Econometrica Supplementary Material

SUPPLEMENT TO "MATCHING WITH COMPLEMENTARY CONTRACTS"
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This supplement to "Matching With Complementary Contracts" gives supporting material for Example 3 in the main text, including an alternative numerical example where firms produce complementary products; shows that bundling preserves complementarity in both TU and NTU environments; and provides a lemma supporting the proof of Proposition 5 in the main text.

APPENDIX A: EXAMPLES

DERIVATION OF EQUILIBRIUM FIRM PROFIT IN EXAMPLE 3 We have

$$\begin{aligned}
&\big((p^* - c)' e_i\big)\big(e_i'(a + Sp^*)\big) \\
&\quad = \big(-(S+\bar{S})^{-1}(a - \bar{S}c) - c\big)' e_i e_i' \big(a - S(S+\bar{S})^{-1}(a - \bar{S}c)\big) \\
&\quad = -\big(a - \bar{S}c + (S+\bar{S})c\big)'(S+\bar{S})^{-1} e_i e_i' S(S+\bar{S})^{-1}\big((S+\bar{S})S^{-1}a - a + \bar{S}c\big) \\
&\quad = -(a+Sc)'(S+\bar{S})^{-1} e_i e_i' S(S+\bar{S})^{-1}\bar{S}\big(S^{-1}a + c\big) \\
&\quad = -(a+Sc)'(S+\bar{S})^{-1} e_i e_i' \bar{S}(S+\bar{S})^{-1}(a+Sc) \\
&\quad = (a+Sc)'(S+\bar{S})^{-1} e_i (-S_{ii}) e_i' (S+\bar{S})^{-1}(a+Sc).
\end{aligned}$$

EXAMPLE A.1—Patent Licensing Among Competing Licensees: As in Example 3, consider a Bertrand–Nash model of differentiated product competition with linear demand. There are three firms: $I = \{1, 2, 3\}$. Each firm $i \in I$ sells a single product. Demand for the firms' products is linear, and given by $Q(p) = a + Sp$, where

$$a = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}, \qquad S = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}.$$

Thus, firms 2 and 3 produce complementary products which are each substitutes for the products for firm 1. Each firm has constant marginal cost c_i, and sets prices to maximize profits $(p_i - c_i)Q(p)_i$ given the pricing decisions of the other firms. As in Example 3, firm i's equilibrium profit is given by $(a+Sc)'(S+\bar{S})^{-1}e_i(\bar{S}_{ii})e_i'(S+\bar{S})^{-1}(a+Sc)$; here, $\bar{S} = -2I$. Now suppose that each firm owns patents on technologies that would lower the costs of firms 2 and 3, if they were to adopt them. In particular, if firm $i \in \{2, 3\}$ licenses the technology of firm $j \neq i$, their costs will be reduced by θ_{ij}.

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We can represent patent license agreements between these firms using the set of primitive contracts $\Omega = \{\omega_{21}, \omega_{23}, \omega_{31}, \omega_{32}\}$, where ω_{ij} represents the license of firm i's technology to firm j and $N(\omega_{ij}) = \{i, j\}$. Then in a transferable utility environment, we have

$$v_i(\Psi) = \big(a + Sc(\Psi)\big)'(S - 2I)^{-1} 2 e_i e_i' (S - 2I)^{-1}\big(a + Sc(\Psi)\big),$$
$$\text{where } c_i(\Psi) = c_i^0 - \sum_{j:\omega_{ij} \in \Psi} \theta_{ij}.$$

Since patent licenses only lower the cost of firms 2 and 3, v_i is supermodular in Ω_i and has increasing differences in (Ω_i, Ω_{-i})—and hence agent i's demand correspondence for primitive contracts satisfies the gross complements condition—if and only if $(a + Sc)'(S - 2I)^{-1} 2 e_i e_i' (S - 2I)^{-1}(a + Sc)$ is convex and has positive cross-partial derivatives in c_2 and c_3. The Hessian matrix of this expression is given by $\hat{S}_i = S(S - 2I)^{-1} 2 e_i e_i' (S - 2I)^{-1} S$; for agents $i \in \{1, 2, 3\}$, these are

$$\hat{S}_1 = \frac{1}{81}\begin{bmatrix} 32 & -8 & -8 \\ -8 & 2 & 2 \\ -8 & 2 & 2 \end{bmatrix}, \qquad \hat{S}_2 = \frac{1}{81}\begin{bmatrix} 2 & -8 & -2 \\ -8 & 32 & 8 \\ -2 & 8 & 2 \end{bmatrix},$$
$$\hat{S}_3 = \frac{1}{81}\begin{bmatrix} 2 & -8 & -2 \\ -8 & 2 & 8 \\ -2 & 8 & 32 \end{bmatrix}.$$

It follows that each agent's demand correspondence for primitive contracts satisfies the gross complements condition, and we can apply Theorem 2 using Lemma 7.

Now we solve for the largest conditionally efficient set of primitive contracts, Ω^*. Let

$$c_1^0 = 30, \qquad c_2^0 = 60, \qquad c_3^0 = 40, \qquad \theta_{21} = 20, \qquad \theta_{23} = 5, \qquad \theta_{31} = 5, \qquad \theta_{32} = 5.$$

We have

$$F_\vee(\Omega) = F_\vee\big(\{\omega_{21}, \omega_{23}, \omega_{31}, \omega_{32}\}\big) = \{\omega_{23}, \omega_{31}, \omega_{32}\}$$
$$F_\vee\big(\{\omega_{23}, \omega_{31}, \omega_{32}\}\big) = \{\omega_{23}, \omega_{31}, \omega_{32}\}$$
$$\Rightarrow \quad \Omega^* = \{\omega_{23}, \omega_{31}, \omega_{32}\}.$$

There are no gains from trade between firms 1 and 2, even when each firm anticipates that all licenses between the other firms will transact. Hence, firm 1 does not license its technology to firm 2. However, Ω^* is inefficient: Because firms 2 and 3 produce complementary products, firm 3 would be willing to subsidize firm 2's license of firm 1's technology. Hence, externalities produce deadweight loss in this example.

Alternatively, let

$$c_1^0 = 30, \qquad c_2^0 = 60, \qquad c_3^0 = 40, \qquad \theta_{21} = 20, \qquad \theta_{23} = 1, \qquad \theta_{31} = 5, \qquad \theta_{32} = 1.$$

Then we have

$$F_\vee(\Omega) = F_\vee\big(\{\omega_{21}, \omega_{23}, \omega_{31}, \omega_{32}\}\big) = \{\omega_{23}, \omega_{31}, \omega_{32}\}$$
$$F_\vee\big(\{\omega_{23}, \omega_{31}, \omega_{32}\}\big) = \{\omega_{23}, \omega_{32}\}$$

$$F_\vee\big(\{\omega_{23}, \omega_{32}\}\big) = \{\omega_{23}, \omega_{32}\}$$
$$\Rightarrow \quad \Omega^* = \{\omega_{23}, \omega_{32}\}.$$

This time, there are gains from trade between firms 1 and 3 when firm 3 takes as given that firm 2 will license firm 1's technology, as it does in the first two rounds of the algorithm. But these gains from trade disappear after the second round of the algorithm, when firm 1 fails to license its technology to firm 2. Once again, Ω^* is inefficient—but this time it contains too many contracts instead of too few: Firm 1 would be willing to pay firms 2 and 3 not to license their technologies to one another.

APPENDIX B: BUNDLING RESULTS

Lemma B.1 shows that bundling preserves complementarity in both the ordinal and cardinal sense.

LEMMA B.1—Bundling Preserves Complementarity: *Suppose that* $\alpha : \hat{\Omega} \to \Omega$ *is surjective*.

i. *If* $\hat{u}_i : 2^{\hat{\Omega}} \to \mathbb{R} \cup \{-\infty\}$ *is quasisupermodular, so is* $\hat{u}_i \circ \alpha^{-1}$.

ii. *If* $\hat{v}_i : 2^{\hat{\Omega}} \to \mathbb{R} \cup \{-\infty\}$ *is supermodular, so is* $\hat{v}_i \circ \alpha^{-1}$.

PROOF: Recall that since α is a function, $\alpha^{-1}(Y \cap Z) = \alpha^{-1}(Y) \cap \alpha^{-1}(Z)$ and $\alpha^{-1}(Y \cup Z) = \alpha^{-1}(Y) \cup \alpha^{-1}(Z)$.

(i) Let Y, Z be two subsets of X with $\hat{u}_i(\alpha^{-1}(Y)) - \hat{u}_i(\alpha^{-1}(Y \cap Z)) \geq 0$ or equivalently $\hat{u}_i(\alpha^{-1}(Y)) - \hat{u}_i(\alpha^{-1}(Y) \cap \alpha^{-1}(Z)) \geq 0$. By quasisupermodularity of $\hat{u}_i$, $\hat{u}_i(\alpha^{-1}(Y) \cup \alpha^{-1}(Z)) - \hat{u}_i(\alpha^{-1}(Z)) \geq 0$ and hence $\hat{u}_i(\alpha^{-1}(Y \cup Z)) - \hat{u}_i(\alpha^{-1}(Z)) \geq 0$, as desired.

(ii) Let Ψ, Φ be two subsets of Ω. Since $\hat{v}_i$ is supermodular,

$$\hat{v}_i\big(\alpha^{-1}(Y) \cup \alpha^{-1}(Z)\big) - \hat{v}_i\big(\alpha^{-1}(Z)\big) \geq \hat{v}_i\big(\alpha^{-1}(Y)\big) - \hat{v}_i\big(\alpha^{-1}(Y) \cap \alpha^{-1}(Z)\big)$$
$$\Leftrightarrow \quad \hat{v}_i\big(\alpha^{-1}(Y \cup Z)\big) - \hat{v}_i\big(\alpha^{-1}(Z)\big) \geq \hat{v}_i\big(\alpha^{-1}(Y)\big) - \hat{v}_i\big(\alpha^{-1}(Y \cap Z)\big),$$

as desired. *Q.E.D.*

Lemma B.2 can be thought of as three sets of results. The first (parts (i) through (v)) can be thought of as set-theoretic accounting, showing that the local bijectivity of α on a set of primitive contracts Ψ makes subsets of Ψ equivalent to their images under α, and subsets of $\alpha(\Psi)$ equivalent to their preimages. In particular, the subset relation is preserved (parts (iii) and (iv)), as are the primitive contracts which name an agent (parts (ii) and (v)). The second (parts (vi) through (ix)) show that this implies that optimization works equivalently on Ψ and $\alpha(\Psi)$: In NTU environments, choice (part (vi)), acceptance (part (vii)), and aggregate acceptance (part (viii)) are equivalent on Ψ and $\alpha(\Psi)$, and in TU environments, the solutions to the conditional social planner's problem are identical when the available sets of primitive contracts are Ψ and $\alpha(\Psi)$ (part (ix)). Finally, parts (x) and (xi) show the relationship between solutions to the conditional social planner's problem in the two environments.

LEMMA B.2—Properties of Locally Bijective Bundling Maps: *Suppose that* $M = \langle I, \Omega, \{T^\omega\}_{\omega \in \Omega}, N, \{u_i\}_{i \in I}\rangle$ *is more bundled than* $\hat{M} = \langle I, \hat{\Omega}, \{\hat{T}^\omega\}_{\omega \in \hat{\Omega}}, \hat{N}, \{\hat{u}_i\}_{i \in I}\rangle$ *with bundling map* α. *Let* $\{C_i\}_{i \in I}$ *and* $\{\hat{C}_i\}_{i \in I}$ *represent the agents' choice functions in* M *and*

$\hat{M}$, respectively; $\{A_i\}_{i\in I}$ and $\{\hat{A}_i\}_{i\in I}$ represent their acceptance functions in M and $\hat{M}$, respectively; and A and $\hat{A}$ represent their aggregate acceptance functions in M and $\hat{M}$, respectively.

If $\alpha^{-1}(\alpha(\omega)) = \omega$ for all $\omega \in \Psi \subseteq \hat{\Omega}$, then

i. *For $Y \subseteq \Omega$, $\alpha(\alpha^{-1}(Y)) = Y$; for $\Phi \subseteq \Psi$, $\alpha^{-1}(\alpha(\Phi)) = \Phi$.*
ii. *For $\Phi \subseteq \Psi$, $\alpha(\Phi_i) = \alpha(\Phi)_i$ and $\Phi_i = \alpha^{-1}(\alpha(\Phi)_i)$; $\alpha(\Phi_{-i}) = \alpha(\Phi)_{-i}$ and $\Phi_{-i} = \alpha^{-1}(\alpha(\Phi)_{-i})$.*
iii. *For $Y \subseteq \Omega$, $Y' \subseteq Y \Leftrightarrow \alpha^{-1}(Y') \subseteq \alpha^{-1}(Y)$.*
iv. *For $\Phi \subseteq \Psi$, $\Phi' \subseteq \Phi \Leftrightarrow \alpha(\Phi') \subseteq \alpha(\Phi)$.*
v. *For $Y \subseteq \alpha(\Psi)$, $\alpha^{-1}(Y_i) = \alpha^{-1}(Y)_i$ and $Y_i = \alpha(\alpha^{-1}(Y)_i)$; $\alpha^{-1}(Y_{-i}) = \alpha^{-1}(Y)_{-i}$ and $Y_{-i} = \alpha(\alpha^{-1}(Y)_{-i})$.*
vi. *If M and $\hat{M}$ are NTU environments, $C_i(\alpha(\Psi)_i|\alpha(\Psi)_{-i}) = \{\alpha(Y)|Y \in \hat{C}_i(\Psi_i|\Psi_{-i})\}$.*
vii. *If M and $\hat{M}$ are NTU environments, $A_i(\alpha(\Psi)) = \alpha(\hat{A}_i(\Psi))$.*
viii. *If M and $\hat{M}$ are NTU environments, $A(\alpha(\Psi)) = \alpha(\hat{A}(\Psi))$.*
ix. *If M and $\hat{M}$ are TU environments with valuations $\{v_i\}_{i\in I}$ and $\{\hat{v}_i\}_{i\in I}$, respectively, then $\max_{\hat{\Phi}\subseteq\Psi}\sum_{i\in I}\hat{v}_i(\hat{\Phi}_i \cup \Psi_{-i}) = \max_{\Phi\subseteq\alpha(\Psi)}\sum_{i\in I} v_i(\Phi_i \cup \alpha(\Psi)_{-i})$ and $\alpha^{-1}(\Phi) \in \arg\max_{\hat{\Phi}\subseteq\Psi}\sum_{i\in I}\hat{v}_i(\hat{\Phi}_i \cup \Psi_{-i}) \Leftrightarrow \Phi \in \arg\max_{\Phi\subseteq\alpha(\Psi)}\sum_{i\in I} v_i(\Phi_i \cup \alpha(\Psi)_{-i})$.*
x. *If M and $\hat{M}$ are TU environments with conditional optimizer correspondences F and $\hat{F}$, respectively; supermodular valuations $\{v_i\}_{i\in I}$ and $\{\hat{v}_i\}_{i\in I}$, respectively; and $\alpha(\Psi) \in F(\alpha(\Psi))$, then there exists $\hat{\Phi} \in \hat{F}(\Psi)$ with $\hat{\Phi} \supseteq \Psi$.*
xi. *If M and $\hat{M}$ are TU environments with conditional optimizer correspondences F and $\hat{F}$, respectively; supermodular valuations $\{v_i\}_{i\in I}$ and $\{\hat{v}_i\}_{i\in I}$, respectively; and $\Psi \in \hat{F}(\Psi)$, then there exists $\Phi \in F(\alpha(\Psi))$ with $\Phi \supseteq \alpha(\Psi)$.*

PROOF: (i) The first conclusion follows from surjectivity of α. Now $\omega \in \alpha^{-1}(\alpha(\Phi)) \Leftrightarrow \alpha(\omega) \in \alpha(\Phi)$ and $\omega \in \Phi \Rightarrow \alpha(\omega) \in \alpha(\Phi)$ are immediate. For $\alpha(\omega) \in \alpha(\Phi) \Rightarrow \omega \in \Phi$, suppose $\alpha(\omega) \in \alpha(\Phi)$. Then there exists $\omega' \in \Phi$ with $\alpha(\omega) = \alpha(\omega')$. Then $\omega, \omega' \in \alpha^{-1}(\alpha(\omega))$; since $\omega = \alpha^{-1}(\alpha(\omega))$, $\omega = \omega' \in \Phi$.

(ii) For $\hat{\omega} \in \Psi$, letting $\omega = \alpha(\hat{\omega})$ in part (i) of the "more bundled than" definition yields $N(\alpha(\hat{\omega})) = \hat{N}(\hat{\omega})$. It follows that for each i and $\hat{\omega} \in \Psi$, $\hat{\omega} \in \hat{\Omega}_i \Leftrightarrow \alpha(\hat{\omega}) \in \Omega_i$. Then $\alpha(\Phi_i) = \{\omega|\omega = \alpha(\hat{\omega})$ for some $\hat{\omega} \in \Phi \cap \hat{\Omega}_i\} = \{\omega|\omega = \alpha(\hat{\omega})$ for some $\hat{\omega} \in \Phi$, $\omega \in \Omega_i\} = \alpha(\Phi)_i$. Then from (i), $\Phi_i = \alpha^{-1}(\alpha(\Phi_i)) = \alpha^{-1}(\alpha(\Phi)_i)$. Similarly, $\alpha(\Phi_{-i}) = \alpha(\Phi)_{-i}$ and $\Phi_{-i} = \alpha^{-1}(\alpha(\Phi_{-i})) = \alpha^{-1}(\alpha(\Phi)_{-i})$.

(iii) The implication follows from preservation of the subset relation under preimages. The reverse implication follows from preservation of the subset relation under images and (i).

(iv) The implication follows from preservation of the subset relation under images. The reverse implication follows from preservation of the subset relation under preimages and (i).

(v) By (iii), $\alpha^{-1}(Y) \subseteq \alpha^{-1}(\alpha(\Psi))$; by (i), $\alpha^{-1}(Y) \subseteq \Psi$. Then letting $\Phi = \alpha^{-1}(Y)$ in (ii) yields $\Phi_i = \alpha^{-1}(Y)_i = \alpha^{-1}(\alpha(\alpha^{-1}(Y))_i)$ and $\alpha(\alpha^{-1}(Y)_i) = \alpha(\alpha^{-1}(Y)) \cap \Omega_i$; by (i), $\alpha^{-1}(Y)_i = \alpha^{-1}(Y_i)$ and $\alpha(\alpha^{-1}(Y)_i) = Y_i$. Similarly, $\alpha^{-1}(Y)_{-i} = \alpha^{-1}(Y_{-i})$ and $\alpha(\alpha^{-1}(Y)_{-i}) = Y_{-i}$.

(vi) $Z \in C_i(\alpha(\Psi)_i|\alpha(\Psi)_{-i}) \Leftrightarrow Z \subseteq \alpha(\Psi)_i$ and $u_i(Z \cup \alpha(\Psi)_{-i}) \geq u_i(S \cup \alpha(\Psi)_{-i})$ for each $S \subseteq \alpha(\Psi)_i \Leftrightarrow Z \subseteq \alpha(\Psi)_i$ and $\hat{u}_i(\alpha^{-1}(Z \cup \alpha(\Psi)_{-i})) \geq \hat{u}_i(\alpha^{-1}(S \cup \alpha(\Psi)_{-i}))$ for each $S \subseteq \alpha(\Psi)_i \Leftrightarrow Z \subseteq \alpha(\Psi)_i$ and $\hat{u}_i(\alpha^{-1}(Z) \cup \alpha^{-1}(\alpha(\Psi)_{-i})) \geq \hat{u}_i(\alpha^{-1}(S) \cup \alpha^{-1}(\alpha(\Psi)_{-i}))$ for each $S \subseteq \alpha(\Psi)_i$. From (iii), this is equivalent to $\alpha^{-1}(Z) \subseteq \alpha^{-1}(\alpha(\Psi)_i)$ and $\hat{u}_i(\alpha^{-1}(Z) \cup \Psi_{-i}) \geq$

$\hat{u}_i(\alpha^{-1}(S) \cup \Psi_{-i})$ for each $S \subseteq \alpha(\Psi)_i$. From (ii), this is equivalent to $\alpha^{-1}(Z) \subseteq \Psi_i$ and $\hat{u}_i(\alpha^{-1}(Z) \cup \Psi_{-i}) \geq \hat{u}_i(\alpha^{-1}(S) \cup \Psi_{-i})$ for each $S \subseteq \alpha(\Psi_i)$.

Now suppose this last statement holds. For any $\hat{S} \subseteq \Psi$, choose $S = \alpha(\hat{S})$; by (iv), we have $\alpha^{-1}(Z) \subseteq \Psi_i$ and so $\hat{u}_i(\alpha^{-1}(Z) \cup \Psi_{-i}) \geq \hat{u}_i(\alpha^{-1}(\alpha(\hat{S})) \cup \Psi_{-i}) = \hat{u}_i(\hat{S} \cup \Psi_{-i})$. Then $\alpha^{-1}(Z) \in \hat{C}_i(\Psi_i | \Psi_{-i})$.

Conversely, suppose $\alpha^{-1}(Z) \in \hat{C}_i(\Psi_i | \Psi_{-i})$. Then $\hat{u}_i(\alpha^{-1}(Z) \cup \Psi_{-i}) \geq \hat{u}_i(\hat{S} \cup \Psi_{-i})$ for each $\hat{S} \subseteq \Psi_i$. For any $S \subseteq \alpha(\Psi_i)$, choose $\hat{S} = \alpha^{-1}(S)$; by (i), $S = \alpha(\hat{S})$, and so by (iv), $\hat{S} \subseteq \Psi_i$. Then $\hat{u}_i(\alpha^{-1}(Z) \cup \Psi_{-i}) \geq \hat{u}_i(\alpha^{-1}(S) \cup \Psi_{-i})$ for each $S \subseteq \alpha(\Psi_i) \Leftrightarrow Z \in C_i(\alpha(\Psi)_i | \alpha(\Psi)_{-i})$.

(vii) From (vi), we have $A_i(\alpha(\Psi)) = \bigcup_{Z \in C_i(\alpha(\Psi)_i | \alpha(\Psi)_{-i})} Z = \bigcup_{Y \in \hat{C}_i(\Psi_i | \Psi_{-i})} \alpha(Y) = \alpha(\bigcup_{Y \in \hat{C}_i(\Psi_i | \Psi_{-i})} Y) = \alpha(\hat{A}_i(\Psi))$.

(viii) From (ii), $A(\alpha(\Psi)) = \bigcap_{i \in I}(A_i(\alpha(\Psi)) \cup \alpha(\Psi)_{-i}) = \bigcap_{i \in I}(A_i(\alpha(\Psi)) \cup \alpha(\Psi_{-i}))$. Then from (vii), $A(\alpha(\Psi)) = \bigcap_{i \in I}(\alpha(\hat{A}_i(\Psi)) \cup \alpha(\Psi_{-i})) = \bigcap_{i \in I} \alpha(\hat{A}_i(\Psi) \cup \Psi_{-i})$. Then by (i), $A(\alpha(\Psi)) = \alpha(\alpha^{-1}(A(\alpha(\Psi)))) = \alpha(\alpha^{-1}(\bigcap_{i \in I} \alpha(\hat{A}_i(\Psi) \cup \Psi_{-i})))$. Since preimages preserve intersections, $A(\alpha(\Psi)) = \alpha(\bigcap_{i \in I} \alpha^{-1}(\alpha(\hat{A}_i(\Psi) \cup \Psi_{-i})))$. Then by (i), $A(\alpha(\Psi)) = \alpha(\bigcap_{i \in I} \hat{A}_i(\Psi) \cup \Psi_{-i}) = \alpha(\hat{A}(\Psi))$.

(ix) For any $\hat{\Phi} \subseteq \Psi$,

$$\begin{aligned}\sum_{i \in I} \hat{v}_i(\hat{\Phi}_i \cup \Psi_{-i}) &= \sum_{i \in I} \hat{v}_i\big(\alpha^{-1}\big(\alpha(\hat{\Phi}_i \cup \Psi_{-i})\big)\big) \quad \text{(by (i))} \\ &= \sum_{i \in I} v_i\big(\alpha(\hat{\Phi}_i) \cup \alpha(\Psi_{-i})\big) \\ &= \sum_{i \in I} v_i\big(\alpha(\hat{\Phi})_i \cup \alpha(\Psi)_{-i}\big) \quad \text{(by (ii))}. \end{aligned} \tag{1}$$

Hence, by (ii) and since images preserve unions,

$$\max_{\hat{\Phi} \subseteq \Psi} \sum_{i \in I} \hat{v}_i(\hat{\Phi}_i \cup \Psi_{-i}) = \max_{\hat{\Phi} \subseteq \Psi} \sum_{i \in I} v_i\big(\alpha(\hat{\Phi})_i \cup \alpha(\Psi)_{-i}\big) \leq \max_{\Phi \subseteq \alpha(\Psi)} \sum_{i \in I} v_i\big(\Phi_i \cup \alpha(\Psi)_{-i}\big),$$

and conversely, by (v) and since preimages preserve unions,

$$\max_{\Phi \subseteq \alpha(\Psi)} \sum_{i \in I} v_i\big(\Phi_i \cup \alpha(\Psi)_{-i}\big) = \max_{\Phi \subseteq \alpha(\Psi)} \sum_{i \in I} \hat{v}_i\big(\alpha^{-1}(\Phi)_i \cup \Psi_{-i}\big) \leq \max_{\hat{\Phi} \subseteq \Psi} \sum_{i \in I} \hat{v}_i(\hat{\Phi}_i \cup \Psi_{-i}).$$

Then

$$\max_{\Phi \subseteq \alpha(\Psi)} \sum_{i \in I} v_i\big(\Phi_i \cup \alpha(\Psi)_{-i}\big) = \max_{\hat{\Phi} \subseteq \Psi} \sum_{i \in I} \hat{v}_i(\hat{\Phi}_i \cup \Psi_{-i}),$$

which, together with (1), implies that

$$\alpha^{-1}(\Phi) \in \arg\max_{\hat{\Phi} \subseteq \Psi} \sum_{i \in I} \hat{v}_i(\hat{\Phi}_i \cup \Psi_{-i}) \quad \Leftrightarrow \quad \Phi \in \arg\max_{\Phi \subseteq \alpha(\Psi)} \sum_{i \in I} v_i\big(\Phi_i \cup \alpha(\Psi)_{-i}\big).$$

(x) First note that by definition, $F(\alpha(\Psi)) = \arg\max_{\Phi \subseteq \Omega} \sum_{i \in I} v_i(\Phi_i \cup \alpha(\Psi)_{-i})$ and $\hat{F}(\Psi) = \arg\max_{\hat{\Phi} \subseteq \hat{\Omega}} \sum_{i \in I} \hat{v}_i(\hat{\Phi}_i \cup \Psi_{-i})$. Since $\alpha(\Psi) \in F(\alpha(\Psi))$, it follows that

$$\max_{\Phi \subseteq \alpha(\Psi)} \sum_{i \in I} v_i\big(\Phi_i \cup \alpha(\Psi)_{-i}\big) = \max_{\Phi \subseteq \Omega} \sum_{i \in I} v_i\big(\Phi_i \cup \alpha(\Psi)_{-i}\big) = \sum_{i \in I} v_i\big(\alpha(\Psi)\big), \tag{2}$$

and

$$\Phi \in \arg\max_{\Phi \subseteq \alpha(\Psi)} \sum_{i \in I} v_i\big(\Phi_i \cup \alpha(\Psi)_{-i}\big) \quad \Leftrightarrow \quad \Phi \in F\big(\alpha(\Psi)\big) \text{ and } \Phi \subseteq \alpha(\Psi). \tag{3}$$

Then from (2) and (ix),

$$\sum_{i \in I} v_i\big(\alpha(\Psi)\big) = \max_{\Phi \subseteq \Omega} \sum_{i \in I} v_i\big(\Phi_i \cup \alpha(\Psi)_{-i}\big) = \max_{\hat{\Phi} \subseteq \Psi} \sum_{i \in I} \hat{v}_i(\hat{\Phi}_i \cup \Psi_{-i}),$$

and from (3) and (ix),

$$\alpha^{-1}(\Phi) \in \arg\max_{\hat{\Phi} \subseteq \Psi} \sum_{i \in I} \hat{v}_i(\hat{\Phi}_i \cup \Psi_{-i}) \quad \Leftrightarrow \quad \Phi \in F\big(\alpha(\Psi)\big) \text{ and } \Phi \subseteq \alpha(\Psi). \tag{4}$$

Now, for each $\hat{\Phi}' \in \hat{F}(\Psi)$, we have by definition $\sum_{i \in I} \hat{v}_i(\hat{\Phi}'_i \cup \Psi_{-i}) \geq \max_{\hat{\Phi} \subseteq \Psi} \sum_{i \in I} \hat{v}_i(\hat{\Phi}_i \cup \Psi_{-i})$. Since $\alpha(\Psi) \in F(\alpha(\Psi))$, (i) and (4) imply $\sum_{i \in I} \hat{v}_i(\Psi) \geq \sum_{i \in I} \hat{v}_i((\Psi \cap \hat{\Phi}')_i \cup \Psi_{-i})$. Since $\{\hat{v}_i\}_{i \in I}$ are supermodular, it follows that $\sum_{i \in I} \hat{v}_i((\Psi \cup \hat{\Phi}')_i \cup \Psi_{-i}) \geq \sum_{i \in I} \hat{v}_i(\hat{\Phi}'_i \cup \Psi_{-i})$; hence $\hat{\Phi}' \cup \Psi \in \hat{F}(\Psi)$. The result follows.

(xi) Since $\Psi \in \hat{F}(\Psi)$, it follows that

$$\sum_{i \in I} \hat{v}_i(\Psi) = \max_{\hat{\Phi} \subseteq \Psi} \sum_{i \in I} \hat{v}_i(\hat{\Phi}_i \cup \Psi_{-i}) = \max_{\hat{\Phi} \subseteq \hat{\Omega}} \sum_{i \in I} \hat{v}_i(\hat{\Phi}_i \cup \Psi_{-i}), \tag{5}$$

and

$$\hat{\Phi} \in \arg\max_{\hat{\Phi} \subseteq \Psi} \sum_{i \in I} \hat{v}_i(\hat{\Phi}_i \cup \Psi_{-i}) \quad \Leftrightarrow \quad \hat{\Phi} \in \hat{F}(\Psi) \text{ and } \hat{\Phi} \subseteq \Psi. \tag{6}$$

Then from (5) and (ix),

$$\sum_{i \in I} \hat{v}_i(\Psi) = \max_{\hat{\Phi} \subseteq \hat{\Omega}} \sum_{i \in I} \hat{v}_i(\hat{\Phi}_i \cup \Psi_{-i}) = \max_{\Phi \subseteq \alpha(\Psi)} \sum_{i \in I} v_i\big(\Phi_i \cup \alpha(\Psi)_{-i}\big),$$

and from (6) and (ix),

$$\Phi \in \arg\max_{\Phi \subseteq \alpha(\Psi)} \sum_{i \in I} v_i\big(\Phi_i \cup \alpha(\Psi)_{-i}\big) \quad \Leftrightarrow \quad \alpha^{-1}(\Phi) \in \hat{F}(\Psi) \text{ and } \alpha^{-1}(\Phi) \subseteq \Psi. \tag{7}$$

Now, for each $\Phi' \in F(\alpha(\Psi))$, we have $\sum_{i \in I} v_i(\Phi'_i \cup \alpha(\Psi)_{-i}) \geq \max_{\Phi \subseteq \alpha(\Psi)} \sum_{i \in I} v_i(\Phi_i \cup \alpha(\Psi)_{-i})$. Since $\Psi \in \hat{F}(\Psi)$ and by (i), $\alpha^{-1}(\alpha(\Psi)) = \Psi$, (7) implies $\alpha(\Psi) \in \arg\max_{\Phi \subseteq \alpha(\Psi)} \sum_{i \in I} v_i(\Phi_i \cup \alpha(\Psi)_{-i})$ and hence $\sum_{i \in I} v_i(\alpha(\Psi)) \geq \sum_{i \in I} v_i((\alpha(\Psi) \cap \Phi')_i \cup \alpha(\Psi)_{-i})$. Since $\{v_i\}_{i \in I}$ are supermodular, it follows that $\sum_{i \in I} v_i((\alpha(\Psi) \cup \Phi')_i \cup \alpha(\Psi)_{-i}) \geq \sum_{i \in I} v_i(\Phi'_i \cup \alpha(\Psi)_{-i})$; hence $\Phi' \cup \alpha(\Psi) \in F(\alpha(\Psi))$. The result follows. *Q.E.D.*

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