Econometrica Supplementary Material

SUPPLEMENT TO “THE EMPIRICAL CONTENT OF BINARY CHOICE MODELS”

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This online appendix contains: (i) the construction of the continuous extension of the choice probability function to a domain containing Ω, as mentioned in footnote 11 in the proof of Theorem 1, and (ii) a version of Theorem 1 (called Theorem S1) with proof that does not require the limit conditions C/C' of Theorem 1, but involves a slight strengthening of the continuity conditions B/B'.

APPENDIX S1: CONSTRUCTION OF CONTINUOUS EXTENSION OF CHOICE PROBABILITY FUNCTION

IN THE PROOF OF THEOREM 1, the definition of $q^{-1}(\cdot, a_1)$ in (12) in the main text implicitly assumes that $\Omega_0(a_1)$ equals (or contains) $[y_L(a_1), y_H(a_1)]$. If however the support of price and income are discrete, then $\Omega_0(a_1)$ can be a strict subset of $[y_L(a_1), y_H(a_1)]$. Then $q(\cdot, \cdot)$ is not defined at the points “in between” the points of support and, therefore, $q^{-1}(\cdot, a_1)$ in (12) is not well-defined. To cover this case, one can extend $q(\cdot, \cdot)$ to a continuous function $q^c(\cdot, \cdot)$ defined on a rectangle Ω^c containing Ω such that (i) $q^c(\cdot, \cdot)$ equals $q(\cdot, \cdot)$ on Ω, (ii) $q^c(\cdot, \cdot)$ satisfies the same shape restrictions on Ω^c that are satisfied by $q(\cdot, \cdot)$ on Ω, and (iii) $q^c(\cdot, \cdot)$ satisfies the limit conditions C of Theorem 1. The proof of Theorem 1 then holds with Ω, $\Omega_0(\cdot)$ and $q(\cdot, \cdot)$ equalling their corresponding extensions in the case where (P, Y) have discrete support. Here, we provide an explicit construction that achieves this extension.[1]

Suppose the support of (P, Y) is the discrete set $\bar{\Omega} = \{p_1, \dots, p_M\} \times \{y_1, \dots, y_N\}$, with $p_1 < p_2 < \cdots < p_M$ and $y_1 < y_2 < \cdots < y_N$. Suppose the choice probability $q(y, y - p)$, which is defined for $(p, y) \in \bar{\Omega}$, satisfies the shape constraints (A) of Theorem 1, i.e. $q(\cdot, \cdot)$ is nonincreasing in the first and nondecreasing in the second argument. We want to construct an extension of $q(\cdot, \cdot)$, denoted by $q^c(y, y - p)$, which is (i) defined for all $(y, y - p)$ with $p_1 \le p \le p_M$ and $y_1 \le y \le y_N$, (ii) equals $q(y, y - p)$ for $(p, y) \in \bar{\Omega}$, and (iii) satisfies all three conditions A, B, C of Theorem 1. The construction proceeds in three steps.

Step 1: First, we extend $q(\cdot, \cdot)$ to the rectangular grid

$$T = \{y_1, \dots, y_N\} \times \bigcup_{j=1}^{N} \bigcup_{k=1}^{M} \{y_j - p_k\}.$$

To do this, define $\tilde{q}(\cdot, \cdot) : T \to [0, 1]$ as

$$\tilde{q}(y, y - p) = \lambda \bar{L}(y, y - p) + (1 - \lambda) \bar{U}(y, y - p), \tag{S1}$$

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[1]Alternatively, one can construct $q^c(\cdot, \cdot)$ as a smooth, tensor-product polynomial spline with coefficients chosen to satisfy the shape restrictions and a high enough degree to guarantee that $q^c(\cdot, \cdot)$ passes through the interpolating points $\{y^j, y^j - p^j, q(y^j, y^j - p^j) : (y^j, y^j - p^j) \in \Omega\}$, along the lines of Costantini and Fontanella (1990).

where $\lambda \in [0,1]$ is arbitrary, and for any $(y, y-p) \in T$,

$$\bar{L}(y, y-p) = \begin{cases} \sup_{(p',y')\in\bar{\Omega}: y'\geq y, y'-p'\leq y-p} q(y', y'-p') \\ \quad \text{if } \{(p', y') \in \bar{\Omega} : y' \geq y, y'-p' \leq y-p\} \neq \phi, \\ 0 \quad \text{if } \{(p', y') \in \bar{\Omega} : y' \geq y, y'-p' \leq y-p\} = \phi, \end{cases}$$

$$\bar{U}(y, y-p) = \begin{cases} \inf_{(p',y')\in\bar{\Omega}: y'\leq y, y'-p'\geq y-p} q(y', y'-p') \\ \quad \text{if } \{(p', y') \in \bar{\Omega} : y' \leq y, y'-p' \geq y-p\} \neq \phi, \\ 1 \quad \text{if } \{(p', y') \in \bar{\Omega} : y' \leq y, y'-p' \geq y-p\} = \phi. \end{cases}$$

Note that $\tilde{q}(\cdot,\cdot)$, which is well-defined on all of T, satisfies the shape constraints (A) of Theorem 1. This is because the set $\{(p', y') \in \bar{\Omega} : y' \geq y, y'-p' \leq y-p\}$ is decreasing in y for fixed $y-p$, and increasing in $y-p$ for fixed y, so $\bar{L}(\cdot,\cdot)$ is decreasing in the first and increasing in the second argument; an analogous argument works for $\bar{U}(\cdot,\cdot)$. Furthermore, if $(p, y) \in \bar{\Omega}$, then

$$(p, y) \in \{(p', y') \in \bar{\Omega} : y' \geq y, y'-p' \leq y-p\},$$
$$(p, y) \in \{(p', y') \in \bar{\Omega} : y' \leq y, y'-p' \geq y-p\},$$

whence the shape restrictions on $q(\cdot,\cdot)$ imply that $\bar{L}(y, y-p) = q(y, y-p) = \bar{U}(y, y-p)$, and hence $\tilde{q}(y, y-p) = q(y, y-p)$. Note, however, that $\tilde{q}(\cdot,\cdot)$ does not satisfy the continuity condition (B) and the limit conditions (C) of Theorem 1.

Step 2: The second step is to extend $\tilde{q}(\cdot,\cdot)$ to a *continuous* function $q^c(\cdot,\cdot)$ on the entire rectangle $[y_1, y_N] \times [y_1 - p_M, y_N - p_1]$, satisfying the shape constraints (A) of theorem 1, while also satisfying the interpolation conditions $q^c(y, y-p) = q(y, y-p)$ for $(p, y) \in \bar{\Omega}$. This is done using bilinear shape-preserving interpolation as follows.

Recall $y_1 < y_2 < \cdots < y_N$, and define $w_1 < w_2 < \cdots < w_J$ with $J \leq MN$ to be the ordered values of the set $\{y_1 - p_1, \ldots, y_1 - p_M, \ldots, y_N - p_1, \ldots, y_N - p_M\}$. We can have $J < MN$ if for some $(j,k) \neq (l,m)$, it holds that $y_j - p_k = y_l - p_m$. For each $i = 1, \ldots N-1$, $j = 1, \ldots, J-1$, and for $(y, y-p) \in [y_i, y_{i+1}] \times [w_j, w_{j+1}]$, let

$$\alpha_i(y) = \frac{y - y_i}{y_{i+1} - y_i}, \quad \beta_j(w) = \frac{w - w_j}{w_{j+1} - w_j},$$

$$\begin{aligned} q^c(y, \underbrace{y-p}_{w}) = {} & (1-\alpha_i(y)) \times (1-\beta_j(w)) \times \tilde{q}(y_i, w_j) \\ & + \alpha_i(y) \times (1-\beta_j(w)) \times \tilde{q}(y_{i+1}, w_j) \\ & + (1-\alpha_i(y)) \times \beta_j(w) \times \tilde{q}(y_i, w_{j+1}) \\ & + \alpha_i(y) \times \beta_j(w) \times \tilde{q}(y_{i+1}, w_{j+1}), \end{aligned} \tag{S2}$$

where $\tilde{q}(\cdot,\cdot)$ is defined in (S1).

Step 3: The last step in the construction is to extend $q^c(\cdot,\cdot)$ beyond $[y_1, y_N] \times [y_1 - p_M, y_N - p_1]$ to ensure that the limit conditions (C) of Theorem 1 are satisfied. To do this, choose any pair of real numbers y_L, y_H s.t. $y_L < y_1$ and $y_H > y_N$. Let

$$D = [y_L, y_H] \times [y_1 - p_M, y_N - p_1].$$

For any $w \in [y_1 - p_M, y_N - p_1]$, define

$$q^c(y,w) = \begin{cases} \dfrac{y - y_L}{y_1 - y_L} \times q^c(y_1, w) + \dfrac{y_1 - y}{y_1 - y_L} & \text{if } y \in [y_L, y_1], \\ \dfrac{y_H - y}{y_H - y_N + p_1} q^c(y_N - p_1, w) & \text{if } y \in [y_N - p_1, y_H]. \end{cases} \tag{S3}$$

That is for $y \in [y_L, y_1]$, $q^c(y, w)$ is the negatively sloped straight line joining $q^c(y_1, w)$ to $1 \equiv q^c(y_L, w)$, and for $y \in [y_N - p_1, y_H]$, $q^c(y, w)$ is the negatively sloped straight line joining $q^c(y_N - p_1, w)$ to $0 \equiv q^c(y_H, w)$.

Proof that $q^c(\cdot,\cdot): D \to [0,1]$ *equals* $q(y, y-p)$ *for* $(p, y) \in \bar{\Omega}$ *and satisfies conditions* (A), (B), (C) *of Theorem 1* To see the first assertion, observe that at the grid points $y = y_i$, $y - p = w_j$, we get from (S2) that $\alpha_i(y) = 0 = \beta_j(w)$, so that $q^c(y, w) = \tilde{q}(y_i, w_j)$. We have already seen that for $(p, y) \in \bar{\Omega}$, $q(y, y-p) = \tilde{q}(y, y-p)$. Now, since $(p, y) \in \bar{\Omega}$ implies $(y, y-p) \in T$, putting these two conclusions together, we get that for $(p, y) \in \bar{\Omega}$, it holds that $q^c(y, y-p) = \tilde{q}(y, y-p) = q(y, y-p)$.

As for the continuity condition (B) of Theorem 1, observe that holding fixed w, as $y \in [y_i, y_{i+1}) \nearrow y_{i+1}-$, we have that $\alpha_i(y) \nearrow 1$ whence from (S2), it follows that

$$q^c(y,w) \searrow \big(1 - \beta_j(w)\big) \times \tilde{q}(y_{i+1}, w_j) + \beta_j(w) \times \tilde{q}(y_{i+1}, w_{j+1}). \tag{S4}$$

On the other hand, for the same w and for $y \in [y_{i+1}, y_{i+2})$, we have that $\alpha_i(y) = \frac{y - y_{i+1}}{y_{i+2} - y_{i+1}}$ which at $y = y_{i+1} \in [y_{i+1}, y_{i+2})$ equals 0, whence from (S2) with i replaced by $i+1$ and $i+1$ replaced by $i+2$, we get

$$q^c(y,w) = \big(1 - \beta_j(w)\big) \times \tilde{q}(y_{i+1}, w_j) + \beta_j(w) \times \tilde{q}(y_{i+1}, w_{j+1}),$$

which equals (S4). Therefore, for fixed w, $\tilde{q}(y, w)$ is simply a piecewise linear function of y joined at the end-points $y_2, \dots, y_{N-1}$ and, therefore, continuous in y for $y \in [y_1, y_N]$. For $y \in [y_L, y_H] \backslash [y_1, y_N]$, continuity is obvious from (S3) and the fact that $\lim_{y \nearrow y_1 -} q^c(y, w) = q^c(y_1, w) = \lim_{y \searrow y_1 +} q^c(y, w)$ and $\lim_{y \nearrow (y_N - p_1) -} q^c(y, w) = q^c(y_N - p_1, w) = \lim_{y \searrow (y_N - p_1) +} q^c(y, w)$. An analogous argument shows that $q^c(y, w)$ is also continuous in w for fixed y (this property is not needed to prove Theorem 1 but is used in Theorem S1, the alternative version of Theorem 1 without the limiting condition, which appears below).

The limiting conditions (C) of Theorem 1 are satisfied, since (S3) implies that $q^c(y_L, w) = 1$ and $q^c(y_H, w) = 0$ for each $w \in [y_1 - p_M, y_N - p_1]$.

Finally, to see that the shape restrictions (A) of Theorem 1 hold on $[y_1, y_N] \times [y_1 - p_M, y_N - p_1]$, note from (S2) that the coefficient of y in $q^c(y, w)$ equals

$$\underbrace{\frac{1}{y_{i+1} - y_i}}_{\geq 0} \times \left\{ \begin{array}{l} \underbrace{\big(1 - \beta_j(w)\big)}_{\geq 0} \times \underbrace{\big[\tilde{q}(y_{i+1}, w_j) - \tilde{q}(y_i, w_j)\big]}_{\leq 0,\ \text{since } y_i \leq y_{i+1}} \\ \underbrace{-\beta_j(w)}_{\leq 0} \times \big[\underbrace{\tilde{q}(y_i, w_{j+1}) - \tilde{q}(y_{i+1}, w_{j+1})}_{\geq 0,\ \text{since } y_i \leq y_{i+1}}\big] \end{array} \right\} \leq 0.$$

Similarly, the coefficient of w in $q^c(y, w)$ equals

$$\underbrace{\frac{1}{w_{j+1} - w_j}}_{\geq 0} \times \left\{ \begin{array}{l} \underbrace{(1 - \alpha_i(y))}_{\geq 0} \times \underbrace{[\tilde{q}(y_i, w_{j+1}) - \tilde{q}(y_i, w_j)]}_{\geq 0,\text{ since } w_j \leq w_{j+1}} \\ \underbrace{+\alpha_i(y)}_{\geq 0} \times \underbrace{[\tilde{q}(y_{i+1}, w_{j+1}) - \tilde{q}(y_{i+1}, w_j)]}_{\geq 0,\text{ since } w_j \leq w_{j+1}} \end{array} \right\} \geq 0.$$

From (S3), it follows that the shape restrictions also hold on $[y_L, y_1] \times [y_1 - p_M, y_N - p_1]$ and on $[y_N, y_H] \times [y_1 - p_M, y_N - p_1]$, and thus condition (A) of Theorem 1 holds on all of $[y_L, y_H] \times [y_1 - p_M, y_N - p_1]$.

Thus $q^c(\cdot, \cdot)$ satisfies all three conditions of Theorem 1.

APPENDIX S2: MAIN RESULT WITHOUT CONDITION (C/C')

The following is a version of Theorem 1 that does not require the technical conditions C and C' of Theorem 1, but involves a slight strengthening of the technical condition B. The proof of this version is considerably longer than that of Theorem 1. The proof works by constructing an extension $Q(\cdot, \cdot)$ of $q(\cdot, \cdot)$ which satisfies properties (A)–(C) of Theorem 1 although $q(\cdot, \cdot)$ itself does not satisfy property (C).[2]

Suppose the support of price P and income Y in the population is $[p_l, p_u] \times [y_l, y_u]$. Correspondingly, the support of $Y - P$ is $\Omega_1 \stackrel{\text{defn}}{=} [y_l - p_u, y_u - p_l]$. Pick any $a_1 \in \Omega_1$. Corresponding to $Y - P = a_1$, the support of $Y = a_1 + P$ is therefore

$$\Omega_0(a_1) \stackrel{\text{defn}}{=} [\underbrace{\max\{p_l + a_1, y_l\}}_{L(a_1)}, \underbrace{\min\{p_u + a_1, y_u\}}_{U(a_1)}].$$

Note that by definition, $L(\cdot)$ and $U(\cdot)$ are nondecreasing and continuous. Let $\Omega = \bigcup_{a_1 \in \Omega_1} \bigcup_{a_0 \in \Omega_0(a_1)} \{a_0, a_1\}$.

THEOREM S1: *For binary choice under general heterogeneity, the following two statements are equivalent*:

(I) *The choice probabilities* $q(y, y - p)$, *defined above, satisfy that* (A) $q(\cdot, y - p)$ *is nonincreasing, and* $q(y, \cdot)$ *is nondecreasing*; (B) $q(\cdot, \cdot)$ *is continuous.*

(II) *There exists a pair of utility functions* $W_0(\cdot, \eta)$ *and* $W_1(\cdot, \eta)$, *where the first argument denotes the amount of numeraire, and* η*denotes unobserved heterogeneity, and a distribution* $G(\cdot)$ *of* η *such that for any* $(y - p) \in \Omega_1$ *and correspondingly* $y \in \Omega_0(y - p)$,

$$q(y, y - p) = \int 1\{W_0(y, \eta) \leq W_1(y - p, \eta)\}\, dG(\eta),$$

where (A') *for each fixed* η, $W_0(\cdot, \eta)$ *and* $W_1(\cdot, \eta)$ *are nondecreasing*; (B') *for each fixed* η, $W_0(\cdot, \eta)$ *and* $W_1(\cdot, \eta)$*are continuous, and for any* $(a_0, a_1) \in \Omega$, *it holds that* $\int 1\{W_0(a_0, \eta) \leq W_1(a_1, \eta)\}\, dG(\eta)$ *is continuous in* (a_0, a_1).

[2]The case where (P, Y) have a discrete support is handled in exactly the same way as in Theorem 1 with two small modifications: (a) Step 3 in the construction immediately above is not required, and (b) continuity of $q^c(\cdot, \cdot)$ in the *second* argument is guaranteed by the construction in Step 2.

Discussion of assumptions: Relative to Theorem 1, conditions (C/C') are omitted, and condition (B/B') is strengthened to continuity in both arguments. Note that under monotonicity in any one argument, the joint continuity of $q(\cdot,\cdot)$ is equivalent to coordinate wise continuity; cf. Kruse and Deely (1969).

To prove Theorem S1, we will utilize several lemmas.

LEMMA S1—Apostol (1974, Ex 4.19): *Suppose* $r(\cdot):[c,b]\to\mathbb{R}$, *is continuous on* $[c,b]$. *For* $x\in[c,b]$, *define* $g(x)=\sup\{r(z):x\le z\le b\}$, *and* $h(x)=\sup\{r(z):c\le z\le x\}$. *Then* $g(\cdot)$ *and* $h(\cdot)$ *are continuous on* $[c,b]$.

PROOF OF LEMMA S1: Fix any $x\in[c,a_1]$.

First, suppose $g(x)>r(x)$. Choose $\varepsilon=g(x)-r(x)>0$. Now by continuity of $r(\cdot)$, there must exist $\delta>0$ s.t. for any $z\in[x-\delta,x+\delta]$, we have that $r(z)<r(x)+\varepsilon=r(x)+g(x)-r(x)=g(x)$. Therefore, $\sup\{r(z):x-\delta\le z\le x+\delta\}<g(x)$. Therefore, $g(x-\delta)=g(x)=g(x+\delta)$, implying continuity of $g(\cdot)$ at x.

Next, suppose the sup is at x, i.e. $g(x)=r(x)$. By continuity, for any $\varepsilon>0$, there exists $\delta>0$, s.t. for all $u\in[x-\delta,x+\delta]$, we have that $r(x)+\varepsilon\ge r(u)\ge r(x)-\varepsilon$. For $u\in[x,x+\delta]$, $g(u)=\sup\{r(z):u\le z\le a_1\}\ge r(u)\ge r(x)-\varepsilon=g(x)-\varepsilon$, since $g(x)=r(x)$, by assumption. But $g(u)\le g(x)$ by definition. Therefore, for all $u\in[x,x+\delta]$, we have that $g(x)\ge g(u)>g(x)-\varepsilon$. Next, for all $u\in[x-\delta,x]$, $r(u)\le r(x)+\varepsilon=g(x)+\varepsilon$ implying

$$\begin{aligned} g(u) &= \sup\{r(z):u\le z\le a_1\} \\ &\le \sup\{r(z):x-\delta\le z\le a_1\} \\ &= \max\{\underbrace{\sup\{r(z):x-\delta\le z\le x\}}_{\le g(x)+\varepsilon},\underbrace{\sup\{r(z):x\le z\le a_1\}}_{g(x)}\} \\ &\le g(x)+\varepsilon. \end{aligned}$$

Thus for all $u\in[x-\delta,x+\delta]$, we have that $g(x)+\varepsilon\ge g(u)>g(x)-\varepsilon$. Therefore, $g(\cdot)$ is continuous at x.

An exactly similar proof works for $h(x)=\sup\{r(z):c\le z\le x\}$. *Q.E.D.*

LEMMA S2—Taylor (1955), Chapter 15.7, Theorem VII: *Suppose the function* $f:\mathbb{R}^2\to\mathbb{R}$ *is continuous, and the function* $g(\cdot):\mathbb{R}\to\mathbb{R}$ *is continuous w.r.t. the* $L1$-*norm. Then the function* $h:\mathbb{R}\to\mathbb{R}$ *defined as* $h(x)=f(g(x),x)$ *is continuous on* $\mathbb{R}$.

PROOF OF LEMMA S2: Pick any $x_0\in\mathbb{R}$, and $\varepsilon>0$. Continuity of $f(\cdot,\cdot)$ implies that there exists $\delta>0$ s.t. $|f(g(x),x)-f(g(x_0),x_0)|\le\varepsilon$, whenever $\|(g(x),x)-(g(x_0),x_0)\|\le\delta$. Now, continuity of $g(\cdot)$ implies that given the above $\delta>0$, there exists $\delta_1>0$ s.t. $|g(x)-g(x_0)|\le\delta/2$ whenever $|x-x_0|\le\delta_1$. Choose $\delta^*=\min\{\delta/2,\delta_1\}$. Then whenever $|x-x_0|\le\delta^*$, we have that $|g(x)-g(x_0)|\le\delta/2$ and $|x-x_0|\le\delta/2$, and thus $\|(g(x),x)-(g(x_0),x_0)\|=|g(x)-g(x_0)|+|x-x_0|\le\delta$ and, therefore,

$$\left|h(x)-h(x_0)\right|=\left|f\big(g(x),x\big)-f\big(g(x_0),x_0\big)\right|<\varepsilon. \qquad \textit{Q.E.D.}$$

Construction: The following construction will be used to prove the theorem. Pick $a_1\in\Omega_1$. Recall the definitions $L(a_1)\equiv\max\{p_l+a_1,y_l\}$, and $U(a_1)\equiv\min\{p_u+a_1,y_u\}$. Let a_{0L},

a_{0H} be any pair of real numbers satisfying $a_{0L} < y_l$ and $a_{0H} > y_u$. For any $a_0 < L(a_1)$ and $a_0 > U(a_1)$, respectively, define

$$H(a_0, a_1) = \sup\{q(L(x), x) : L(x) \in [a_0, L(a_1)]\},$$
$$h(a_0, a_1) = \inf\{q(U(x), x) : U(x) \in [U(a_1), a_0]\}.$$

Note that as a_0 decreases with a_1 fixed, or a_1 increases with a_0 fixed, the set $[a_0, L(a_1)]$ expands and, therefore, the sup over it weakly increases; thus $H(\cdot, a_1)$ is nonincreasing and $H(a_0, \cdot)$ is nondecreasing. Similarly, $h(\cdot, a_1)$ is nonincreasing and $h(a_0, \cdot)$ is nondecreasing. Now, define the function $Q(\cdot,\cdot) : [a_{0L}, a_{0H}] \to [0, 1]$ as follows. For any $a_1 \in \Omega_1$,

$$Q(a_0, a_1) = \begin{cases} H(y_l, a_1) + (1 - H(y_l, a_1))\dfrac{y_l - a_0}{y_l - a_{0L}} & \text{if } a_{0L} \le a_0 < y_l, \\ H(a_0, a_1) & \text{if } y_l \le a_0 < L(a_1), \\ q(a_0, a_1) & \text{if } a_0 \in [L(a_1), U(a_1)], \\ h(a_0, a_1) & \text{if } U(a_1) < a_0 \le y_u, \\ \dfrac{a_{0H} - a_0}{a_{0H} - y_u} h(y_u, a_1) & \text{if } y_u < a_0 \le a_{0H}. \end{cases} \tag{S5}$$

CLAIM S1: *Suppose* $q(\cdot,\cdot)$ *satisfies* (A) *and* (B) *of Theorem* S1. *Then the function* $Q(\cdot,\cdot)$ *defined in* (S5) *satisfies the following properties*:

(1) $Q(\cdot, a_1)$ *is nonincreasing*, *and* $Q(a_0, \cdot)$ *is nondecreasing for all* $(a_0, a_1) \in [a_{0L}, a_{0H}] \times \Omega_1$
(2) $Q(\cdot,\cdot)$ *is continuous in each argument*, *holding the other argument fixed*.
(3) *For any* $a_1 \in \Omega_1$, *there exist real numbers* a_{0L} *and* a_{0H} *such that* $\lim_{a_0 \searrow a_{0L}} Q(a_0, a_1) = 1$ *and* $\lim_{a_0 \nearrow a_{0H}} Q(a_0, a_1) = 0$.

PROOF: Property (3) is obvious because $Q(a_{0L}, a_1) = 1$ and $Q(a_{0H}, a_1) = 0$, by construction. To show (1) and (2), fix $a_1 \in \Omega_1$. Since $q(\cdot,\cdot)$ satisfies (A) and (B) on $a_0 \in [L(a_1), U(a_1)]$, we only need to establish the properties over the range $a_0 < L(a_1)$ and $a_0 > U(a_1)$.

Property (1): First, we show that the shape restrictions hold for $Q(\cdot,\cdot)$. We have already noted that $H(\cdot, a_1)$ and $h(\cdot, a_1)$ are both nonincreasing; further since $H(y_l, a_1) \le 1$ and $h(y_u, a_1) \ge 0$, we have that $H(y_l, a_1) + (1 - H(y_l, a_1))\frac{y_l - a_0}{y_l - a_{0L}}$ is nonincreasing in a_0 for $a_{0L} \le a_0 < y_l$, and $\frac{a_{0H} - a_0}{a_{0H} - y_u} h(y_u, a_1)$ is nonincreasing in a_0 for $y_u < a_0 \le a_{0H}$. Thus $Q(a_0, a_1)$ is nonincreasing in a_0 for all $a_0 < L(a_1)$ and $a_0 > U(a_1)$.

Next, pick $a_0 \in [a_{0L}, a_{0H}]$, and consider monotonicity of $Q(a_0, \cdot)$. Let $a_1^1, a_1^2 \in \Omega_1$ with $a_1^1 < a_1^2$, implying $L(a_1^1) \le L(a_1^2)$ and $U(a_1^1) \le U(a_1^2)$. Now there are 10 cases to consider, labeled (a)–(j) below, depending on the ordering of $L(a_1^2)$ and $U(a_1^1)$, and where a_0 lies.
Case (a) $a_{0L} \le a_0 < y_l$, then

$$\begin{aligned} Q(a_0, a_1^1) &= H(a_0, a_1^1) \\ &= \frac{y_l - a_0}{y_l - a_{0L}} + H(y_l, a_1^1)\frac{a_0 - a_{0L}}{y_l - a_{0L}} \\ &\le \frac{y_l - a_0}{y_l - a_{0L}} + H(y_l, a_1^2)\frac{a_0 - a_{0L}}{y_l - a_{0L}}, \quad \text{since } H(y_l, \cdot) \text{ nondecreasing} \\ &= Q(a_0, a_1^2). \end{aligned}$$

Case (b) $y_l \leq a_0 \leq L(a_1^1)$, that is, $[a_0, L(a_1^1)] \sqsubseteq [a_0, L(a_1^2)]$, and so $H(a_0, a_1^1) \leq H(a_0, a_1^2)$ and, therefore, $Q(a_0, a_1^1) = H(a_0, a_1^1) \leq H(a_0, a_1^2) = Q(a_0, a_1^2)$. Case (c): $y_u < a_0 \leq a_{0H}$, and Case (d) $U(a_1^2) < a_0 \leq y_u$, the proofs are exactly analogous to respectively (a) and (b) above.

So we are left with the following cases, where Cases (e)–(g) correspond to $U(a_1^1) < L(a_1^2)$, and (h)–(j) to $U(a_1^1) \geq L(a_1^2)$.

For Case (e) $L(a_1^1) \leq a_0 \leq U(a_1^1) < L(a_1^2)$, since $L(a_1^1) < a_0 < L(a_1^2)$, by continuity of $L(\cdot)$ and the intermediate value theorem, there exists $c \in [a_1^1, a_1^2]$ s.t. $a_0 = L(c)$. Therefore,

$$\begin{aligned} Q(a_0, a_1^1) &= q(a_0, a_1^1) = q(L(c), a_1^1) \\ &\overset{(1)}{\leq} q(L(c), c) \\ &\overset{(2)}{\leq} \sup\{q(L(x), x) : L(x) \in [L(c), L(a_1^2)]\} \\ &= \sup\{q(L(x), x) : L(x) \in [a_0, L(a_1^2)]\}, \quad \text{since } a_0 = L(c) \\ &= Q(a_0, a_1^2), \end{aligned}$$

where $\overset{(1)}{\leq}$ holds because $a_1^1 \leq c$ and condition (A) of Theorem 1, and $\overset{(2)}{\leq}$ holds by definition of sup. Next, suppose case (f) $L(a_1^1) \leq U(a_1^1) \leq a_0 < L(a_1^2) \leq U(a_1^2)$, then by continuity of $L(\cdot)$ and the intermediate value theorem, there exists $c \in [a_1^1, a_1^2]$ s.t. $a_0 = L(c)$; and by continuity of $U(\cdot)$ and the intermediate value theorem, there exists $d \in [a_1^1, a_1^2]$ s.t. $a_0 = U(d)$, with $d \leq c$. Then

$$\begin{aligned} &Q(a_0, a_1^1) \\ &\quad = \inf\{q(U(x), x) : U(a_1^1) \leq U(x) \leq a_0\}, \quad \text{by (S5)} \\ &\quad = \inf\{q(U(x), x) : U(a_1^1) \leq U(x) \leq U(d)\}, \quad \text{by } a_0 = U(d) \\ &\quad \leq q(U(d), d), \quad \text{since } d \in \{x : U(a_1^1) \leq U(x) \leq U(d)\} \\ &\quad \leq q(L(c), c), \quad \text{by (Aii) since } U(d) = a_0 = L(c) \text{ and } d \leq c \\ &\quad \leq \sup\{q(L(x), x) : L(c) \leq L(x) \leq L(a_1^2)\}, \quad \text{since } c \in \{x : L(c) \leq L(x) \leq L(a_1^2)\} \\ &\quad = \sup\{q(L(x), x) : a_0 \leq L(x) \leq L(a_1^2)\}, \quad \text{since } a_0 = L(c) \\ &\quad = Q(a_0, a_1^2), \quad \text{by definition (S5).} \end{aligned}$$

Next, for case (g) $L(a_1^1) \leq U(a_1^1) < L(a_1^2) \leq a_0 \leq U(a_1^2)$, using continuity of $U(\cdot)$ and the intermediate value theorem, we have $a_0 = U(c)$ for some $c \in [a_1^1, a_1^2]$ so that

$$\begin{aligned} Q(a_0, a_1^2) &= Q(U(c), a_1^2) \\ &= q(U(c), a_1^2), \quad \text{since } a_0 = U(c) \in [L(a_1^2), U(a_1^2)] \\ &\geq q(U(c), c), \quad \text{since } c \leq a_1^2 \text{ and condition (A)} \\ &\geq \inf\{q(U(x), x) : U(a_1^1) \leq U(x) \leq U(c)\} \\ &= Q(U(c), a_1^1), \quad \text{by (S5)} \end{aligned}$$

$$= Q(a_0, a_1^1).$$

Next, consider case (h) $L(a_1^1) \le a_0 \le L(a_1^2) \le U(a_1^1)$. Since $L(a_1^1) \le a_0 \le L(a_1^2)$, by continuity and the intermediate value theorem, we have that $a_0 = L(c)$ for some $c \in [a_1^1, a_1^2]$, whence we have

$$\begin{aligned} Q(a_0, a_1^1) &= q(a_0, a_1^1) = q(L(c), a_1^1) \\ &\le q(L(c), c), \quad \text{since } c \ge a_1^1 \\ &\le \sup\{q(L(x), x) : L(c) \le L(x) \le L(a_1^2)\} \\ &= Q(L(c), a_1^2) \\ &= Q(a_0, a_1^2). \end{aligned}$$

Next, if case (i) $L(a_1^1) \le L(a_1^2) \le a_0 \le U(a_1^1)$, we have that $Q(a_0, a_1^1) = q(a_0, a_1^1) \le q(a_0, a_1^2) = Q(a_0, a_1^2)$.

Finally, for the Case (j) $L(a_1^1) \le L(a_1^2) \le U(a_1^1) \le a_0 \le U(a_1^2)$, the same argument as in (g) applies.

This establishes the requisite shape restrictions, that is, Property (1).

Property (2): First, consider continuity of $Q(\cdot, a_1)$. Note that $H(y_l, a_1) + (1 - H(y_l, a_1)) \times \frac{y_l - a_0}{y_l - a_{0L}}$ is obviously continuous at a_0 for $a_{0L} \le a_0 < y_l$; next, at $a_0 = y_l$, $Q(a_0, a_1) = H(y_l, a_1) + (1 - H(y_l, a_1))\frac{y_l - y_l}{y_l - a_{0L}} = H(y_l, a_1)$, while at $a_0 = L(a_1) > y_l$,

$$Q(a_0, a_1) = \sup\{q(L(x), x) : L(x) \in [L(a_1), L(a_1)]\} = q(L(a_1), a_1),$$

and thus $Q(\cdot, a_1)$ is continuous at $a_0 = y_l$ and at $a_0 = L(a_1)$. Finally, if $a_0 \in (y_l, L(a_1))$, then we can have $L(x) \in [a_0, L(a_1)]$ only if $L(x) > y_l$ in which case $L(x) = x + p_l$ and thus $q(L(x), x) = q(x + p_l, x)$ implying

$$\begin{aligned} Q(a_0, a_1) &= \sup\{q(L(x), x) : a_0 \le L(x) \le L(a_1)\} \\ &= \sup\{q(x + p_l, x) : x + p_l \in [a_0, L(a_1)]\} \\ &= \sup\{q(x + p_l, x) : x \in [a_0 - p_l, L(a_1) - p_l]\}. \end{aligned} \tag{S6}$$

By Lemma S3, $q(x + p_l, x)$ is continuous in x, and therefore, by Lemma S2, $Q(a_0, a_1)$ is continuous in a_0 for fixed a_1. Thus we have that $Q(\cdot, a_1)$ is continuous on all of $[a_{0L}, U(a_1)]$. An exactly analogous argument works for $a_0 > U(a_1)$.

Finally, consider continuity in a_1 for fixed a_0. If (a) $a_1 \le y_l - p_l$, then $L(a_1) = y_l$ and, therefore,

$$H(a_0, a_1) = \sup\{q(L(x), x) : L(x) \in [a_0, y_l]\}, \tag{S7}$$

which does not depend on a_1 and, therefore, trivially continuous in a_1. So consider (b) $a_1 > y_l - p_l$, so that $L(a_1) = a_1 + p_l$. Therefore, at $a_0 = y_l$, $H(a_0, a_1) = H(y_l, a_1)$ equals

$$\begin{aligned} &\sup\{q(L(x), x) : a_0 \le L(x) \le L(a_1)\} \\ &\quad = \sup\{q(L(x), x) : L(x) \in [y_l, a_1 + p_l]\} \\ &\quad \overset{(2)}{=} \sup\{q(L(x), x) : x \in [y_l - p_u, a_1]\}. \end{aligned} \tag{S8}$$

The last equality $\overset{(2)}{=}$ follows because $\underbrace{L(x)}_{=\max\{p_l+x,y_l\}} \in [y_l, a_1 + p_l]$ if and only if $x \in [y_l - p_u, a_1]$.

Now, since $L(\cdot)$ is continuous, and so is $q(\cdot,\cdot)$, the function $x \mapsto q(L(x), x)$ is continuous in x (see Lemma S3 above), and therefore, it follows from Lemma S2 that $\sup\{q(L(x), x) : x \in [y_l - p_u, a_1]\}$ is continuous in a_1. In particular, as $a_1 \searrow (y_l - p_l)_+$, $L(a_1)$ approaches y_l and so (S8) tends to (S7).

Finally, for any $a_0 > y_l$, (recall $a_1 > y_l - p_l$, so that $L(a_1) = a_1 + p_l$), we have that

$$\begin{aligned} H(a_0, a_1) &= \sup\{q\big(L(x), x\big) : L(x) \in [a_0, a_1 + p_l]\} \\ &= \sup\{q\big(L(x), x\big) : x \in [a_0 - p_l, a_1]\}, \end{aligned}$$

which is continuous in a_1 by Lemmas S2 and S3. Exactly analogous arguments hold for (a') $a_1 \geq y_u - p_u$ and (b') $a_1 < y_u - p_u$, respectively. Thus, we have that $Q(a_0, \cdot)$ is continuous at each a_0. *Q.E.D.*

LEMMA S3: *Suppose the function* $Q(\cdot,\cdot) : [a_{0L}, a_{0H}] \times \Omega_1 \sqsubseteq \mathbb{R}^2 \to [0, 1]$ *satisfies on its domain that* (1) $Q(\cdot, a_1)$ *is nonincreasing, and* $Q(a_0, \cdot)$ *is nondecreasing*; (2) $Q(\cdot, a_1)$ *is continuous, and* (3) *for any* $a_1 \in \Omega_1$, $\lim_{a_0 \searrow a_{0L}} Q(a_0, a_1) = 1$ *and* $\lim_{a_0 \nearrow a_{0H}} Q(a_0, a_1) = 0$. *For any fixed* $a_1 \in \Omega_1$, *define for each* $u \in [0, 1]$,

$$Q^{-1}(u, a_1) \overset{\text{def}}{=} \sup\{a_0 \in [a_{0L}, a_{0H}] : Q(a_0, a_1) \geq u\}. \tag{S9}$$

Then we must have that $Q(Q^{-1}(v, a_1), a_1) = v$, *for any* $v \in [0, 1]$.

PROOF OF LEMMA S3: Since $Q(\cdot,\cdot)$ satisfies the same properties as $q(\cdot,\cdot)$ of Theorem 1(A)–(C), the proof of this lemma is identical to the proof of Claim (i) used to prove Theorem 1 in the main paper. *Q.E.D.*

PROOF OF THEOREM S1: That (II) implies (I) is straightforward, since

$$q(y, y - p) = \int 1\{W_0(y, \eta) \leq W_1(y - p, \eta)\}\, dG(\eta)$$

whence (B') implies (B), and (A') implies (A).

We now show that (I) implies (II). To do so, recall the definition of $Q^{-1}(v, a_1)$ in (S9). Now, consider a random variable $V \simeq \text{Uniform}(0, 1)$. Define $W_0(a_0, V) \overset{\text{defn}}{=} a_0$ and $W_1(a_1, V) \overset{\text{defn}}{=} Q^{-1}(V, a_1)$. We will now show that for $y - p \in \Omega_1$ and correspondingly, $y \in [L(y - p), U(y - p)]$, the functions $W_0(y, V)$ and $W_1(y - p, V)$ will rationalize the choice-probabilities $q(y, y - p)$.

To prove this, note that for any $v \in [0, 1]$, and $(a_0, a_1) \in \Omega$,

$$a_0 \leq Q^{-1}(v, a_1) \quad \overset{\text{by } Q(\cdot, a_1) non\uparrow}{\Longrightarrow} \quad Q(a_0, a_1) \geq \underbrace{Q\big(Q^{-1}(v, a_1), a_1\big)}_{=v, \text{ by Lemma S3}} \quad \Longrightarrow \quad Q(a_0, a_1) \geq v. \tag{S10}$$

Also, by definition of $Q^{-1}(\cdot, a_1)$ as the supremum in (S9), we have that

$$Q(a_0, a_1) \geq v \quad \Longrightarrow \quad a_0 \leq Q^{-1}(v, a_1). \tag{S11}$$

Therefore, by (S10) and (S11), we have that $Q(a_0, a_1) \geq v \Longleftrightarrow a_0 \leq Q^{-1}(v, a_1)$. Thus, for $V \simeq U(0, 1)$, it follows that

$$\Pr\big(Q^{-1}(V, a_1) \geq a_0\big) = \Pr\big(V \leq Q(a_0, a_1)\big) = Q(a_0, a_1). \tag{S12}$$

Recall that for $y - p \in \Omega_1$ and correspondingly $y \in [L(y - p), U(y - p)]$, we have that $Q(y, y - p) = q(y, y - p)$ by definition. Therefore, it follows from (S12) that the utility functions $W_0(y, V) \equiv y$ and $W_1(y - p, V) \equiv Q^{-1}(V, y - p)$ with heterogeneity $V \simeq \text{Uniform}(0, 1)$ rationalize the choice probability function $q(\cdot, \cdot)$ on its domain.

Next, note that $Q^{-1}(v, a_1') \leq Q^{-1}(v, a_1)$ whenever $a_1' < a_1$. To see this, suppose $a_1 > a_1'$ and yet $Q^{-1}(v, a_1) < Q^{-1}(v, a_1')$. Choose c s.t. $Q^{-1}(v, a_1) < c < Q^{-1}(v, a_1')$. Then by conclusion (i) of the previous lemma and by definition (S9) of $Q^{-1}(v, \cdot)$, we must have $Q(c, a_1) < v \leq Q(c, a_1')$. But since $a_1 > a_1'$, this contradicts conclusion (1) of the Claim S1.

Next, it follows from (A) and (B) that $Q^{-1}(v, \cdot)$ is continuous. To see this, fix $v \in [0, 1]$, and suppose to the contrary that $Q^{-1}(v, \cdot)$ is discontinuous at a_1; suppose there exists $\epsilon > 0$ such that for any $\delta > 0$, $Q^{-1}(v, a_1) > Q^{-1}(v, a_1') + \varepsilon$ for all a_1' satisfying $a_1' < a_1 < a_1' + \delta$. For any such a_1' satisfying $Q^{-1}(v, a_1) > Q^{-1}(v, a_1') + \varepsilon$, it follows from the definition of $Q^{-1}(\cdot, a_1')$ that there exists $\varepsilon' = \varepsilon'(\varepsilon) > 0$ s.t.

$$\begin{aligned} Q\big(Q^{-1}(v, a_1), a_1'\big) &\overset{(1)}{\leq} Q\big(Q^{-1}\big(v, a_1'\big), a_1'\big) - \varepsilon' \overset{\text{by Lemma S3}}{=} v - \varepsilon' \\ &\overset{\text{by Lemma S3}}{=} Q\big(Q^{-1}(v, a_1), a_1\big) - \varepsilon'. \end{aligned} \tag{S13}$$

Inequality (1) follows because $Q(Q^{-1}(v, a_1'), a_1') \leq Q(Q^{-1}(v, a_1), a_1')$ since $Q^{-1}(v, a_1) > Q^{-1}(v, a_1')$, and if $Q(Q^{-1}(v, a_1'), a_1') = Q(Q^{-1}(v, a_1), a_1')$ with $Q^{-1}(v, a_1) > Q^{-1}(v, a_1') + \varepsilon$, then that contradicts the definition of $Q^{-1}(v, a_1')$ as the sup. Therefore, it follows from (S13) that

$$Q\big(Q^{-1}(v, a_1), a_1\big) - Q\big(Q^{-1}(v, a_1), a_1'\big) \geq \varepsilon',$$

which contradicts that $Q(\cdot, \cdot)$ is continuous in its second argument for fixed value of its first argument (see property (2) in Claim S1 above), since a_1' can be made arbitrarily close to a_1 by choosing δ small enough.

Finally, $W_0(y, \eta) = y$ is obviously continuous and strictly increasing in y, thus (A') holds. Finally, (B) ensures that (B') is satisfied. *Q.E.D.*

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