Econometrica Supplementary Material

SUPPLEMENT TO "THE FOLK THEOREM IN REPEATED GAMES WITH ANONYMOUS RANDOM MATCHING"

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SUPPLEMENTARY APPENDIX: OMITTED PROOFS

THE OMITTED PROOFS rely on two simple lemmas, which are used to adjust the reward functions to correct for unlikely errors in communication. Let $M \subset \mathbb{N}$ be a finite set, let $F \in \mathbb{R}_{++}$, let $f : M \to [-F, F]$ be a function of $m_i \in M$, and let $\tilde{m}_i \in M \cup \{0\}$ be a random variable such that, for each $m_i \in M$, $\Pr(\tilde{m}_i = m_i | m_i) = p(m_i)$ and $\Pr(\tilde{m}_i = 0 | m_i) = 1 - p(m_i)$. Applied to the remainder of the proof, M will be a message set, f will be a reward function bounded by F, and $p(m_i)$ will be the probability that message m_i is received when message m_i is sent.

LEMMA 24: *With* $\hat{\varepsilon} = \max_{m_i \in M} \frac{1-p(m_i)}{p(m_i)}$, *there exists a function* $g : M \cup \{0\} \to [-(1+\hat{\varepsilon})F, (1+\hat{\varepsilon})F]$ *such that* $\max_{m_i \in M} |f(m_i) - g(m_i)| \leq \hat{\varepsilon}F$, *and* $\mathbb{E}[g(\tilde{m}_i)|m_i] = f(m_i)$ *for all* $m_i \in M$.

PROOF: Define $g(0) = 0$ and $g(m_i) = \frac{1}{p(m_i)} f(m_i)$ $\forall m_i \in M$. The claims follow directly. *Q.E.D.*

A similar result holds if we account for self-generation. For $x_{i-1} \in \{G, B\}$, recall that $\text{sign}(x_{i-1}) = -1$ if $x_{i-1} = G$ and $\text{sign}(x_{i-1}) = 1$ if $x_{i-1} = B$. For each $x_{i-1} \in \{G, B\}$, let $f^{x_{i-1}} : M \to [-F, F]$ be a function of $m_i \in M$ such that there exists $c \geq 0$ satisfying

$$\max_{m_i \in M, x_{i-1} \in \{G,B\}} \text{sign}(x_{i-1}) f^{x_{i-1}}(m_i) \geq -c. \tag{75}$$

LEMMA 25: *With* $\hat{\varepsilon} = \max_{m_i \in M} \frac{1-p(m_i)}{p(m_i)}$, *for all* $x_{i-1} \in \{G, B\}$, *there exists a function* $g^{x_{i-1}} : M \cup \{0\} \to [-(1+2\hat{\varepsilon})F, (1+2\hat{\varepsilon})F]$ *such that*

(i) $\max_{x_{i-1} \in \{G,B\}, m_i \in M} |f^{x_{i-1}}(m_i) - g^{x_{i-1}}(m_i)| < \hat{\varepsilon}F$,
(ii) $\mathbb{E}[g^{x_{i-1}}(\tilde{m}_i)|m_i] = f^{x_{i-1}}(m_i)$ *for all* $m_i \in M$,
(iii) $\min_{m_i \in M} \text{sign}(x_{i-1}) g^{x_{i-1}}(m_i) \geq -(1+\hat{\varepsilon})c - \hat{\varepsilon}F$, *and*
(iv) $\min_{m_i \in M} g^{x_{i-1}}(m_i) \geq g^{x_{i-1}}(0)$.

Applied to the remainder of the proof, condition (iii) helps satisfy self-generation, and condition (iv) helps satisfy the premises for the secure and verified modules.

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PROOF: Without loss, assume $F \geq (1+\hat{\varepsilon})c$ (otherwise, $F := (1+\hat{\varepsilon})c$).[1] For $x_{i-1} = G$, define

$$g^{x_{i-1}}(0) = -F \quad \text{and} \quad g^{x_{i-1}}(m_i) = \frac{1}{p(m_i)} f^{x_{i-1}}(m_i) + \frac{1-p(m_i)}{p(m_i)} F \quad \forall m_i \in M_i.$$

Then, for all m_i, we have (1) $\mathbb{E}[g^{x_{i-1}}(\tilde{m}_i)|m_i] = f^{x_{i-1}}(m_i)$, (2) $g_T^{x_{i-1}}(m_i) \in [-(1+2\hat{\varepsilon})F, (1+2\hat{\varepsilon})F]$, (3) $|f^{x_{i-1}}(m_i) - g^{x_{i-1}}(m_i)| \leq 2\hat{\varepsilon}F$, (4) $\text{sign}(x_{i-1}) g^{x_{i-1}}(\tilde{m}_i) \geq -(1+\hat{\varepsilon})c - \hat{\varepsilon}F$, and (5) $g^{x_{i-1}}(m_i) - g^{x_{i-1}}(0) = \frac{1}{p(m_i)}(f^{x_{i-1}}(m_i) + F) \geq 0$.

For $x_{i-1} = B$, define

$$g^{x_{i-1}}(0) = -(1+\hat{\varepsilon})c, \quad \text{and}$$
$$g^{x_{i-1}}(m_i) = \frac{1}{p(m_i)} f^{x_{i-1}}(m_i) + \frac{1-p(m_i)}{p(m_i)}(1+\hat{\varepsilon})c \quad \forall m_i \in M_i.$$

Then, for all m_i, we have (1) $\mathbb{E}[g^{x_{i-1}}(\tilde{m}_i)|m_i] = f^{x_{i-1}}(m_i)$, (2) $g_T^{x_{i-1}}(m_i) \in [-(1+2\hat{\varepsilon})F, (1+2\hat{\varepsilon})F]$, (3) $|f^{x_{i-1}}(m_i) - g^{x_{i-1}}(m_i)| \leq 2\hat{\varepsilon}F$, (4) $\text{sign}(x_{i-1}) g^{x_{i-1}}(\tilde{m}_i) \geq -(1+\hat{\varepsilon})c$, and (5) $g^{x_{i-1}}(m_i) - g^{x_{i-1}}(0) = \frac{1}{p(m_i)}(f^{x_{i-1}}(m_i) + c) \geq 0$ (the last inequality follows from the condition (75)). *Q.E.D.*

S.1. *Proof of Lemma 5*

Let $\mathbf{a}^1 \in A^N$ be the action profile where player i plays a^1 and all other players play a^0. Let $\mathbf{a}^0 \in A^N$ be the action profile where all players play a^0. Let $\mathbb{T}^{1\text{st}} := \bigcup_{k=1}^{b(M_i)} \{2(k-1)T + 1, \dots, 2(k-1)T + T\}$ denote the set of periods in the first half of each interval. For $n \neq i$, define

$$\hat{\pi}_n(h_{n-1}) = \sum_{t \in \mathbb{T}} \frac{2K \mathbf{1}_{\{\omega_{n-1,t} = a^0\}}}{p_{n-1,n}} + \sum_{t \in \mathbb{T}^{1\text{st}}} \frac{\mathbf{1}_{\{\omega_{n-1,t} = a^1\}} (1-\delta^T) \delta^{t-1} (\hat{u}_n(\mathbf{a}^0) - \hat{u}_n(\mathbf{a}^1))}{p_{n-1,i}}$$

and $\pi_n(x_{n-1}, h_{n-1}) = \hat{\pi}_n(h_{n-1}) + v_n(x_{n-1}) - c_n$, where c_n is a constant to be determined. We will show that, for $n \neq i$, Claims 1 and 3 of the lemma hold for any c_n, and that $\mathbb{E}[\sum_{t \in \mathbb{T}} \delta^{t-1} \hat{u}_n(\mathbf{a}_t) + \hat{\pi}_n(h_{n-1})]$ is a constant independent of m_i.

Setting $c_n = \mathbb{E}[\sum_{t \in \mathbb{T}} \delta^{t-1} \hat{u}_n(\mathbf{a}_t) + \hat{\pi}_n(h_{n-1})]$ then implies that Claim 2 also holds.

For Claim 1, we require that playing a^0 throughout the module is optimal with payoff function (20). This follows immediately from the facts that $K \geq \frac{2\bar{u}}{\bar{\varepsilon}}$ and $\max_{h,\tilde{h}} |w_n(h) - w_n(\tilde{h})| < K$, which imply that the first term of $\hat{\pi}_n(h_{n-1})$ dominates any difference in $\sum_{t \in \mathbb{T}} \delta^{t-1} \hat{u}_n(\mathbf{a}_t)$ and $w_n(h)$. Claim 3 is also immediate.

To see that $\mathbb{E}[\sum_{t \in \mathbb{T}} \delta^{t-1} \hat{u}_n(\mathbf{a}_t) + \hat{\pi}_n(h_{n-1})]$ is independent of m_i, note that player i plays a^1 the same number of times regardless of m_i. Therefore, $\mathbb{E}[\sum_{t \in \mathbb{T}} \frac{2K \mathbf{1}_{\{\omega_{n-1,t} = a^0\}}}{p_{n-1,n}}]$ is independent of m_i. It remains to show that

$$\sum_{t \in \mathbb{T}} \delta^{t-1} \hat{u}_n(\mathbf{a}_t) + \sum_{t \in \mathbb{T}^{1\text{st}}} \frac{\mathbb{E}[\mathbf{1}_{\{\omega_{n-1,t} = a^1\}}] (1-\delta^T) \delta^{t-1} (\hat{u}_n(\mathbf{a}^0) - \hat{u}_n(\mathbf{a}^1))}{p_{n-1,i}} \tag{76}$$

is independent of m_i.

[1] Wherever Lemma 25 is applied, we have $F \geq (1+\hat{\varepsilon})c$.

We show that payoff (76) is independent of m_i for each interval, that is, for each $k \in \{1, \dots, b(M_i)\}$, when the sums in (76) are restricted to $\tau \in \{2(k-1)T+1, \dots, 2kT\}$, they are the same when player i plays a^1 in the first half of the kth interval as when she plays a^1 in the second half. When player i plays a^1 in the second half of the kth interval, (76) equals

$$\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T} \delta^{\tau-1}\hat{u}_n(\mathbf{a}^0) + \sum_{\tau=2(k-1)T+T+1}^{2kT} \delta^{\tau-1}\hat{u}_n(\mathbf{a}^1),$$

while when player i plays a^1 in the first half of the kth interval, the payoff (76) equals

$$\begin{aligned}
&\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T} \delta^{\tau-1}\hat{u}_n(\mathbf{a}^1) + \sum_{\tau=2(k-1)T+T+1}^{2kT} \delta^{\tau-1}\hat{u}_n(\mathbf{a}^0) \\
&\quad + \left(1-\delta^T\right) \sum_{\tau=2(k-1)T+1}^{2(k-1)T+T} \delta^{\tau-1}\left(\hat{u}_n(\mathbf{a}^0) - \hat{u}_n(\mathbf{a}^1)\right) \\
&= \sum_{\tau=2(k-1)T+1}^{2(k-1)T+T} \delta^{\tau-1}\hat{u}_n(\mathbf{a}^1) + \delta^T \sum_{\tau=2(k-1)T+1}^{2(k-1)T+T} \delta^{\tau-1}\hat{u}_n(\mathbf{a}^0) \\
&\quad + \left(1-\delta^T\right) \sum_{\tau=2(k-1)T+1}^{2(k-1)T+T} \delta^{\tau-1}\left(\hat{u}_n(\mathbf{a}^0) - \hat{u}_n(\mathbf{a}^1)\right) \\
&= \delta^T \sum_{\tau=2(k-1)T+1}^{2(k-1)T+T} \delta^{\tau-1}\hat{u}_n(\mathbf{a}^1) + \sum_{\tau=2(k-1)T+1}^{2(k-1)T+T} \delta^{\tau-1}\hat{u}_n(\mathbf{a}^0) \\
&= \sum_{\tau=2(k-1)T+1}^{2(k-1)T+T} \delta^{\tau-1}\hat{u}_n(\mathbf{a}^0) + \sum_{\tau=2(k-1)T+T+1}^{2kT} \delta^{\tau-1}\hat{u}_n(\mathbf{a}^1).
\end{aligned}$$

Finally, for player i, define $\hat{\pi}_i(h_{i-1}) = \sum_{t\in\mathbb{T}} \frac{1}{p_{i-1,i}}(\delta^{t-1}\mathbf{1}_{\{\omega_{i-1,t}=a^1\}}(\hat{u}_i(\mathbf{a}^1) - \hat{u}_i(\mathbf{a}^0)) + \mathbf{1}_{\{\omega_{i-1,t}\in\{a^0,a^1\}\}}2\bar{u})$. The first term in the sum makes player i indifferent between playing a^0 and a^1, and the second term makes her not want to play $a \notin \{a^0, a^1\}$. Since player i is indifferent between a^0 and a^1, it follows that $c_i = \mathbb{E}[\sum_{t\in\mathbb{T}} \delta^{t-1}\hat{u}_i(\mathbf{a}_t) + \hat{\pi}_i(h_{i-1})]$ is independent of m_i. Hence, letting $\pi_{i,t}(x_{i-1}, h_{i-1}) = \hat{\pi}_{i,t}(h_{i-1}) + v_i(x_{i-1}) - c_i$, Claims 1–3 of the lemma hold for $n = i$.

S.2. *Proof of Lemma 6*

By Lemma 1, it suffices to show that, for sufficiently large $\delta < 1$, there exist $(\sigma_i^{**}(x_i))_{i,x_i}$, β^{**}, $(v_i^{**}(x_{i-1}))_{i,x_{i-1}}$, and $(\pi_i^{**}(x_{i-1}, h_{i-1}^{T^{**}}))_{i,x_{i-1},h_{i-1}^{T^{**}}}$ such that (1)–(4) are satisfied in the T^{**}-period discounted repeated game.

Construction of $\sigma_i^{**}(x_i)$

Play within the first T_3 periods is given by $(\sigma_i^*(x_i))_{i\in I}$. Play from periods T_3+1 to T^{**} is given by the phase (final, 4, i)$_{i\in I}$ strategies defined in Section E of the Appendix. Denote player i's strategy for periods $T_3+1,\dots,T^{**}$ by $\sigma_i^{T^{**}}|_{h_i^{T_3}}$ (indicating its dependence on $h_i^{T_3}$).

At the end of phase (final, 4, i), for each $n\neq i, i-1$, denote player $i-1$'s inferences of $t_{i-1}(n)$ and $h_{n,t_{i-1}(n)}$ by $t_{i-1}(n)(i-1)\in\{0,1,\dots,T_3\}$ and $h_{n,t_{i-1}(n)}(i-1)\in A^2\cup\{0\}$, respectively. We say that *communication succeeds* if $t_{i-1}(n)(i-1)=t_{i-1}$ and $h_{n,t_{i-1}(n)}(i-1)\neq 0$ for all $n\neq i, i-1$. Denote the event that communication succeeds (resp., fails) by $s_{i-1}=1$ (resp., $s_{i-1}=0$). Note that, if $s_{i-1}=1$ and all players follow $\sigma^{T^{**}}|_{h^{T_3}}$, then $h_{-i,t_{i-1}}(i-1)=h_{-i,t_{i-1}}$.

Construction of β^{**}

As will be seen, for periods $T_3+1,\dots,T^{**}$, the equilibrium is belief-free. Hence, any consistent beliefs suffice. For periods $1,\dots,T_3$, let $\beta^{**}=\beta^*$.

Construction of $\pi_i^{**}(x_{i-1},h_{i-1}^{T^{**}})$

Since $h_{-i,t_{i-1}}$ uniquely identifies $a_{i,t_{i-1}}$ by Lemma 2, there exists $\tilde{\pi}_{i,t}^{\delta}(t_{i-1},h_{-i,t_{i-1}})$ such that, for all $\mathbf{a}_t\in A^N$ and $t\in\{1,\dots,T_3\}$,

$$\tilde{\pi}_{i,t}^{\delta}(t_{i-1},h_{-i,t_{i-1}})=\mathbf{1}_{\{t_{i-1}=t\}}T_3\big(1-\delta^{t-1}\big)\hat{u}_i(\mathbf{a}_t). \tag{77}$$

Note that

$$\lim_{\delta\to 1}\max_{t,t_{i-1},h_{-i,t_{i-1}}}\tilde{\pi}_{i,t}^{\delta}(t_{i-1},h_{-i,t_{i-1}})=0. \tag{78}$$

We use Lemma 24 to adjust $\tilde{\pi}_{i,t}^{\delta}(t_{i-1},h_{-i,t_{i-1}})$ to account for errors in communication.

CLAIM 1: *There exist* $(\pi_{i,t}^{\delta}(t_{i-1},s_{i-1},h_{-i,t_{i-1}}(i-1)))_{i,t,t_{i-1},s_{i-1},h_{-i,t_{i-1}}(i-1)}$ *such that*

1. *For all* $i\in I$, $t_{i-1}\in\{1,\dots,T_3\}$, *and* $h^{T_3}\in H^{T_3}$,

$$\mathbb{E}\big[\pi_{i,t}^{\delta}\big(t_{i-1},s_{i-1},h_{-i,t_{i-1}}(i-1)\big)|h^{T_3},t_{i-1}\big]=\tilde{\pi}_{i,t}^{\delta}(t_{i-1},h_{-i,t_{i-1}}). \tag{79}$$

2. $\lim_{\delta\to 1}\max_{i,t,t_{i-1},s_{i-1},h_{-i,t_{i-1}}(i-1)}\pi_{i,t}^{\delta}(t_{i-1},s_{i-1},h_{-i,t_{i-1}}(i-1))=0$.

PROOF: Let $\tilde{h}_{-i,t_{i-1}}=h_{-i,t_{i-1}}(i-1)$ if $s_{i-1}=1$ and $\tilde{h}_{-i,t_{i-1}}=0$ otherwise. Since $s_{i-1}=1$ implies $h_{-i,t_{i-1}}(i-1)=h_{-i,t_{i-1}}$, we have $\Pr(\tilde{h}_{-i,t_{i-1}}=h_{-i,t_{i-1}}|t_{i-1})+\Pr(\tilde{h}_{-i,t_{i-1}}=0|t_{i-1})=1$. Moreover, by Lemma 3, we have

$$\Pr(\tilde{h}_{-i,t_{i-1}}=h_{-i,t_{i-1}}|t_{i-1})\geq 1-\big(b(T_3)+(N-2)\big(b(T_3+1)+b\big(A^2\big)\big)\big)\exp(-\bar{\varepsilon}T_0).$$

The right-hand side is no less than 1/2 by the definition (17). Hence, the claim follows from (77), (78), and Lemma 24 (with $\hat{\varepsilon}\leq 1$). *Q.E.D.*

Given (77) and (79), since t_{i-1} is drawn uniformly at random from $\{1,\dots,T_3\}$, we have

$$\mathbb{E}\left[\sum_{t=1}^{T_3}\pi_{i,t}^{\delta}\big(t_{i-1},s_{i-1},h_{-i,t_{i-1}}(i-1)\big)|h^{T_3}\right]=\sum_{\tau=1}^{T_3}\big(1-\delta^{t-1}\big)\hat{u}_i(\mathbf{a}_t). \tag{80}$$

Let $\pi_i^{\delta}(x_{i-1},h_{i-1}^{T^{**}}):=\sum_{t=1}^{T_3}\pi_{i,t}^{\delta}(t_{i-1},s_{i-1},h_{-i,t_{i-1}}(i-1))$. Let $\mathbb{T}(\text{final},4)=\bigcup_{i\in I}\mathbb{T}(\text{final}, 4,i)$. Note that, for all $j\neq i$, $\pi_j^{\delta}(x_{j-1},h_{j-1}^{T^{**}})$ does not depend on the outcome of phase (final, 4, i). Hence, by Lemma 5, there exist $(\pi_t(h_{i-1}^{\mathbb{T}(\text{final},4)}))_{i\in I}$ such that $\sigma^{T^{**}}|_{h^{T_3}}$ is a BFE in $\mathbb{T}(\text{final}, 4)$ conditional on each realized h^{T_3}, when payoffs are given by

$$\mathbb{E}\Big[\sum_{n\in I}\sum_{t\in\mathbb{T}(\text{final},4,n)}\delta^{t-1}\hat{u}_i(\mathbf{a}_t)+\pi_i^{\delta}\big(x_{i-1},h_{i-1}^{\mathbb{T}(\text{final},4)}\big)+\pi_t\big(h_{i-1}^{\mathbb{T}(\text{final},4)}\big)|h_i^{T_3}\Big]. \tag{81}$$

Moreover, since $\lim_{\delta\to 1}\max_{x_{i-1},h_{i-1}^{T^{**}}}|\pi_i^{\delta}(x_{i-1},h_{i-1}^{T^{**}})|=0$, we have

$$\lim_{\delta\to 1}\max_{h_{i-1}^{\mathbb{T}(\text{final},4)}}\big|\pi_t\big(h_{i-1}^{\mathbb{T}(\text{final},4)}\big)\big|\leq\left(\bar{u}+2\frac{\bar{u}}{\bar{\varepsilon}}\right)\big(T^{**}-T_3\big)\leq\frac{\varepsilon^*}{2}T_3, \tag{82}$$

where the last inequality follows from (17). Finally, we define

$$\begin{aligned}\pi_i^{**}\big(x_{i-1},h_{i-1}^{T^{**}}\big):=\pi_i^{*}\big(x_{i-1},h_{i-1}^{T_3}\big)+\pi_i^{\delta}\big(x_{i-1},h_{i-1}^{\mathbb{T}(\text{final},4)}\big)+\pi_t\big(h_{i-1}^{\mathbb{T}(\text{final},4)}\big)\\ +\operatorname{sign}(x_{i-1})8\varepsilon^*T_3.\end{aligned} \tag{83}$$

We now verify conditions (1)–(4).

[*Sequential Rationality*:] Ignoring sunk payoffs and the constant term $\operatorname{sign}(x_{i-1})8\varepsilon^*T_3$, player i maximizes the payoff (81) in $\mathbb{T}(\text{final}, 4)$. By construction of $(\pi_t(h_{i-1}^{\mathbb{T}(\text{final},4)}))_{i\in I}$, (1) holds for all $t\in\mathbb{T}(\text{final}, 4)$ for any consistent belief system, since by Lemma 5 the basic protocol is a BFE.

Next, by Lemma 5, the expected payoff $\mathbb{E}[\sum_{t\in\mathbb{T}_1}\delta^{t-1}\hat{u}_i(\mathbf{a}_t)+\pi_t(h_{i-1}^{\mathbb{T}(\text{final},4)})|h^{T_3}]$ does not depend on h^{T_3}. Therefore, in period $t\leq T_3$, player i maximizes

$$\begin{aligned}&\mathbb{E}\left[\sum_{\tau=1}^{T_3}\delta^{t-1}\hat{u}_i(\mathbf{a}_\tau)+\pi_i^{*}\big(x_{i-1},h_{i-1}^{T_3}\big)+\pi_i^{\delta}\big(x_{i-1},h_{i-1}^{\mathbb{T}(\text{final},4)}\big)|h_i^{t-1}\right]\\ &\quad=\mathbb{E}\left[\sum_{\tau=1}^{T_3}\delta^{t-1}\hat{u}_i(\mathbf{a}_\tau)+\pi_i^{*}\big(x_{i-1},h_{i-1}^{T_3}\big)+\mathbb{E}\big[\pi_i^{\delta}\big(x_{i-1},h_{i-1}^{\mathbb{T}(\text{final},4)}\big)|h^{T_3}\big]|h_i^{t-1}\right]\\ &\quad=\mathbb{E}\left[\sum_{\tau=1}^{T_3}\hat{u}_i(\mathbf{a}_\tau)+\pi_i^{*}\big(x_{i-1},h_{i-1}^{\mathbb{T}(\text{final},4)}\big)|h_i^{t-1}\right],\end{aligned} \tag{84}$$

where the first equality follows by iterated expectation, and the second follows from (80). Since (84) equals the objective in (22), (22) implies (1).

[*Promise Keeping*:] Equation (2) is satisfied with $v_i^{**}(x_{i-1})$ defined by

$$\begin{aligned}v_i^{**}(x_{i-1})=\frac{1-\delta}{1-\delta^{T_1}}\mathbb{E}\Bigg[\sum_{t=1}^{T_1}\delta^{t-1}\hat{u}_i(\mathbf{a}_t)+\pi_i^{*}\big(x_{i-1},h_{i-1}^{T_3}\big)+\pi_i^{\delta}\big(x_{i-1},h_{i-1}^{\mathbb{T}(\text{final},4)}\big)\\ +\pi_t\big(h_{i-1}^{\mathbb{T}(\text{final},4)}\big)+\operatorname{sign}(x_{i-1})8\varepsilon^*T_3\Bigg]\end{aligned}$$

$$= \frac{1-\delta}{1-\delta^{T_1}} \mathbb{E}\Bigg[\sum_{t=1}^{T_3} \hat{u}_i(\mathbf{a}_t) + \sum_{t=T_3+1}^{T_4} v_i(x_{i-1}) + \pi_i^*\big(x_{i-1}, h_{i-1}^{T_3}\big) + \operatorname{sign}(x_{i-1}) 8\varepsilon^* T_3 \Bigg] \tag{85}$$

for $x_{i-1} \in \{G, B\}$, where we have used the fact that the expected value of $\sum_{t=T_3+1}^{T_4} \delta^{t-1} \times \hat{u}_i(\mathbf{a}_t) + \pi_t(h_{i-1}^{\mathbb{T}(\text{final},4)})$ equals $\sum_{t=T_3+1}^{T_4} v_i(x_{i-1})$, by Lemma 5.

[*Self-Generation*:] Since $\lim_{\delta\to 1} \max_{x_{i-1}, h_{i-1}^{\mathbb{T}(\text{final},4)}} |\pi_i^\delta(x_{i-1}, h_{i-1}^{\mathbb{T}(\text{final},4)})| = 0$, we have

$$\begin{aligned} &\lim_{\delta\to 1} \operatorname{sign}(x_{i-1})\big(\pi_i^*\big(x_{i-1}, h_{i-1}^{T_3}\big) + \pi_i^\delta\big(x_{i-1}, h_{i-1}^{\mathbb{T}(\text{final},4)}\big) + \pi_t\big(h_{i-1}^{\mathbb{T}(\text{final},4)}\big) + \operatorname{sign}(x_{i-1}) 8\varepsilon^* T_3\big) \\ &\quad \geq \operatorname{sign}(x_{i-1}) \pi_i^*\big(x_{i-1}, h_{i-1}^{T_3}\big) - \lim_{\delta\to 1} \big|\pi_t\big(h_{i-1}^{\mathbb{T}(\text{final},4)}\big)\big| + 8\varepsilon^* T_3 > 0, \end{aligned}$$

where the first inequality follows by (21), and the second by (24) and (82). Hence, for sufficiently large δ, (3) holds.

[*Full Dimensionality*:] Since $\frac{1-\delta}{1-\delta^{T^{**}}} \to \frac{1}{T^{**}}$ as $\delta \to 1$ and $T^{**} > T_3$, (85) implies

$$\begin{aligned} \lim_{\delta\to 1} v_i^{**}(x_{i-1}) &\to \frac{1}{T^{**}} \mathbb{E}\Bigg[\sum_{t=1}^{T_3} \hat{u}_i(\mathbf{a}_t) + \pi_i^*\big(x_{i-1}, h_{i-1}^{T_3}\big) + \operatorname{sign}(x_{i-1}) 8\varepsilon^* T_3 \Bigg] \\ &\begin{cases} \geq \dfrac{T_3}{T^{**}} v_i(x_{i-1}) - 8\varepsilon^* & \text{if } x_{i-1} = G, \\ \leq \dfrac{T_3}{T^{**}} v_i(x_{i-1}) + 8\varepsilon^* & \text{if } x_{i-1} = B, \end{cases} \\ &\begin{cases} \geq v_i(x_{i-1}) - \dfrac{T^{**} - T_3}{T^{**}} 2\bar{u} - 8\varepsilon^* & \text{if } x_{i-1} = G, \\ \leq v_i(x_{i-1}) + \dfrac{T^{**} - T_3}{T^{**}} 2\bar{u} + 8\varepsilon^* & \text{if } x_{i-1} = B. \end{cases} \end{aligned}$$

The second line follows from (23), and the third follows from $\bar{u} \geq \max_{\mathbf{a}\in A^N} |\hat{u}(\mathbf{a})|$ and $v(x) \in F^*$. By (5), we have $v_i(B) + 9\varepsilon^* < v_i < v_i(G) - 9\varepsilon^*$. With (17), the last line implies

$$\lim_{\delta\to 1} v_i^{**}(x_{i-1}) \begin{cases} \geq v_i(x_{i-1}) - 9\varepsilon^* & \text{if } x_{i-1} = G, \\ \leq v_i(x_{i-1}) + 9\varepsilon^* & \text{if } x_{i-1} = B. \end{cases}$$

Hence, for sufficiently large δ, we have $v_i^{**}(B) < v_i < v_i^{**}(G)$.

S.3. *Proof of Lemma 7*

By Lemma 6, it suffices to show that there exist $(\sigma_i^{**}(x_i))_{i,x_i}$, β^{**}, $(v_i^{**}(x_{i-1}))_{i,x_{i-1}}$, and $(\pi_i^{**}(x_{i-1}, h_{i-1}^{T_3}))_{i,x_{i-1},h_{i-1}^{T_3}}$ such that Conditions (22)–(24) are satisfied in the T_3-period discounted repeated game.

Construction of $\sigma_i^{**}(x_i)$

Play within the first T_2 periods is given by $(\sigma_i^*(x_i))_{i\in I}$. Play from periods $T_2 + 1$ to T_3 is given by the phase $(\text{final}, 3, i)_{i\in I}$ strategies defined in Section E. Denote player $i-1$'s inference of $(a_{n,t}, \omega_{n,t})_{t\in\bigcup_{j\in I}\mathbb{T}(\text{final},2,j)}$ by $(a_{n,t}(i-1), \omega_{n,t}(i-1))_{t\in\bigcup_{j\in I}\mathbb{T}(\text{final},2,j)}$. Note that, by (17) and Lemma 3, for each $t \in \bigcup_{j\in I}\mathbb{T}(\text{final}, 2, j)$, we have

$$\Pr\bigl((a_{n,t}(i-1), \omega_{n,t}(i-1))_{n\neq i,i-1} = (a_{n,t}, \omega_{n,t})_{n\neq i,i-1} | (a_{n,t}, \omega_{n,t})_{n\neq i,i-1}\bigr) \geq \frac{1}{2}. \tag{86}$$

Construction of β^{**}

As will be seen, for periods $T_2 + 1, \dots, T_3$, the equilibrium is belief-free. Hence, any consistent beliefs suffice. For periods $1, \dots, T_2$, let $\beta^{**} = \beta^*$.

Construction of $\pi_i^{**}(x_{i-1}, h_{i-1}^{T_3})$

Since $(a_{-i,t}, \omega_{-i,t})$ uniquely identifies $a_{i,t}$ by Lemma 2, there exists $\tilde{\pi}_{i,t}(a_{-i,t}, \omega_{-i,t})$ such that, for all $\mathbf{a}_t \in A^N$ and $t \in \bigcup_{n\in I}\mathbb{T}(\text{final}, 2, n)$,

$$\tilde{\pi}_{i,t}(x_{i-1}, a_{-i,t}, \omega_{-i,t}) = \begin{cases} v_i(x_{i-1}) - \hat{u}_i(\mathbf{a}_t) & \text{if } t \notin \mathbb{T}(\text{final}, 2, n), \\ v_i(x_{i-1}) - \hat{u}_i(\mathbf{a}_t) - 1_{\{a_{i,t}\neq a^0\}} & \text{if } t \in \mathbb{T}(\text{final}, 2, n). \end{cases}$$

We use Lemma 24 to adjust $\tilde{\pi}_{i,t}(x_{i-1}, a_{-i,t}, \omega_{-i,t})$ to account for errors in communication.

CLAIM 2: *There exist* $(\pi_{i,t}(x_{i-1}, a_{-i,t}(i-1), \omega_{-i,t}(i-1)))_{i,t\in\bigcup_{n\in I}\mathbb{T}(\text{final},2,n),x_{i-1},a_{-i,t}(i-1),\omega_{-i,t}(i-1)}$ *such that*

1. *For all* $i \in I$, $t \in \bigcup_{n\in I}\mathbb{T}(\text{final}, 2, n)$, x_{i-1}, *and* $h^{T_2} \in H^{T_2}$,

$$\mathbb{E}\bigl[\pi_{i,t}\bigl(x_{i-1}, a_{-i,t}(i-1), \omega_{-i,t}(i-1)\bigr)|x_{i-1}, h^{T_2}\bigr] = \tilde{\pi}_{i,t}(x_{i-1}, a_{-i,t}, \omega_{-i,t}). \tag{87}$$

2. $\max_{i,t,a_{-i,t}(i-1),\omega_{-i,t}(i-1)} |\pi_{i,t}(x_{i-1}, a_{-i,t}(i-1), \omega_{-i,t}(i-1))| \leq 2(\bar{u}+1)$.

PROOF: We construct $\pi_{i,t}$ from $\tilde{\pi}_{i,t}$ as we constructed $\pi_{i,t}^{\delta}$ from $\tilde{\pi}_{i,t}^{\delta}$ in Claim 1. The bound (86) and Lemma 24 imply the result. *Q.E.D.*

Let $\pi_i^{T_3}(x_{i-1}, h_{i-1}^{T_3}) := \sum_{t\in\bigcup_{n\in I}\mathbb{T}(\text{final},2,n)} \pi_{i,t}(x_{i-1}, a_{-i,t}(i-1), \omega_{-i,t}(i-1))$. Let $\mathbb{T}(\text{final}, 3)$ be the set of periods in $(\text{final}, 3, i)_{i\in I}$. Note that, for all $j \neq i$, the reward $\pi_j^{T_3}(x_{j-1}, h_{j-1}^{T_3})$ does not depend on the outcome in phase $(\text{final}, 3, i)$. Hence, by Lemma 5, there exist $(\pi_t(h_{i-1}^{\mathbb{T}(\text{final},3)}))_{i\in I}$ such that σ^{T_3} is a BFE in $\mathbb{T}(\text{final}, 3)$ when payoffs are given by

$$\mathbb{E}\Biggl[\sum_{t\in\mathbb{T}(\text{final},3)} \hat{u}_i(\mathbf{a}_t) + \pi_i^{T_3}\bigl(x_{i-1}, h_{i-1}^{T_3}\bigr) + \pi_i\bigl(h_{i-1}^{\mathbb{T}(\text{final},3)}\bigr)|h_i^{T_2}\Biggr]. \tag{88}$$

Moreover, since the reward $\pi_i^{T_3}$ is additively separable across $t \in \bigcup_{n\in I}\mathbb{T}(\text{final}, 2, n)$, we have

$$\max_{i,h_{i-1}^{\mathbb{T}(\text{final},3)}} \bigl|\pi_i\bigl(h_{i-1}^{\mathbb{T}(\text{final},3)}\bigr)\bigr| \leq 2\frac{\bar{u}+2(\bar{u}+1)}{\bar{\varepsilon}}(T_3 - T_2).$$

Together with Claim 2, we have

$$\max_{i,h_{i-1}^{T_3}} \left|\pi_i^{T_3}\left(x_{i-1}, h_{i-1}^{T_3}\right)\right| + \max_{i,h_{i-1}^{\mathbb{T}(\text{final},3)}} \left|\pi_i\left(h_{i-1}^{\mathbb{T}(\text{final},3)}\right)\right|$$
$$\leq 2(\bar{u}+1)(T_2-T_1) + 2\frac{\bar{u}+2(\bar{u}+1)}{\bar{\varepsilon}}(T_3-T_2) \leq \varepsilon^* T_3, \tag{89}$$

where the last inequality follows from (17).

Finally, we define

$$\pi_i^{**}\left(x_{i-1}, h_{i-1}^{T_3}\right) := \pi_i^*\left(x_{i-1}, h_{i-1}^{T_2}\right) + \pi_i^{T_3}\left(x_{i-1}, h_{i-1}^{T_3}\right) + \pi_i\left(h_{i-1}^{\mathbb{T}(\text{final},3)}\right) + \text{sign}(x_{i-1})7\varepsilon^* T_3.$$

The verification of Conditions (1)–(4) is now the same as in Lemma 6.

S.4. *Proof of Lemma 9*

We construct strategies $\sigma_i^{**}(x_i)$, beliefs β^{**}, and reward functions $\pi_i^{**}(x_{i-1}, h_{i-1}^{T_2})$ in the T_2-period game that satisfy the premise of Lemma 7.

Construction of $\sigma_i^{**}(x_i)$

Play within the first T_1 periods is given by $(\sigma_i^*(x_i))_{i\in I}$. Play from periods T_1+1 to T_2 is given by the phase $(\text{final}, 2, i)_{i\in I}$ strategies defined in Section E, with $\mathcal{I}_{\text{jam}} = \{i-1\}$ in phase $(\text{final}, 2, i)$. For each $i \in I$ and $n \neq i, i-1$, denote player $i-1$'s inference of $m_{i-1}(n)$ by $m_{i-1}(n)(i-1)$. If $m_{i-1}(n)(i-1) = 0$ for some $n \neq i, i-1$, or if player $i-1$ plays JAM during a round where she receives a message via the secure protocol, let $s_{i-1} = 0$ ("communication fails"). Otherwise, $s_{i-1} = 1$ ("communication succeeds").

Construction of β^{**}

For periods $T_1+1, \dots, T_2$, specify beliefs as in Lemma 8 given the sender's equilibrium message. For periods $1, \dots, T_1$, let $\beta^{**} = \beta^*$.

Construction of $\pi_i^{**}(x_{i-1}, h_{i-1}^{T_2})$

Fix $x_{i-1} \in \{G, B\}$ arbitrarily. If $s_{i-1} = 1$, denote player $i-1$'s inference of player n's message during phase $(\text{final}, 2, i)$ by $(x_n(i-1), h_n^{\mathbb{T}''}(i-1))$. We first construct a function $\tilde{\pi}_i^*(x_{-i}(i-1), h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}(i-1))$ as follows: Define $(\tilde{x}_{-i}, \tilde{h}_{-i}^{\mathbb{T}''}) = (x_{-i}(i-1), h_{-i}^{\mathbb{T}''}(i-1))$ if $s_{i-1} = 1$ and $(\tilde{x}_{-i}, \tilde{h}_{-i}^{\mathbb{T}''}) = 0$ otherwise. Note that (i) (10) implies

$$\min_{x_{-i},h_{-i}^{\mathbb{T}''}} \Pr\left(s_{i-1} = 1 | x_{-i}, h_{-i}^{\mathbb{T}''}\right) \geq 1 - Nb\left(2|A|^{2(T_1 - L(T_0)^3)}\right)\left(\exp(-\bar{\varepsilon}T_0) + 2\exp\left(-(T_0)^{\frac{1}{2}}\right)\right), \tag{90}$$

(ii) $s_{i-1} = 1$ implies $(x_{-i}(i-1), h_{-i}^{\mathbb{T}''}(i-1)) = (x_{-i}, h_{-i}^{\mathbb{T}''})$, and (iii) π_i^* satisfies (37). Hence, in the notation of Lemma 25,

$$\hat{\varepsilon} = \frac{Nb\left(2|A|^{2(T_1-L(T_0)^3)}\right)\left(\exp(-\bar{\varepsilon}T_0) + 2\exp\left(-(T_0)^{\frac{1}{2}}\right)\right)}{1 - Nb\left(2|A|^{2(T_1-L(T_0)^3)}\right)\left(\exp(-\bar{\varepsilon}T_0) + 2\exp\left(-(T_0)^{\frac{1}{2}}\right)\right)},$$

$$F = \max_{\tilde{x}_{-i},h_{i-1}^{T^*},\tilde{h}_{-i}^{\mathbb{T}''}} \left|\pi_i^*\left(\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}\right)\right| \leq_{\text{by (34)}} 8\bar{u}T_1, \qquad c = 5\varepsilon^* T_1.$$

Lemma 25 implies that there exists $\tilde{\pi}_i^*(\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''})$ such that

$$\max_{\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}} \left| \tilde{\pi}_i^*\left(\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}\right)\right| \leq (1+2\hat{\varepsilon})F, \tag{91}$$

$$\mathbb{E}\left[\tilde{\pi}_i^*\left(\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}\right)| x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right] = \pi_i^*\left(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right), \tag{92}$$

$$\operatorname{sign}(x_{i-1})\tilde{\pi}_i^*\left(\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}\right) \geq -(1+\hat{\varepsilon})c - \hat{\varepsilon}F, \quad \text{and} \tag{93}$$

$$\tilde{\pi}_i^*\left(\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}\right) \text{ is minimized when } s_{i-1} = 0. \tag{94}$$

Finally, we define the reward function $\pi_i^{**}(x_{i-1}, h_{i-1}^{T_2}) = \tilde{\pi}_i^{**}(x_{i-1}, h_{i-1}^{T_2})$. It remains to verify the premise of Lemma 7.

[*Sequential Rationality*:] We verify (22) for all $t = 1, \dots, T_2$ by backward induction. In phase (final, 2, i), player i maximizes the conditional expectation of

$$-\sum_{t\in\mathbb{T}(\text{final},2,i)} 1_{\{a_{i,t}\neq a^0\}} + \tilde{\pi}_i^*\left(\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}\right).$$

Given (91) and (94), the premise for secure communication with magnitude $(1+2\hat{\varepsilon})F$ for player i is satisfied, for each $x \in \{G, B\}^N$. Moreover, (32) holds by inequalities (17) and (34). Hence, Lemma 8 implies (27) for $t = T_1 + 1, \dots, T_2$.

Since (92) implies that π_i^* and $\tilde{\pi}_i^*$ are equal in expectation given $\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}$ (assuming players follow σ^{**} in phases (final, 3, i)$_{i\in I}$, as we have shown is optimal), (35) implies (27).

[Promise Keeping:] Let

$$\hat{v}_i(x_{i-1}) := \frac{1}{T_2}\mathbb{E}^{\sigma^*(x)}\left[\sum_{t=1}^{T_1}\hat{u}_i(\mathbf{a}_t) + \sum_{t=T_1+1}^{T_2} v_i(x_{i-1}) - \sum_{t\in\mathbb{T}(\text{final},2,i)} 1_{\{a_{i,t}\neq a^0\}} + \tilde{\pi}_i^{**}\left(x_{-i}, h_{i-1}^{T_2}\right)\right].$$

Equation (36) implies $\hat{v}_i(x_{i-1}) = v_i(x_{i-1})$.

[*Self-Generation*:] By (17), (93) implies (29).

S.5. *Proof of Lemma 11*

Claim 1: If $\operatorname{susp}(h_n) = 1$ for some $n \neq j$, then (ii) holds. If $\theta_j(h_{-j}, \zeta, j') = E$ for some $j' \in I$, then (iii) holds. So assume otherwise.

In light of the definition of FAIL, this implies that, for each $j' \neq j$ and $n \neq j'$, player n observes a^1 in each half-interval in $\mathbb{T}(j')$ where player j' plays a^1. For $n = j$, since players $-j$ follow the equilibrium strategy and take REG, we have $(a_{j',t}(j), \omega_{j',t}(j))_{t\in\mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$. Moreover, for each player $n \neq j, j'$, since $\operatorname{susp}(h_n) = 0$, she does not observe a^1 in any other half-interval in $\mathbb{T}(j')$ than those in which player j' takes a^1. Hence, $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$. Combining these observations, we have $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$ for each $j', n \in I$. Therefore, $m_i(n) = m_i(n')$ for all $n \in I$. Finally, as player i follows the protocol, this message must equal m_i.

For the last part of the claim, consider each event that induces $\operatorname{susp}(h_j) = 1$: If $(a_{n,t}(j), \omega_{n,t}(j))_{t\in\mathbb{T}(\text{msg})} = 0$ for some $n \neq j$, then $(a_{n,t}(j), \omega_{n,t}(j))_{t\in\mathbb{T}(\text{msg})} \neq (a_{n,t}, \omega_{n,t})_{t\in\mathbb{T}(\text{msg})}$. Hence, either some player $j' \neq n, j$ played JAM or player j did not match with player n in a half-interval in $\mathbb{T}(n)$ where player n played a^1. In either case, $\theta_j(h_{-j}, \zeta, n) = E$. If $(a_{j,t}(n), \omega_{j,t}(n))_{t\in\mathbb{T}(\text{msg}), j\in I}$ is not feasible, then again there exists $n \neq j$ with $(a_{n,t}(j), \omega_{n,t}(j))_{t\in\mathbb{T}(\text{msg})} \neq (a_{n,t}, \omega_{n,t})_{t\in\mathbb{T}(\text{msg})}$.

Claim 2: Same as Claim 1, except that the commonly inferred $\hat{m}_i$ may differ from m_i.

Claim 3: Follows from Claim 3 of Lemma 10.

Claim 4: Given Claim 3, it suffices to show $\Pr^{\sigma^*,m_i}(\theta_j(h_{-j},\zeta)=E)\le\exp(-T^{\frac{1}{3}})$. For each $j'\in I$, if no one plays JAM in $\mathbb{T}(j')$, then $\theta_j(h_{-j},\zeta,j')=E$ only if some player $n\neq j'$ fails to observe a^1 in a half-interval in $\mathbb{T}(j')$ where player j' plays a^1. By Lemma 3, this event occurs with probability at most $(N-1)b(A^{4b(M_i)})\exp(-\bar{\varepsilon}T)$. In total, $\theta_j(h_{-j},\zeta)=E$ occurs with probability at most

$$\underbrace{2N(N-1)b\big(A^{4b(M_i)}\big)\exp\big(-T^{\frac{1}{2}}\big)}_{\exists j'\in I,n\neq j:\text{ player } n \text{ plays JAM in } \mathbb{T}(j')}+\underbrace{N(N-1)b\big(A^{4b(M_i)}\big)\exp(-\bar{\varepsilon}T)}_{\exists j'\in I,n\neq i: n \text{ fails to observe } a^1 \text{ in } \mathbb{T}(j')}. \tag{95}$$

By (38), this sum is at most $\exp(-T^{\frac{1}{3}})$.

Claim 5: Follows from Claim 1 of Lemma 10.

S.6. *Proof of Lemma 13*

We prove the first part of the lemma by backward induction. We assume throughout that $\zeta_j=\text{reg}$; if instead $\zeta_j=\text{jam}$, then (42) equals $w_j(h,\zeta)$ and $\theta_j(h_{-j},\zeta)=E$, so player j is indifferent over all protocol strategies by Condition 1 of the premise for communication.

Final Checking Round

Let j' be the index of the final checking round. Fix $h\in H^{<j'}$. The following lemma verifies the receivers' incentives, since both $\hat{u}_j(\mathbf{a}_\tau)-\mathbf{1}_{\{a_{j,\tau}\neq a^0\}}$ and $\hat{u}_j(\mathbf{a}_\tau)-\mathbf{1}_{\{a_{j,\tau}\neq a^*_{j,\tau}(h_{-j})\}}$ for $\tau\notin\mathbb{T}(j')$ are sunk.

LEMMA 26: *Assume $j\neq j'$ and $\zeta_j=\text{reg}$. For every history $h^{<j'}\in H^{<j'}$ and h_j^{t-1} with $t\in\mathbb{T}(j')$, and every action $a_{j,t}\in A$, when player j follows her optimal continuation strategy after taking action $a_{j,t}$, we have*

$$\mathbb{E}\Big[-\sum_{\tau\in\mathbb{T}(j')}1_{\{a_{j,\tau}\neq a^0\}}+w_j(h,\zeta)|h^{<j'},h_j^{t-1},a_{j,t}=a^0\Big]$$
$$\ge\mathbb{E}\Big[-\sum_{\tau\in\mathbb{T}(j')}1_{\{a_{j,\tau}\neq a^0\}}+w_j(h,\zeta)|h^{<j'},h_j^{t-1},a_{j,t}\neq a^0\Big]. \tag{96}$$

PROOF: If $\theta_j(h_{-j},\zeta,j'')=E$ for some $j''\neq j'$, the result follows immediately from (8) and (42), given $\zeta_j=\text{reg}$. So suppose $\theta_j(h_{-j},\zeta,j'')=R$ for all $j''\neq j'$. Since a deviation by any player $j''\neq j$ induces $\theta_j(h_{-j},\zeta)=E$, we also assume players $-j$ follow σ^*_{-j} in every checking round. Hence, $\theta_j(h_{-j},\zeta,j')=E$ if and only if (i) some player $n\neq j'$ does not observe a^1 in a half-interval where player j' plays a^1 or (ii) some player $n\neq j,j'$ plays JAM in $\mathbb{T}(j')$. In particular, let $R_{j',-j}$ denote the event that each player $n\neq j,j'$ is matched with player j' in every half-interval where player j' takes a^1. Then $\Pr(\theta_j(h_{-j},\zeta,j')=E|R_{j',-j},h^{<j'},h_j^{t-1})$ is independent of σ_j.

With i replaced by j', i^* replaced with j, $\mathbb{T}$ replaced with $\mathbb{T}(j')$, and Lemma 4 replaced with Lemma 12, by the same argument as for Lemma 8, with probability at least

$$1-Nb\big(A^{4b(M_i)}\big)\exp\big(-\bar{\eta}T+2T^{\frac{1}{2}}\big), \tag{97}$$

conditional on $(a_{j,\tau}, \omega_{j,\tau})_{\tau\in\mathbb{T}(j')}$, either $\theta_j(h_{-j}, \zeta, j') = E$ or [for each $n \neq j$, $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} \in \{(a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}, 0\}$, and $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$ if and only if $a_{j,\tau} = a^0$ for each $\tau \in \mathbb{T}$ such that $\mu_\tau(j) = n$ and τ is in a half-interval where player j' plays a^0]. The latter event implies $R_{j',-j}$.

Since $\Pr(\theta_j(h_{-j}, \zeta, j') = E|R_{j',-j}, h^{<j'}, h_j^{t-1})$ is independent of σ_j and $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} = 0$ induces $\text{susp}_n(h_n) = 1$, playing $a_{j,\tau} = a^0$ for each $\tau \geq t$ maximizes $w_j(h, \zeta)$ with probability at least (97). Together with (44), this implies that the reward term $-1_{\{a_{j,t}\neq a^0\}}$ outweighs any possible benefit to player j from playing $a \neq a^0$ in an attempt to manipulate $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg}),n\neq j}$. *Q.E.D.*

We next verify the sender's incentive:

LEMMA 27: *Assume* $\zeta_{j'} = \text{reg}$. *For every history* $h^{<j'} \in H^{<j'}$ *and* $h_{j'}^{t-1}$ *with* $t \in \mathbb{T}(j')$, *and every action* $a_{j',t} \in A$, *when player* j' *follows her optimal continuation strategy after taking action* $a_{j',t}$, *we have*

$$\mathbb{E}\Bigg[-\sum_{\tau\in\mathbb{T}(j')} 1_{\{a_{j',\tau}\neq a^*_{j',\tau}(h_{-j'})\}} + w_{j'}(h, \zeta)|h^{<j'}, h_{j'}^{t-1}, a_{j',t} = a^*_{j',t}(h_{-j'})\Bigg]$$
$$\geq \mathbb{E}\Bigg[-\sum_{\tau\in\mathbb{T}(j')} 1_{\{a_{j',\tau}\neq a^*_{j',\tau}(h_{-j'})\}} + w_{j'}(h, \zeta)|h^{<j'}, h_{j'}^{t-1}, a_{j',t} \neq a^*_{j',t}(h_{-j'})\Bigg].$$

PROOF: Again, we assume $\theta_{j'}(h_{-j'}, \zeta, j'') = R$ for all $j'' \neq j'$ and players $-j'$ follow $\sigma^*_{-j'}$ in all checking rounds. In addition, assume $\text{REG}_{j',-j'}$, as otherwise $\theta_{j'}(h_{-j'}, \zeta, j') = E$. Given the reward $-1_{\{a_{j',t}\neq a^*_{j',t}(h_{-j'})\}}$, it suffices to show that following $\sigma^*_{j'}$ maximizes $w_{j'}(h, \zeta)$.

By Claims 4 and 5 of Lemma 10, for each $j'' \neq j'$, since we have assumed $\theta_{j'}(h_{-j'}, \zeta, j'') = R$, we have $(a_{j'',t}(n), \omega_{j'',t}(n))_{t\in\mathbb{T}(\text{msg})} \in \{(a_{j'',t}, \omega_{j'',t})_{t\in\mathbb{T}(\text{msg})}, 0\}$ for all $n \in I$.

Fix $t \in \mathbb{T}(j')$, $h^{<j'}$, and $h_{j'}^{t-1}$. If $(a_{j'',t}(n), \omega_{j'',t}(n))_{t\in\mathbb{T}(\text{msg})} = 0$ for some $j'' \neq j'$ and $n \in I$, then Claim 1 of Lemma 11 implies that $\text{susp}_{n'}(h_{n'}) = 1$ for some $n' \neq j$. Hence, maximizing $w_{j'}(h, \zeta)$ is equivalent to maximizing the probability that $\theta_j(h_{-j}, \zeta, j') = E$. If player j' followed $\sigma^*_{j'}$ until period $t-1$ within $\mathbb{T}(j')$, then following $\sigma^*_{j'}$ maximizes $\theta_{j'}(h_{-j'}, \zeta, j') = E$, by Claim 1 of Lemma 10. Otherwise, $\theta_{j'}(h_{-j'}, \zeta, j') = R$ given $\text{REG}_{j',-j'}$ and any strategy maximizes $w_{j'}(h, \zeta)$. In total, it is optimal to follow $\sigma^*_{j'}$.

Now suppose $(a_{j'',t}(n), \omega_{j'',t}(n))_{t\in\mathbb{T}(\text{msg})} = (a_{j'',t}, \omega_{j'',t})_{t\in\mathbb{T}(\text{msg})}$ for each $j'' \neq j'$ and $n \in I$. Suppose player j' followed $\sigma^*_{j'}$ until period $t-1$ within $\mathbb{T}(j')$. On the one hand, if player j' deviates from $\sigma^*_{j'}$ in period t, then $\theta_{j'}(h_{-j'}, \zeta, j') = R$ given $\text{REG}_{j',-j'}$. Since $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$ for some $n \neq j'$ induces $\text{susp}(h_n) = 1$, player j''s payoff is $P(\sigma_{j'}|h^{<j'}, h_{j'}^{t-1})v_{j'}^{m_i} + (1 - P(\sigma_{j'}|h^{<j'}, h_{j'}^{t-1}))v_{j'}^0$, where m_i corresponds to $(a_{i,t})_{t\in\mathbb{T}(\text{msg})}$ and $P(\sigma_{j'}|h^{<j'}, h_{j'}^{t-1})$ is the probability that $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$ for all $n \neq j'$. On the other hand, if player j' follows $\sigma^*_{j'}$ in period t, then her equilibrium payoff is $P(\sigma^*_{j'}|h^{<j'}, h_{j'}^{t-1})v_{j'}^{m_i} + (1 - P(\sigma^*_{j'}|h^{<j'}, h_{j'}^{t-1}))v_{j'}^E$, since $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$ implies $\theta_{j'}(h_{-j'}, \zeta, j') = E$. As $\min\{v_{j'}^{m_i}, v_{j'}^E\} \geq v_{j'}^0$ by premise and $P(\sigma^*_{j'}|h^{<j'}, h_{j'}^{t-1}) \geq P(\sigma_{j'}|h^{<j'}, h_{j'}^{t-1})$ by definition, it is optimal to play $\sigma^*_{j'}$.

Suppose instead player j' deviated from σ^*_j within $\mathbb{T}(j')$ before period $t-1$. Then $\theta_{j'}(h_{-j'}, \zeta, j') = R$ given $\text{REG}_{j',-j'}$, so her payoff is $P(\sigma_{j'}|h^{<j'}, h^{t-1}_{j'})v^{m_i}_{j'} + (1 - P(\sigma_{j'}|h^{<j'}, h^{t-1}_{j'}))v^0_{j'}$. Again, following $\sigma^*_{j'}$ for the rest of the round maximizes $P(\sigma_{j'}|h^{<j'}, h^{t-1}_{j'})$. *Q.E.D.*

Backward Induction: Given that players will follow σ^* in subsequent rounds and Claim 1 of Lemma 10, we can assume $\theta_j(h_{-j}, \zeta, j'') = R$ for each j'' for which the j''-checking round follows the current round. Hence, the same proof as for Lemmas 26 and 27 establishes each player's incentive to follow σ^* after any history.

Message Round: Again, given that players will follow σ^* in the checking rounds and Claim 1 of Lemma 10, we can assume $\theta_j(h_{-j}, \zeta, j') = R$ for each $j' \in I$, and therefore assume $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$ and $\text{susp}_n(h_n) = 0$ for all $n, j' \in I$. Given this, the strategy of each player $j \neq i$ does not affect $w_j(h, \zeta)$, so incentives are satisfied. For player i, given $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$ for all $n, j' \in I$, $m_i(n)$ will equal $\hat{m}_i$ if player i plays $(a_{i,t})_{t\in\mathbb{T}(\text{msg})}$ corresponding to the binary expansion of $\hat{m}_i$ (with the interpretation that, if $(a_{i,t})_{t\in\mathbb{T}(\text{msg})}$ does not correspond to the binary expansion of any $\hat{m}_i \in M_i$, then $m_i(n) = 1$). Hence, σ_i^{*,m_i^*} is optimal after any history.

i^*-QBFE: The last part of the lemma is immediate: Since $v^E_j = v^{m_i}_j = v^{\text{punish}}_j$ for each $m_i \in M_i$ and $j \neq i^*$, players $-i^*$'s incentives are satisfied. For player i^*, the proof of the first part of the lemma applies.

S.7. *Proof of Lemma 14*

We construct strategies $(\sigma^{**}_i(x_i))_{i,x_i}$ and reward functions $(\pi^{**}_i(x_{-i}, h^{\mathbb{T}'}_{-i}))_{i,x_{-i},h^{\mathbb{T}'}_{-i}}$ in the T_1-period game that satisfy the premise of Lemma 9.

Construction of $\sigma^{**}_i(x_i)$

Play for the first T^* periods is given by $(\sigma^*_i(x_i))_i$. Play from periods T^*+1 to T_1 is given by the phase $(\text{final}, 1, i)_{i\in I}$ strategies outlined in Section E. More precisely:

- Player $i-1$ (mod N) sends $t_{i-1}(1), \dots, t_{i-1}(L)$ using the verified protocol with repetition T_0 and $\mathcal{I}_{\text{jam}} = -i$. Each player $n \in I$ infers a message $(t_{i-1}(1)(n), \dots, t_{i-1}(L)(n))$.
- Sequentially, each player $n \neq i, i-1$ sends $h_{n,t_{i-1}(l)(n)} = (a_{n,t_{i-1}(l)(n)}, \omega_{n,t_{i-1}(l)(n)})_{l=1,\dots,L}$ and $\chi_n \in \{0, 1\}$ using the secure protocol with repetition T_0 and $\mathcal{I}_{\text{jam}} = \{i-1\}$. For each $n \neq i, i-1$, player $i-1$ infers a message $(h_{n,t_{i-1}(l)(n)}(i-1), \chi_n(i-1))$.
- If there exists a player $n \neq i$ with $\text{susp}(h_n) = 1$ or $\theta_i(h_{-i}) = E$ in the verified protocol, or if player $i-1$ infers 0 or plays JAM during a round where she receives a message in the secure protocol, let $s_{i-1} = 0$ ("communication fails"). Otherwise, $s_{i-1} = 1$ ("communication succeeds"). Note that s_{i-1} is a function of $h^{\mathbb{T}'}_{-i}$. Here, ζ_n is assumed to equal jam for each $n \neq i$ and reg for i, and so is omitted from θ_i.

Construction of β^{**}

In periods where player n sends a message via the secure protocol, specify trembles as in Lemma 8. In periods where players use the verified protocol, any consistent belief system suffices. For periods $1, \dots, T^*$, let $\beta^{**} = \beta^*$.

Construction of $\pi^{**}_i(x_{-i}, h^{\mathbb{T}'}_{-i})$

Fix x_{i-1} arbitrarily. If $s_{i-1} = 1$, we denote player $i-1$'s inference of player n's message during phase $(\text{final}, 1, i)$ by $h^{\mathbb{L}_{i-1}}_n(i-1)$ and $\chi_n(i-1)$. As in the proof of Lemma 9, define $\tilde{h}^{\mathbb{L}_{i-1}}_{-i} = h^{\mathbb{L}_{i-1}}_{-i}(i-1)$ and $\tilde{\chi}_{-i} = \chi_{-i}(i-1)$ if $s_{i-1} = 1$, and define $\tilde{h}^{\mathbb{L}_{i-1}}_{-i} = 0$ and $\tilde{\chi}_{-i} = 0$

otherwise. Since $M_i = \{1, \dots, (T_0)^3\}^L$ for the verified communication, Condition (17) implies (38), and therefore Claim 4 of Lemma 11 holds for verified communication. In addition, (10) implies that the secure communication is successful with probability at least $(N-2)b(2A^{2L})(\exp(-T_0) + 2\exp(-(T_0)^{\frac{1}{2}}))$. In total, we have

$$\begin{aligned}\min_{h_{-i}^{\mathbb{L}_{i-1}}} \Pr\bigl(s_{i-1} = 1 | h_{-i}^{\mathbb{L}_{i-1}}\bigr) &\geq 1 - \exp\bigl(-(T_0)^{\frac{1}{3}}\bigr) - (N-2)b\bigl(2A^{2L}\bigr)\bigl(\exp(-T_0) + 2\exp\bigl(-(T_0)^{\frac{1}{2}}\bigr)\bigr) \\ &:= 1 - p^1_{\text{error}}(T_0). \end{aligned} \tag{98}$$

Moreover, the event $s_{i-1} = 1$ implies $h_{-i}^{\mathbb{L}_{i-1}}(i-1) = h_{-i}^{\mathbb{L}_{i-1}}$ and $\chi_{-i}(i-1) = \chi_{-i}$, and the reward π_i^* satisfies the condition (49). Hence, in the notation of Lemma 25,

$$\begin{aligned}&\hat{\varepsilon} = \frac{p^1_{\text{error}}(T_0)}{1 - p^1_{\text{error}}(T_0)}, \quad \text{and} \\ &F = \max_{x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}} \bigl|\pi_i^*\bigl(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}\bigr)\bigr| \leq_{\text{by (46)}} 7\bar{u}T^*, \qquad c = 2\varepsilon^* T^*.\end{aligned} \tag{99}$$

Therefore, Lemma 25 implies that there exists $\tilde{\pi}_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i})$ such that

$$\max_{x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}} \bigl|\tilde{\pi}_i^*\bigl(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i}\bigr)\bigr| \leq (1 + 2\hat{\varepsilon})F, \tag{100}$$

$$\mathbb{E}\bigl[\tilde{\pi}_i^*\bigl(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i}\bigr) | x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}} \chi_{-i}\bigr] = \pi_i^*\bigl(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}} \chi_{-i}\bigr), \tag{101}$$

$$\begin{aligned}&\operatorname{sign}(x_{i-1})\tilde{\pi}_i^*\bigl(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i}\bigr) \\ &\quad \geq -(1+\hat{\varepsilon})c + \hat{\varepsilon}F \quad \forall x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i}, \quad \text{and}\end{aligned} \tag{102}$$

$$\tilde{\pi}_i^*\bigl(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i}\bigr) \text{ is minimized when } s_{i-1} = 0. \tag{103}$$

We define the reward function

$$\begin{aligned}\pi_i^{**}\bigl(x_{-i}, h_{-i}^{\mathbb{T}''}\bigr) &= \tilde{\pi}_i^*\bigl(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i}\bigr) + \sum_{\substack{t=1,\dots,T_1 \\ t \notin \bigcup_{l=1}^{L} \mathbb{T}(\text{main}(l))}} \tilde{\pi}_i^{\text{cancel}}(x_{i-1}, a_{-i,t}, \omega_{-i,t}) \\ &\quad + \sum_{\substack{t \in \mathbb{T}(\text{final},1,i): \\ \text{verified protocol}}} \pi_i^{\text{verify}}\bigl(h_{-i}^{\mathbb{T}''}\bigr) + \sum_{\substack{t \in \mathbb{T}(\text{final},1,i): \\ \text{secure protocol}}} \pi_i^{\text{secure}}\bigl(h_{-i}^{\mathbb{T}''}\bigr).\end{aligned}$$

Here, the rewards $\pi_i^{\text{verify}}(h_{-i}^{\mathbb{T}''})$ and $\pi_i^{\text{secure}}(h_{-i}^{\mathbb{T}''})$ are defined analogously to (42) and (30) for the periods where players $-i$ communicate by the verified and secure communication modules in phase (final, 1, i). Note that these rewards depend only on the history in phase (final, 1, i), and the per-period reward is bounded by 1. Also, $\tilde{\pi}_i^{\text{cancel}}$ is bounded by $[-\bar{u}, \bar{u}]$. So, we have

$$\bigl|\pi_i^{**}\bigl(x_{-i}, h_{-i}^{\mathbb{T}''}\bigr)\bigr| \leq \bigl|\mathbb{T}''\bigr|(1+\bar{u}) + (1+2\hat{\varepsilon})7\bar{u}T^* \leq_{\text{by (17)}} 8\bar{u}T^*. \tag{104}$$

Since $\pi_i^{**}(x_{-i}, h_{-i}^{\mathbb{T}''})$ satisfies (37) (given (104)), it remains to verify the other three conditions of Lemma 9.

[*Sequential Rationality*:] We verify (35) for $t = 1, \dots, T_1$ by backward induction. Given $\tilde{\pi}_i^{\text{cancel}}$, for periods $t' = T^* + 1, \dots, T_1$, player i maximizes the conditional expectation of

$$\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i}\right) + \sum_{\substack{t \in \mathbb{T}(\text{final},1,i):\\ \text{verified}}} \pi_i^{\text{verify}}\left(h_{-i}^{\mathbb{T}''}\right) + \sum_{\substack{t \in \mathbb{T}(\text{final},1,i):\\ \text{secure}}} \pi_i^{\text{secure}}\left(h_{-i}^{\mathbb{T}''}\right).$$

Since the reward $\tilde{\pi}_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i})$ depends only on the histories in $\mathbb{T}(\text{final}, 1, i)$, player i follows the equilibrium strategy in phases $(\text{final}, 1, j)_{j \neq i}$.

For phase $(\text{final}, 1, i)$, given (100) and (103), the premise for secure communication with magnitude $(1 + 2\hat{\varepsilon})F$ for player i is satisfied for all $x \in \{G, B\}^N$. In addition, as $v_i^E = v_i^0 = [\text{value of } \tilde{\pi}_i^* \text{ given } s_{i-1} = 0]$, the premise for verified communication with magnitude $(1 + 2\hat{\varepsilon})F$ for player i is satisfied for all $x \in \{G, B\}^N$. Since $\mathcal{I}_{\text{jam}} = -i$ for verified communication, Condition (17) implies Conditions (38), (43), and (44) (for verified communication), as well as Condition (32) (for secure communication). In total, Lemmas 8 and 13 imply sequential rationality for $t' \in \mathbb{T}(\text{final}, 1, i)$. Finally, since π_i^* and $\tilde{\pi}_i^*$ are equal in expectation given x_{-i}, $h_{-i}^{\mathbb{T}'}$, $h_{-i}^{\mathbb{L}_{i-1}}$, χ_{-i}, (47) implies (35) for $t = 1, \dots, T^*$.

[*Promise Keeping*:] Since π_i^* and $\tilde{\pi}_i^*$ are equal in expectation given x_{-i}, $h_{-i}^{\mathbb{T}'}$, $h_{-i}^{\mathbb{L}_{i-1}}$, (48) holds.

[*Self Generation*:] By (17) and (102), $\text{sign}(x_{i-1})\tilde{\pi}_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \tilde{\chi}_{-i}) \geq -3\bar{u}T^*$. Other terms in $\pi_i^{**}(x_{-i}, h_{-i}^{\mathbb{T}''})$ are bounded by $(1 + \bar{u})|\mathbb{T}''| + 2\varepsilon^* L(T_0)^3 \leq_{\text{by (17)}} -2\bar{u}T^*$. So, (37) holds.

S.8. *Proof of Lemma 15*

Compared to Lemma 14, we introduce (50) and replace (46) with (51) (a more restrictive condition), (47) with (52) (less restrictive), and (48) with (53) (less restrictive). We show that the third replacement is without loss, and then show the same for the second. Given (52), let $\hat{v}_i(x_{-i}) := \frac{1}{L(T_0)^3}\mathbb{E}^{\sigma^*(x)}[\sum_{t \in \bigcup_{l=1}^L \mathbb{T}(\text{main}(l))} \hat{u}_i(\mathbf{a}_t) + \pi_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i})]$.

Since $v_i(x_{i-1}) \in [-\bar{u}, \bar{u}]$, Conditions (49) and (53) imply

$$\hat{v}_i(x_{-i}) - \left(v_i(x_{i-1}) + 2\,\text{sign}(x_{i-1})\varepsilon^*\right) \in [-2\bar{u}, 2\bar{u}]. \tag{105}$$

Define $\tilde{\pi}_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}) = \pi_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}) - (\hat{v}_i(x_{-i}) - (v_i(x_{i-1}) + 2\,\text{sign}(x_{i-1})\varepsilon^*))T^*$. Note that changing the reward function from π_i^* to $\tilde{\pi}_i^*$ only subtracts a constant and thus does not affect sequential rationality. In addition, since $\text{sign}(x_{i-1})(\hat{v}_i(x_{-i}) - (v_i(x_{i-1}) + \text{sign}(x_{i-1})2\varepsilon^*)) \geq 0$ by (53), (49) implies $\text{sign}(x_{i-1})\tilde{\pi}_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}) \geq -2\varepsilon^* T^*$. Hence, self-generation also holds with reward function $\tilde{\pi}_i^*$. Finally, since $(\hat{v}_i(x_{-i}) - (v_i(x_{i-1}) + 2\,\text{sign}(x_{i-1})\varepsilon^*))T^*$ is bounded by $2\bar{u}T^*$ by (105), (51) implies

$$\sup_{x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}} \left|\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}\right)\right| \leq 7\varepsilon^* T^*.$$

Hence, (46) also holds with reward function $\tilde{\pi}_i^*$. Therefore, the premise of Lemma 14 holds. We now show that it is also without loss to replace (47) with (52). We assume that, before the end of main phase l, player i believes that $t_{i-1}(l)$ is uniformly distributed over $\mathbb{T}(\text{main}(l))$.[2] Define $\tilde{\pi}_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) := \pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) +$

[2]This belief results if trembles in periods $t = 1, \dots, T^*$ are independent of $(\mathbb{L}_i, h_i^{t-1})$, and thus is consistent.

$\text{sign}(x_{i-1}) \max_{\tilde{x}_{i-1},\tilde{a}_{-i},\tilde{\omega}_{-i}} \pi_i^{\text{cancel}}(\tilde{x}_{i-1}, \tilde{a}_{-i}, \tilde{\omega}_{-i})$. We have $\tilde{\pi}_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) \in [-2\bar{u}, 2\bar{u}]$, by (7). Note that

$$\begin{aligned}&\mathbb{E}\big[\hat{u}_i(\mathbf{a}) + \pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i})|a\big] \\ &\quad = v_i(x_{i-1}) + \text{sign}(x_{i-1}) \max_{\tilde{x}_{i-1},\tilde{a}_{-i},\tilde{\omega}_{-i}} \pi_i^{\text{cancel}}(\tilde{x}_{i-1}, \tilde{a}_{-i}, \tilde{\omega}_{-i})\end{aligned} \tag{106}$$

and $\text{sign}(x_{i-1})\tilde{\pi}_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) \geq 0$. Since $T^* \in \mathbb{T}'$, we can define

$$\begin{aligned}&\tilde{\pi}_i^*\big(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}\big) \\ &:= \begin{cases} \pi_i^*\big(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\big) & \text{if } \chi_n = 0 \text{ for all } n \neq i, \\ \displaystyle\sum_{t\in\mathbb{T}'} \tilde{\pi}_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) & \\ \quad + (T_0)^3 \displaystyle\sum_{l=1}^{L} \tilde{\pi}_i^{\text{cancel}}(x_{i-1}, a_{-i,t_{i-1}(l)}, \omega_{-i,t_{i-1}(l)}) & \text{if } \chi_n = 1 \text{ for some } n \neq i. \end{cases}\end{aligned}$$

The $(T_0)^3$ term cancels the probability that $t_{i-1}(l) = t$ for each $t \in \mathbb{T}(\text{main}(l))$, so with this reward function player i is indifferent over all action profiles when $\chi_n = 1$ for some $n \neq i$.

Given reward function $\tilde{\pi}_i^*$, (47) and (49) hold. Moreover, given (51) for $\pi_i^*(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}})$,

$$\sup_{x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}},\chi_{-i}} \big|\tilde{\pi}_i^*\big(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}\big)\big| \leq \max\big\{7\bar{u}T^*, 2\bar{u}T^*\big\} \leq 7\bar{u}T^*.$$

Therefore, the premise of Lemma 14 holds.

S.9. *Proof of Lemma 17*

Definition of the Reward Function

We must define $\pi_{i,t}^{\text{indiff}}(h_{-i})$. Given h_{-i}, fix h_i uniquely identified from h_{-i} by Lemma 2. Let H_i^0 be the set of histories for player i with $\omega_{i,1} \neq a^1$ and $\omega_{i,2} \neq a^1$. Given the resulting profile $h = (h_i, h_{-i})$, for $t = 2$, we define $\Delta v_{i,t}(h_{-i})$ as follows:

1. If $\omega_{i,t-1} = a^1$, then $\Delta v_{i,t}(h_{-i}) := 0$.
2. Otherwise, define $\Pr(\mathcal{I}_{\text{jam}} \setminus \{i\}|h^{t-1}, H_i^0, a_i)$ as the conditional probability that the realized set of jamming players other than i at the end of the protocol equals $\mathcal{I}_{\text{jam}} \setminus \{i\}$, given that players $-i$ follow the protocol, $h_i \in H_i^0$, and player i plays a_i in period t. Let

$$\Delta v_{i,t}(h_{-i}) = \sum_{\mathcal{I}_{\text{jam}}\setminus\{i\}} \big(\Pr\big(\mathcal{I}_{\text{jam}} \setminus \{i\}|h^{t-1}, H_i^0, a^1\big) - \Pr\big(\mathcal{I}_{\text{jam}} \setminus \{i\}|h^{t-1}, H_i^0, a^0\big)\big)v_i\big(\mathcal{I}_{\text{jam}} \setminus \{i\}\big).$$

Note that $|\Delta v_{i,t}(h_{-i})| \leq K$, by the bound (63).
Finally, for $t = 2$, we define

$$\pi_{i,t}^{\text{indiff}}(h_{-i}) = -\mathbf{1}_{\{a_{i,t}=a^1\}}\Delta v_{i,t}(h_{-i}). \tag{107}$$

For $t = 1$, define $\Pr(\mathcal{I}_{\text{jam}} \setminus \{i\}|h^{t-1}, H_i^0, a_i)$ as the conditional probability that the realized set of jamming players other than i at the end of the protocol equals $\mathcal{I}_{\text{jam}} \setminus \{i\}$, given that players $-i$ follow the protocol, $h_i \in H_i^0$, and player i plays a_i in period t and a^0 in period $t+1$. The resulting definitions of $\Delta v_{i,t}(h_{-i})$ and $\pi_{i,t}^{\text{indiff}}(h_{-i})$ are the same as for $t=2$.

Note that $|\pi_{i,t}^{\text{indiff}}(h_{-i})| \leq K$ for $t=1,2$. Hence, condition (i) holds.

Incentive Compatibility

We show that, for every player i and period $t=1,2$, it is optimal to follow the protocol in period t given that she will follow the protocol in every later period.

Recall that $\Pr(h_i \in H_i^0)$ is independent of player i's strategy, and Condition 2 of the premise implies that $w_i(h) = w_i(\tilde{h})$ for all h and $\tilde{h}$ satisfying $h_i \notin H_i^0$ and $\tilde{h}_i \notin H_i^0$. Moreover, $w_i(h) = v_i(\mathcal{I}_{\text{jam}} \setminus \{i\})$ if $h_i \in H_i^0$. Hence, player i maximizes her payoff by maximizing $\sum_{t=1}^{2} \pi_{i,t}^{\text{indiff}}(h_{-i}) + v_i(\mathcal{I}_{\text{jam}} \setminus \{i\})$ conditional on $h_i \in H_i^0$.

For $t=2$, ignoring sunk payoffs, player i maximizes $\pi_{i,t}^{\text{indiff}}(h_{-i}) + v_i(\mathcal{I}_{\text{jam}} \setminus \{i\})$ conditional on $h_i \in H_i^0$. By (107), player i is indifferent between a^0 and a^1. Moreover, she is also indifferent between a^0 and any $a \notin \{a^0, a^1\}$, since (i) the distribution of $\mathcal{I}_{\text{jam}} \setminus \{i\}$ is the same whether she takes a^0 or $a \notin \{a^0, a^1\}$, and (ii) (107), $\pi_i^{\text{indiff}}(h_{-i})$ is the same as well.

For $t=1$, noting that her period 1 action does not affect the distribution of anyone's action in period 2, player i again maximizes payoff $\pi_{i,t}^{\text{indiff}}(h_{-i}) + v_i(\mathcal{I}_{\text{jam}} \setminus \{i\})$ conditional on $h_i \in H_i^0$. Again, (107) implies she is indifferent among all actions.

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