(x_{n-l})}{m'(x_{n-l})}\right)\right]^k \frac{k^k}{(E_{n+1} - E_{n+k+1})^k} \\ &= \delta^k \left[\min_{0\leq l\leq k-1}\left(\frac{x_{n-l+1} - m(x_{n-l})}{m'(x_{n-l})}\right)\right]^k \frac{k^k}{(x_{n+1} - E_{n+k+1})^k}, \end{aligned}$$

where we have used the AM-GM inequality. Now, if δ is close to 1 and the equilibrium is close to a CLS in the sense of Proposition 6, then $\frac{x - m(x)}{m'(x)(x - E(S(s(x))))} \approx 1$ (see Appendix B) and $\det(M_n \dots M_{n-k+1}) \approx k^k$. (In particular, these statements hold with equality for the linear case we have solved above.) Hence there must be an eigenvalue of absolute value at least $k^{\frac{k}{2k+1}} > 1$. Thus, any deviation from an equilibrium resulting from a local perturbation of m which adds a nonzero component to a generalized eigenvector of this eigenvalue (in the Jordan form decomposition of the matrix) will grow exponentially.

Similarly, in the example given above of a linear m with a bump, a continuous equilibrium constructed up to $x' - c$ would have a discontinuity at $s^{-1}(x')$. This insight extends more generally. For brevity, we show this for smooth (C^∞) equilibria.

PROPOSITION 10: *Assume $u_a(x) = C - (\alpha - x)^2$. Let $s : [x^*, x^{**}) \to [x^*, x^{**})$ be a smooth equilibrium for a given m. Let $x_0 \in (x^*, x^{**})$. A perturbation $\tilde{m}$ of m is an increasing function $\tilde{m} = m + \rho\kappa$ where $\kappa : [x^*, x^{**}) \to [x^*, x^{**})$ has support $(x_0 - \epsilon, x_0 + \epsilon)$. For each $\tilde{m}$, let $\tilde{s}$ be an equilibrium under $\tilde{m}$ such that $\tilde{s}|_{[x^*, x_0 - \epsilon)} = s|_{[x^*, x_0 - \epsilon)}$.*

Suppose m is C^∞. Then, if κ is C^k but its $(k+1)$th derivative has a discontinuity in $(x_0 - \epsilon, x_0 + \epsilon)$, $\tilde{s}$ has a discontinuity in $[x^*, s^{-k-1}(x_0 + \epsilon)]$ for all $\rho \neq 0$.

PROOF: Denote $E(y) = E(S(y))$, $W(y) = W(S(y))$, and $\frac{1}{2}\frac{\partial W(y)}{\partial E(y)} = L^{-1}(y)$. Then

$$s(x) = \arg\max_y -m(x)^2 + 2m(x)E(y) - W(y) \quad \Rightarrow \quad m(x) = \frac{1}{2}\frac{\partial W(y)}{\partial E(y)}\bigg|_{s(x)},$$

$$s(x) = \left(\frac{1}{2}\frac{\partial W(y)}{\partial E(y)}\right)^{-1}\big(m(x)\big) = L\big(m(x)\big),$$

$$(W, E)(x) = \Big((1-\delta)x^2 + \delta W\big(L\big(m(x)\big)\big), (1-\delta)x + \delta E\big(L\big(m(x)\big)\big)\Big).$$

In particular, $W(E)$ must be a strictly convex function so that s is surjective, and it must have no kinks, that is, s must be strictly increasing (if s is locally constant at x, it will be discontinuous at $s^{-1}(x)$ as long as $s(y) > m(y)$ in this area), so L^{-1} and L are strictly increasing and well-defined.

Now suppose E is C^{l+1} around $s(s(x))$ but s has a $(l+1)$-kink at $s(x)$, that is, it is C^{l+1} in $(s(x)-\eta, s(x)) \cup (s(x), s(x)+\eta)$ but only C^l in $(s(x)-\eta, s(x)+\eta)$. Then s' has a l-kink at $s(x)$. Since

$$\frac{\partial W(y)}{\partial E(y)} = \frac{\dfrac{\partial W(y)}{\partial y}}{\dfrac{\partial E(y)}{\partial y}} = \frac{2(1-\delta)y + \delta W'\big(s(y)\big)s'(y)}{(1-\delta) + \delta E'\big(s(y)\big)s'(y)}$$

$$= \frac{W'\big(s(y)\big)}{E'\big(s(y)\big)} + \frac{(1-\delta)\left(2y - \dfrac{W'\big(s(y)\big)}{E'\big(s(y)\big)}\right)}{1-\delta+\delta E'\big(s(y)\big)s'(y)} = 2m(y) + \frac{(1-\delta)\big(2y - 2m(y)\big)}{1-\delta+\delta E'\big(s(y)\big)s'(y)},$$

L^{-1} has a l-kink at $s(x)$; L has a l-kink at $m(x)$; and s has a l-kink at x.

Then, if κ has a $(k+1)$-kink at x, then $\tilde{m}$ has a $(k+1)$-kink at x; $\tilde{s}$ has a $(k+1)$-kink at x; $\tilde{s}$ has a k-kink at $\tilde{s}^{-1}(x)$; ... and $\tilde{s}$ has a discontinuity at $\tilde{s}^{-k-1}(x)$. *Q.E.D.*

A similar result holds even for smooth perturbations.

PROPOSITION 11: *Assume $u_a(x) = C - (\alpha - x)^2$. Let $s : [x^*, x^{**}) \to [x^*, x^{**})$ be a smooth equilibrium for a given m with $m' \geq A > \frac{1}{2}$ everywhere. Let $z \in (x^*, x^{**})$ and $z' = s(z)$. Given a smooth function $\kappa : [x^*, x^{**}) \to [x^*, x^{**})$ with support contained in $[z', z]$ and $\rho \in \mathbb{R}$, let $m_\rho = m + \rho\kappa$, and let s_ρ be the unique equilibrium under m_ρ that equals s within $[x^*, z']$, if it exists. Then there exists κ such that s_ρ is discontinuous for all $\rho \neq 0$.*

PROOF: Pick $K \in \mathbb{N}$ and define a set of $2K$ sequences (y_{ni}) ($n \in \mathbb{Z}$, $i \in \{1, \dots, 2K\}$) as follows. $y_{01} = z'$, $y_{0(2K)} = z$, $(y_{0i})_i$ is an arithmetic sequence, and, for all n, i, $y_{(n+1)i} = s(y_{ni})$.

Let $\tilde{\kappa}$ be a nonnegative C^∞ function with support $[0, 1]$ such that $\tilde{\kappa}(\frac{1}{2}) = 1$. Define $\kappa(x) = \sum_{i \leq 2K \text{ odd}} \tilde{\kappa}(\frac{x - y_{0i}}{y_{0(i+1)} - y_{0i}})$, so that κ has a copy of $\tilde{\kappa}$ "squeezed" into each interval $[y_{0i}, y_{0(i+1)}]$ for i odd. Any such κ will work.

We write $W(E)$ to denote $W(E^{-1}(E))$, W_ρ to denote W with perturbation $\rho\kappa$, etc. Assume $\rho > 0$. It is easy to show that $s_\rho(x) \equiv s(x)$ for $x \in [y_{ni}, y_{n(i+1)}]$ for all n and even i, and $s_\rho(x) > s(x)$ for $x \in [y_{0i}, y_{0(i+1)}]$ for odd i (because $s_\rho(x) = s(y)$ whenever $m_\rho(x) = m(y)$). In addition, $W_\rho(E) < W(E)$ for $x \in [E(y_{0i}), E(y_{0(i+1)})]$ for odd i, since $\frac{\partial W}{\partial E}(y) = 2m(y)$

but, for a path $(x_t)_t$, decreasing x_0 by ϵ decreases $E((x_t)_t)$ by $(1-\delta)\epsilon$ and $W((x_t)_t))$ by $(1-\delta)\epsilon(2x_0-\epsilon)$, that is, $\frac{\Delta W}{\Delta E}\approx 2x_0 > 2m(x_0)$.

For odd i, let $\Delta W_{0i} = \max_{E\in[E(y_{0i}),E(y_{0(i+1)})]}[W(E)-W_\rho(E)]$, and denote by $\tilde{E}_{0i}$ the argmax. Then there must be a point $\hat{E}_{0i}\in[E(y_{0i}),E(y_{0(i+1)})]$ for which $\frac{\partial W}{\partial E}(\hat{E}_{0i})-\frac{\partial W_\rho}{\partial E_\rho}(\hat{E}_{0i})\geq \frac{\Delta W_{0i}}{E(y_{0(i+1)})-E(y_{0i})}$.

Let $\acute{y}$ be such that $2m(\acute{y})=\frac{\partial W}{\partial E}(\hat{E}_{0i})$ and $\grave{y}$ be such that $2m(\grave{y})=\frac{\partial W_\rho}{\partial E_\rho}(\hat{E}_{0i})$. Then $\acute{y}-\grave{y}\geq\frac{1}{2\overline{m}'}\frac{\Delta W_{0i}}{E(y_{0(i+1)})-E(y_{0i})}$. Then, denoting $\grave{E}=E_\rho(\grave{y})$ and $\acute{E}=E_\rho(\acute{y})$, $W_\rho(\grave{E})=(1-\delta)\grave{y}^2+\delta W_\rho(\hat{E}_{0i})\leq(1-\delta)\grave{y}^2+\delta W(\hat{E}_{0i})$, and $W(\grave{E})\geq W(\acute{E})-(\acute{E}-\grave{E})\frac{\partial W}{\partial E}(\acute{E})=(1-\delta)\acute{y}^2+\delta W(\hat{E}_{0i})-(\acute{E}-\grave{E})2m(\acute{y})$. Also note that $\grave{y}>s(\acute{y})>m(\acute{y})$ and $y-E(s(y))\geq 2(y-m(y))$ for all y. Hence

$$\begin{aligned}
\Delta W_{1i} &\geq W(\grave{E})-W_\rho(\grave{E})\geq(1-\delta)(\acute{y}-\grave{y})\big(\acute{y}+\grave{y}-2m(\acute{y})\big)\\
&\geq\frac{1-\delta}{2\overline{m}'}\left[\frac{\acute{y}-m(\acute{y})}{E(y_{0(i+1)})-E(y_{0i})}\right]\Delta W_{0i},\\
\prod_i\Delta W_{1i} &\geq\left[\frac{1-\delta}{2\overline{m}'}\right]^K\prod_i\left[\frac{\acute{y}_i-m(\acute{y}_i)}{E(y_{0(i+1)})-E(y_{0i})}\right]\prod_i\Delta W_{0i}\\
&\geq\left[\frac{(1-\delta)K}{2\overline{m}'}\right]^K\prod_i\left[\frac{\acute{y}_i-m(\acute{y}_i)}{E(y_{0(2K)})-E(y_{01})}\right]\prod_i\Delta W_{0i}\\
&\geq\left[\frac{K}{4\overline{m}'}\right]^K\prod_i\left[\frac{\acute{y}_i-m(\acute{y}_i)}{y_{0(2K)}-m(y_{0(2K)})}\right]\prod_i\Delta W_{0i}\geq\left[\frac{K}{8\overline{m}'}\right]^K\prod_i\Delta W_{0i}.
\end{aligned}$$

If we choose $K>8\overline{m}'$, iterating this argument, we find that there must be n, i (possibly functions of ρ) for which $\Delta W_{ni}>(x^{**}-x^*)^2$, a contradiction. *Q.E.D.*

It follows that, if $m|_{[x^*,z)}$ admits a smooth equilibrium s, then the set of extensions of m to $[x^*,x^{**})$ that admit a smooth extension of s to $[x^*,x^{**})$ is shy. I conjecture that $\forall\epsilon>0$, the set of m's admitting a smooth equilibrium on $[x^*,x^*+\epsilon]$ is also shy.

Non-Monotonic Equilibria

Proposition 3 shows that MVEs must be monotonic in a neighborhood of a stable steady state, and 1Es are monotonic everywhere. However, non-monotonic MVEs may exist; we provide an example here. Assume that $u_\alpha(x)=C-(\alpha-x)^2$ and let $d=\sqrt{C}=d_x^-=d_x^+$ for all x. In addition, suppose that $m(x)=x-\rho d$ for all $x\in\mathbb{R}$, where $\rho\in[\frac{1}{2},1)$ is a parameter.[15] For simplicity, we will take the MVT as a primitive, that is, we will analyze the game in which $m(x)$ picks $s(x)$.

Assume $\delta=0$. Then $s(x)=m(x)$, so $S(s(x))=(x-\rho d,x-2\rho d,x-3\rho d,\ldots)$. Crucially, $x-\rho d, x-2\rho d\in(m(x)-d,m(x)+d)$ but $x-3\rho d\notin(m(x)-d,m(x)+d)$.

[15] m can be obtained as a median voter function if $f(x)=ke^{-\hat{\rho}x}$ for an appropriately chosen $\hat{\rho}>0$. This is a degenerate example in the sense that there is an infinite mass of voters distributed on the real line, as opposed to a unit mass with support $[-1,1]$, but it allows for a simpler construction.

Now, suppose that δ is small but positive. Assume a successor function s_1 of the form $s_1(x) = x - \rho d + \eta_1$ with η_1 small, such that $s_1^2(x) > m(x) - d$ but $s_1^3(x) < m(x) - d$. For s_1 to be an equilibrium, η_1 must maximize

$$u_{m(x)}(x - \rho d + \eta) + \delta u_{m(x)}\big(s_1(x - \rho d + \eta)\big) = C(1+\delta) - \eta^2 - \delta(\rho d - \eta_1 - \eta)^2$$
$$\implies \quad \eta_1 = \frac{\delta}{1+2\delta}\rho d.$$

Note that, since $\eta_1 > 0$, this calculation will be invalid for ρ close enough to $\frac{1}{2}$, as in fact we will then have $s_1^3(x) > m(x) - d$.

Next, assume a successor function $s_2(x) = x - \rho d + \eta_2$ with η_2 such that $s_2^3(x) > m(x) - d$ but $s_2^4(x) < m(x) - d$. For s_2 to be an equilibrium, we must have

$$\eta_2 = \arg\max_{\eta'} u_{m(x)}\big(x - \rho d + \eta'\big) + \delta u_{m(x)}\big(s_2\big(x - \rho d + \eta'\big)\big) + \delta^2 u_{m(x)}\big(s_2^2\big(x - \rho d + \eta'\big)\big)$$
$$\implies \quad \eta_2 = \frac{\delta + 2\delta^2}{1 + 2\delta + 3\delta^2}\rho d.$$

Now suppose that s_2 is being played, and x considers deviating to some η_3 such that $s_2^2(x - \rho d + \eta_3) < m(x) - d$. The (locally) optimal η_3 must satisfy

$$-2\eta_3 - 2\delta(\eta_3 + \eta_2 - \rho d) = 0 \quad \implies \quad \eta_3 = \frac{\delta}{1+\delta}(\rho d - \eta_2) = \frac{\delta}{1+\delta}\frac{1+\delta+\delta^2}{1+2\delta+3\delta^2}\rho d.$$

Note that $\eta_2 > \eta_1 > \eta_3 > 0$. In particular, since $\eta_2 > \eta_3$, we can choose ρ, δ so that $x - 3\rho d + 3\eta_2 > m(x) - d > x - 3\rho d + \eta_3 + 2\eta_2$. Furthermore, we can choose them so that $U_{m(x)}(S_2(x - \rho d + \eta_3)) = U_{m(x)}(S_2(x - \rho d + \eta_2))$.[16] If we choose our parameters this way, then $m(x)$ is indifferent about deviating (this is true for all x). Now construct a successor function s as follows: $s(x) = s_2(x)$ for all $x < x_0$; $s(x_0) = x - \rho d + \eta_3$; and $s(x)$ for $x > x_0$ is defined by backward induction. This is a non-monotonic equilibrium by construction. While this example relies on indifference, we can adjust m to make it strict.

APPENDIX E: Additional Results

E.1. *Supermajority Requirements and Other Decision Rules*

We assume as part of our solution concept that chosen policies are Condorcet winners. The analysis readily extends to other decision rules. We briefly discuss two.

First, suppose that, given a policy x and a set of members $I(x)$, an unmodeled political process gives some agent $n(x)$ the right to choose tomorrow's policy. (For example, the function $n(x)$ might reflect the notion that the policy x affects the relative power of different agents within the organization.) All the results extend to this case, substituting $n(x)$ for $m(x)$, even if $n(x)$ is not a median voter function.

Second, consider an organization with a bias toward inaction in which policy changes require a supermajority $\rho > \frac{1}{2}$. Define $m_p(x)$ as the pth percentile-member of $I(x)$, and

[16]This follows from a continuity argument: when $s_2^2(x - \rho d + \eta_3) = m(x) - d$, $U_{m(x)}(S_2(x - \rho d + \eta_3)) < U_{m(x)}(S_2(x - \rho d + \eta_2))$, whereas when $s_2^2(x - \rho d + \eta_2) = m(x) - d$, then $U_{m(x)}(S_2(x - \rho d + \eta_3)) > U_{m(x)}(S_2(x - \rho d + \eta_2))$, so we can choose intermediate values of ρ, δ for which $U_{m(x)}(S_2(x - \rho d + \eta_3)) = U_{m(x)}(S_2(x - \rho d + \eta_2))$.

assume that a policy $y > x$ can only be chosen over the current policy x if $m_{1-\rho}(x)$ votes for it, while a change to $y < x$ is only possible if m_ρ votes for it. It follows that, in intervals where $x > m_\rho(x)$, the game is equivalent to the main model with $n(x) = m_\rho(x)$; in intervals where $x < m_{1-\rho}(x)$, it is equivalent to setting $n(x) = m_{1-\rho}(x)$; and in intervals where $m_{1-\rho}(x) < x < m_\rho(x)$, no policy changes are possible. In other words, steady states are now intervals rather than points, and we will observe lower policy drift, but the gist of the results is unchanged.

E.2. *Positive Entry and Exit Costs*

The main model assumes free entry and exit. This assumption adds a lot of tractability, but is rarely exactly true in a descriptive sense. Here, we demonstrate that our results are also relevant in a setting with positive entry and exit costs. We begin by considering a variant of the game with only entry costs: every time an outsider chooses to join the club, she must pay a cost $c > 0$.

A full extension of the results to this case is difficult because the introduction of entry costs adds intertemporal concerns to entry and exit decisions: agents considering entry now need to think about what the policy will be several periods from now, whether they will want to leave later, etc. Relatedly, agents with identical preferences may behave differently depending on their current status: if the policy x is stable over time, and an agent α's flow payoff from membership, $u_\alpha(x)$, is positive but very small, then α would choose to remain in the club if she is already a member but not bother entering otherwise. As a result, the club's current policy is no longer the only payoff-relevant state variable; I_t is now an (infinite-dimensional) state variable as well.

In spite of this, the main thrust of the paper's results—namely, that the club should converge to a myopically stable policy—still carries over in this model if we impose some reasonable simplifications. Concretely, we will assume the following:

(i) As in Section 4, we restrict the analysis to $[x^*, x^{**}]$, the right side of the basin of attraction of a stable steady state x^*.

(ii) Assume $x_{t+1} \leq x_t$ for all $x \in [x^*, x^{**}]$, that is, the policy cannot move to the right.

(iii) Assume an initial $x_0 \in [x^*, x^{**}]$ such that $u_{x_0-d_{x_0}}(x^*) \geq (1-\delta)c$. In other words, x_0 is close enough to x^* that all agents who might consider joining the club as the policy moves left from x_0 will strictly prefer not to quit later.[17]

(iv) Assume an initial set of members $I_0 = I(x_0)$.

(v) Voting behavior at time t can condition on I_t only up to a set of measure zero, that is, if I_1, I_2 differ by a set of Lebesgue measure zero, then $s(I_1) = s(I_2)$.

An MVE of this game is given by mappings $s(I)$ and $I(x, I')$ satisfying the above conditions such that I reflects optimal entry and exit decisions given current policy x, an existing set of members I', and the expected continuation; and $S(s(I))$ is a Condorcet winner among all $S(y)$ for the set of voters I.

It turns out that the set of equilibria of this game corresponds exactly to the set of equilibria of a game with free entry and exit but modified utility functions. Let $G(u, c)$ denote the game just described, with cost of entry c and utility functions $u_\alpha(x)$. Let $G(v, 0)$ denote the game with free entry and exit and utility functions $v_\alpha(x)$ given by $v_\alpha = u_\alpha$ for $\alpha \geq x_0 - d_{x_0}$ and $v_\alpha = u_\alpha - (1-\delta)c$ for $\alpha < x_0 - d_{x_0}$.

[17]Effectively, this means we find equilibria restricted to $[x^*, x_0]$.

PROPOSITION 12: *In any MVE* (s, I) *of* $G(u, c)$, $I(x, I') = I_v(x)$ *for all* (x, I') *on the equilibrium path*.

For any MVE (s, I) *of* $G(u, c)$, *there is an MVE* $\tilde{s}$ *of* $G(v, 0)$ *given by* $\tilde{s}(x) = s(I_v(x))$. *Conversely, for any MVE* $\tilde{s}$ *of* $G(v, 0)$, *there is an MVE* $(s, \tilde{I})$ *of* $G(u, c)$ *given by* $s(I) = \tilde{s}(m_v(I))$ *and* $\tilde{I}(x, I') = I_v(x)$.

PROOF: For the first claim, suppose $(x_t, I_t)_t$ is an equilibrium path, and assume that $I_t = I_v(x_t)$. We aim to show that $I_{t+1} = I_v(x_{t+1})$.

There are four types of agents to consider. First, suppose $\alpha \notin I_t$ and $\alpha > x_t$, that is, α is an outsider with a policy preference to the right of x_t. Then $\alpha \notin I_v(x_t)$, that is, $u_\alpha(x_t) < 0$, so $u_\alpha(x_{t+1}) \le u_\alpha(x_t) < 0$, whence $\alpha \notin I_v(x_{t+1})$. Moreover, since $u_\alpha(x_s) < 0$ for all $s \ge t$, α should not join the club at time $t + 1$, that is, $\alpha \notin I_{t+1}$.

Second, suppose $\alpha \in I_t$ and $\alpha \ge x_0 - d_{x_0}$. Then α is an incumbent member at time $t + 1$, and will choose to remain a member iff $u_\alpha(x_{t+1}) \ge 0$, which is also the condition that determines whether $\alpha \in I_v(x_{t+1})$ as $u_\alpha = v_\alpha$ for this agent.

Third, suppose $\alpha \in I_t$ and $\alpha < x_0 - d_{x_0}$. Since $\alpha \in I_t$, α is an incumbent member at time $t + 1$. Since $\alpha < x_0 - d_{x_0}$, we have $u_\alpha(x_s) \ge (1 - \delta)c > 0$ for all $s \ge t$. This means both that α will choose to remain a member forever (in particular, at time $t + 1$) and that $\alpha \in I_v(x_{t+1})$.

Fourth, suppose $\alpha \notin I_t$ and $\alpha < x_t$, that is, α is an outsider with a policy preference to the left of x_t. Then α should join at time $t + 1$ iff $u_\alpha(x_{t+1}) \ge (1 - \delta)c$, that is, iff $\alpha \in I_v(x_{t+1})$.[18] Since $G(u, c)$ has the same membership behavior as $G(v, 0)$, the two games are equivalent, so the sets of equilibrium successor functions are the same. *Q.E.D.*

As for the case of positive exit costs, it can be shown that, with a positive exit cost $c' > 0$, the game $G(u, c, c')$ is still equivalent to a game with free entry and exit, except that now the relevant utility functions are $v_\alpha = u_\alpha + (1 - \delta)c'$ for $\alpha \ge x_0 - d_{x_0}$ and $v_\alpha = u_\alpha - (1 - \delta)c$ for $\alpha < x_0 - d_{x_0}$.

We can apply Propositions 2 and 3 to $G(u, c, c')$ to determine the organization's long-run policy in this setting. Let $m_v(y) = m(I_v(y))$, and note that $m_v(y) \ge m_u(y)$ for all $y \in [x^*, x_0]$. Let $y^*(c, c', \delta)$ be the highest $y \in [x^*, x_0]$ for which $m_v(y^*) = y^*$. Then it follows that $x_t \to y^*$ for any equilibrium path $(x_t, I_t)_t$.

A few interesting observations can be made. First, $y^* > x^*$: the existence of entry and exit costs affects the long-run policy, as some marginal agents near $x^* - d^-_{x^*}$ never enter, or some agents near $x^* + d^+_{x^*}$ never exit. Second, y^* is a function of δ, unlike x^*; this is because the entry and exit decisions of marginal agents now involve intertemporal trade-offs. Third, it can be shown that $y^*(c, c', \delta) \to x^*$ as $c, c \to 0$, or as $\delta \to 1$ if we take c, c' as fixed. That is, small entry and exit costs only have a small effect on the organization's long-run policy, and even sizable costs matter less and less as agents become more patient, as they only have to be paid once.

E.3. *Non-Markov Equilibria*

The restriction to Markov equilibria may appear restrictive: allowing strategies to condition only on the current policy prevents agents from doling out history-dependent rewards and punishments in ways that might be plausible in some applications. This section discusses the set of non-Markov equilibria of the game.

[18]In all of these arguments, α expects the equilibrium path not to change as a function of her behavior, because her joining or leaving the club amounts to a measure zero change to I_{t+1}.

We make two main points. First, if non-Markov equilibria are allowed, many outcomes are possible; under some conditions, an "anything goes" result is obtained. Thus, no strong predictions can be made if we take SPE as our solution concept. Second, there are several substantively plausible perturbations of the game which rule out all non-Markov equilibria. Taken together, the results suggest that Markov equilibria are the most sensible to study in this setting.

For simplicity, we restrict our analysis in the following ways. First, as in Section 4, we restrict our analysis to $[x^*, x^{**})$, the right side of the basin of attraction of a stable steady state x^*. Second, we write the results for the framework given in Section 5, which avoids some non-essential technical issues related to entry and exit. Third, we assume the MVT as a primitive, that is, we study a game in which, given a current policy x, $m(x)$ directly chooses tomorrow's policy.[19] Recall our definition of reluctant agents from Section 4, and say $m(x)$ is *very reluctant* if she is reluctant and $(1-\delta)u_{m(x)}(x') + \delta u_{m(x)}(z(x')) \leq u_{m(x)}(x)$ for all $x' \in (z(x), x)$.

PROPOSITION 13: *If every $x \in [x^*, x^{**})$ is very reluctant, then, for every weakly decreasing path $(y_t)_{t\in\mathbb{N}_0} \subseteq [x^*, x^{**})$ such that $U_{m(y_t)}((y_{t+1}, y_{t+2}, \ldots)) \geq \frac{u_{m(y_t)}(y_t)}{1-\delta}$ for all t, there is an SPE with policy path $(y_t)_t$.*

PROOF: We construct a suitable successor function $s(x, T)$, where T is a payoff-irrelevant function of the history. T can take on the values 0, 1, or 2. s is defined as follows:
(i) $s(y_t, 0) = y_{t+1}$ for all t, and $s(x, 0) = x$ for all $x \notin (y_t)_t$;
(ii) $s(x, 1) = x$;
(iii) $s(x, 2) = z(x)$.
Define $T_0 = 0$, and T_τ for $\tau > 0$ according to the following mapping H:
(i) If $T = 0$ and $x = y_t$, $x' = y_{t+1}$ for some t, then $H(x, T, x') = 0$;
(ii) else, if $x' \notin (z(x), x)$, $H(x, T, x') = 1$;
(iii) else $H(x, T, x') = 2$.
In other words, in state 0, the policy follows the intended equilibrium path, $(y_t)_t$, and T_τ remains equal to zero. In state 1, the policy path is constant and T_τ remains equal to 1. In state 2, the current decision-maker, $m(x)$, chooses the lowest policy that she weakly prefers to x, and the state then changes to 1. Deviations to myopically attractive policies (policies that $m(x)$ strictly prefers to x) are punished by switching to state 2, while deviations to myopically unattractive policies are punished by switching to state 1.

We can verify that this is an SPE. If $(x_\tau, T_\tau) = (y_t, 0)$ for some t, then $m(x_\tau)$'s equilibrium continuation utility is $U_{m(y_t)}(S(y_{t+1})) \geq \frac{u_{m(y_t)}(y_t)}{1-\delta}$. If she deviates to $x' \notin (z(x_\tau), x_\tau)$, then $T_{\tau+1} = 1$ and the continuation is given by $x_{\tau'} = x'$ for all $\tau' > \tau$, yielding utility $\frac{u_{m(y_t)}(x')}{1-\delta} \leq \frac{u_{m(y_t)}(y_t)}{1-\delta}$. If she deviates to $x' \in (z(x_\tau), x_\tau)$, then $T_{\tau+1} = 2$ and the continuation is $x_{\tau+1} = x'$, $x_{\tau'} = z(x')$ for all $\tau' > \tau + 1$, yielding utility $u_{m(y_t)}(x') + \frac{\delta}{1-\delta} u_{m(y_t)}(z(x')) \leq \frac{u_{m(y_t)}(y_t)}{1-\delta}$.

If $(x_\tau, T_\tau) = (x, T)$ with $T = 1$ or $T = 2$, then $m(x)$'s equilibrium continuation utility is $u_{m(x)}(x)$. If she deviates to $x' \notin (z(x), x)$, she gets utility $\frac{u_{m(x)}(x')}{1-\delta} \leq \frac{u_{m(x)}(x)}{1-\delta}$. If she deviates to $x' \in (z(x), x)$, she gets $u_{m(x)}(x') + \frac{\delta}{1-\delta} u_{m(x)}(z(x')) \leq \frac{u_{m(x)}(x)}{1-\delta}$. *Q.E.D.*

[19]In the model from Section 5, the MVT always holds, so this assumption is only for brevity.

The condition $U_{m(y_t)}((y_{t+1}, y_{t+2}, \ldots)) \geq u_{m(y_t)}(y_t)$ is clearly necessary—otherwise, $m(y_t)$ would deviate to staying at y_t forever. What this result shows is that, aside from this common-sense restriction, anything goes.[20] In particular, for each $x \in [x^*, x^{**}]$, there is an SPE with policy path constantly equal to x, so any policy can become an intrinsic steady state in the right SPE.

However, there are several arguments for focusing on Markov equilibria.

(i) Non-Markovian behavior must be supported by non-Markovian behavior arbitrarily close to x^*. Moreover, assuming any particular form of Markov behavior in a neighborhood of x^* collapses the set of equilibria to a single (Markov) equilibrium:

LEMMA 10: *Let $\epsilon > 0$ and $\tilde{s} : [x^*, x^* + \epsilon] \to [x^*, x^* + \epsilon]$ such that $\tilde{s}(x) < x$ for all x. Let s, s' be two SPEs on $[x^*, x^{**}]$ such that $s(x, h) = s'(x, h) = \tilde{s}(x)$ for all $x \in [x^*, x^* + \epsilon]$ and h, and assume that s and s' obey the following tie-breaking rule: if the set $\arg\max_{y \leq x} U_{m(x)} S(y, h)$ has multiple elements, then $s(y)$ is the highest element of the set. Then $s \equiv s'$ and $s(x, h)$ is independent of h.*

PROOF OF LEMMA 10: The intuition behind this result is a simple unraveling argument: suppose two equilibria coincide up to some point $x^* + \epsilon$. Then, for y slightly above $x^* + \epsilon$, $I(y)$ will be choosing between successors in $[x^*, x^* + \epsilon]$, which have the same continuation in both equilibria, so the same choice will be made. Formally, let

$$A = \{x \in [x^*, x^{**}] : \exists \hat{s} \text{ s.t. } \forall h, \forall y \in [x^*, x],\ s(y, h) = s'(y, h) = \hat{s}(y)\},$$

and $x_0 = \sup(A)$. By assumption, $x_0 \geq x^* + \epsilon$. Suppose $x_0 < x^{**}$.

There are two cases. First, suppose $x_0 \notin A$. Then the same proof as in Proposition 2 shows that $u_{m(x_0)}(x_0) < \max_{y \in [x^*, x_0)} U_{m(x_0)} \hat{S}(y)$, so $s(x_0, h), s'(x_0, h) < x_0$ $\forall h$. The tie-breaking rule then implies $s(x_0, h) = s'(x_0, h) = \max(\arg\max_{y \in [x^*, x_0)} U_{m(x_0)} \hat{S}(y))$ $\forall h$, whence $\hat{s}$ can be extended to x_0, a contradiction.

Second, suppose $x_0 \in A$. Then there is a sequence $(x_n)_n$ such that $x_n \to x_0$ and $x_n > x_0$ $\forall n$ such that, for each n, $s(x_n, h_n) > x_0$ for some history h_n, as otherwise the tie-breaking rule would guarantee that $s(x_n, h) \equiv s'(x_n, h)$ are independent of h.

Note that, for each n, $m(x_n)$ always has the option of jumping to any policy $z \in [x^*, x_0]$, and that the continuation would be the history-independent path $\hat{S}(z)$; hence, the optimality of s requires that $U_{m(x_n)}(S(s(x_n, h_n), h_n)) \geq U_{m(x_n)}(\hat{S}(z))$.

For each n, label the continuation path starting at $s(x_n, h_n)$ as $S_n = (s_0^n, s_1^n, \ldots)$, where $s_0^n = s(x_n, h_n)$. Let $s_{k_n}^n$ be the first policy in this path that is in $[x^*, x_0]$. Note that $(s_{k_n}^n, s_{k_n+1}^n, \ldots) = \hat{S}(s_{k_n}^n)$ is history-independent, and $m(x_n)$ always has the option of jumping directly to $s_{k_n}^n$, so

$$\begin{aligned} U_{m(x_n)}(\hat{S}(s_{k_n}^n)) &\leq U_{m(x_n)}(S_n) \\ &\leq \frac{1}{1 - \delta^{k_n+1}} \sum_{t=0}^{k_n-1} \delta^t u_{m(x_n)}(s_t^n) \\ &\leq \frac{u_{m(x_n)}(x_0)}{1 - \delta}. \end{aligned}$$

[20]The condition that all x be very reluctant is a joint condition on u and δ, which is not hard to satisfy. For example, in the quadratic-linear case given by $u_\alpha(x) = C - (\alpha - x)^2$ and $m(x) = ax$, it holds if $2a\delta \geq 1$, that is, if $a > \frac{1}{2}$ and δ is high enough.

As $m(x_n)$ can also choose any $z \in [x^*, x_0]$, it must be that

$$\max_{z\in[x^*,x_0]} U_{m(x_n)}\big(\hat{S}(z)\big) \leq U_{m(x_n)}(S_n) \leq u_{m(x_n)}(x_0).$$

By continuity, $U_{m(x_0)}(\hat{S}(\hat{s}(x_0))) \leq u_{m(x_0)}(x_0)$, which contradicts Proposition 2. *Q.E.D.*

(ii) Consider a discrete approximation of the problem in which the policy space is restricted to a finite set X. Then, for a generic choice of $X = \{x_1, \dots, x_N\} \subseteq [-1, 1]$, there is a unique subgame perfect equilibrium, which is Markov.[21] Hence, if we are interested in equilibria that can be obtained as limits of discrete policy-space equilibria,[22] we need only to study Markov equilibria.

(iii) Consider a variant of the game with a finite number of periods $t = 0, 1, \dots, T$. For each T, the game has a unique equilibrium s_T, which is Markov in (x, t). A limit of such equilibria as we take $T \to \infty$ may not be Markov in x exclusively, but Propositions 2 and 3 can still be extended to this case, so intrinsic steady states are also ruled out under this refinement.

Finally, it is worth noting that the MVEs we construct in the main text, which converge to a myopically stable policy, are strictly preferred by all pivotal decision-makers to an SPE in which the policy never changes. In other words, the fall down the slippery slope is *desired* by agents. More precisely, assume an initial policy x_0, and consider an MVE s such that $s^t(x_0) \xrightarrow[t\to\infty]{} x^*$, and an SPE $\tilde{s}$ where the policy remains constant at x_0. By Lemma 2, there is α_0 such that all agents to the left of α_0 strictly prefer the continuation under s; since $m(x_0)$ has this preference, $\alpha_0 \geq m(x_0)$. Consequently, all agents in $[x^*, m(x_0)]$—in particular, all agents who will be pivotal on the equilibrium path—have the same preference. Hence, by focusing on Markov equilibria, we are not ruling out preferable equilibria that just require mutually desirable coordination on the part of the players to arise.

E.4. *Explicit Voting Protocols*

Our solution concept assumes that, for a policy y to be chosen by a set of voters $I(x)$, $S(y)$ must be a Condorcet winner. This assumption is agnostic about the actual voting process taking place. Here, we discuss two natural microfoundations.

The first is Downsian competition. Suppose that, at each voting stage, there are two politicians A_t, B_t who simultaneously propose policies x_{At}, x_{Bt}; voters observe the two proposals and then vote for the candidate proposing the policy whose continuation path they prefer. Assume either that the politicians are short-lived (they are replaced every period) or that they play Markov strategies, and they are office-motivated, that is, they obtain $R > 0$ from winning and zero from losing. An equilibrium of the voting stage is given by policy proposal strategies $x_A(I)$, $x_B(I)$ and voting strategies $v_\alpha(x_A, x_B)$ such that: for each candidate i, offering $x_i(I)$ maximizes i's winning probability given a set of voters I, and $v_\alpha(x_A, x_B) = i$ if $U_\alpha(S(x_i)) > U_\alpha(S(x_{-i}))$, where $S(x_i)$ is the equilibrium continuation starting at x_i. Then the following holds:

[21]This can be shown by proving Proposition 1 in the discrete case, and then applying backward induction (Acemoglu, Egorov, and Sonin, 2015). The equilibrium is unique so long as there are no indifferences.

[22]Formally, denoting s_X to be the equilibrium for policy space s_X, an equilibrium s is a limit of discrete policy-space equilibria if there is a sequence $(X_n)_n$ such that $\max_{y\in[-1,1]} d(X_n, y) \xrightarrow[n\to\infty]{} 0$, that is, the sets X_n become arbitrarily fine, and $s_{X_n}(x_n) \to s(x)\ \forall (x_n)_n$ s.t. $x_n \in X_n \forall n$ and $x_n \to x$.

REMARK 3: Given an MVE s from the main model, we can construct an MPE of the Downsian model as follows: $x_A(I(x)) = x_B(I(x)) = s(x)$ and $v_\alpha(x_A, x_B) = \mathbb{1}_{\{U_\alpha(S(x_A))>U_\alpha(S(x_B))\}}$ for all x, x_A, x_B.

Conversely, for any MPE of the Downsian model in pure strategies, $x_i(I)$ must be a Condorcet winner for every i, I. Moreover, if $x_A(I) = x_B(I)$ for all I, then $s(x) = x_i(I(x))$ constitutes an MVE of the main model.

In other words, requiring $S(s(x))$ to be a Condorcet winner among $I(x)$ is equivalent to assuming Downsian competition at the voting stage, except that we implicitly rule out situations where there is no Condorcet winner (in which case the Downsian model might still have equilibria with mixed proposal strategies), and we rule out mixed policy choices by voters ($s(x)$ is assumed to be deterministic).

Another possible microfoundation is a sequential proposal protocol similar to that used in Acemoglu, Egorov, and Sonin (2008, 2012). Suppose that, at each voting stage, there is a period of continuous time $[0, 1]$ (this is measured on a different scale from the time that passes between periods, and no discounting accrues during the voting stage). At each instant $y \in [0, 1]$, policy y is proposed to the organization and each voter α casts a vote $v_\alpha(y, I) \in \{0, 1\}$. If a majority votes 1 when y is under consideration, the voting stage ends and y is chosen; if no policy receives a majority, then the policy next period is equal to the current policy. Voters are strategic and voting strategies are pure and weakly undominated. Any MVE s from the main model can also be implemented as an equilibrium of this model.

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