*Econometrica Supplementary Material*

# SUPPLEMENT TO "ON THE EFFICIENCY OF SOCIAL LEARNING"

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This file contains the proofs of the statements of "on the efficiency of online learning," with the exception of Theorem 2, which is proven in the main paper.

## S1. THE BENCHMARK: PROOF OF THEOREM 1

WE HERE PROVIDE THE MISSING ARGUMENTS in the analysis of the benchmark case.

LEMMA S1: *One has $\psi_*(0) > 0$.*

PROOF: Recall that, for $\lambda \geq 0$, $\psi(\lambda) = \ln \mathbf{E}_L[\exp(\lambda \ln \frac{\tilde{q}}{1-\tilde{q}})]$, which, by virtue of Claim 1 in the paper, is equal to

$$\psi(\lambda) = \ln \mathbf{E}_L\left[\left(\frac{f_H}{f_L}(\tilde{q})\right)^\lambda\right] = \ln \int_0^1 f_H^\lambda(q) f_L^{1-\lambda}(q)\, dq.$$

This readily yields $\psi(0) = \psi(1) = 0$.

Since $t \mapsto e^{tx}$ is convex, the set $\Lambda := \{\lambda \geq 0, \psi(\lambda) < +\infty\}$ is an interval. Since the private belief $\tilde{q}$ is not a.s. constant, and since $t \mapsto e^{tx}$ is strictly convex whenever $x \neq 0$, $\psi$ is strictly convex on $\Lambda$. Since $\psi(0) = \psi(1) = 0$, this implies that $\psi(\lambda) < 0$ for each $\lambda \in (0, 1)$ and $\psi(\lambda) > 0$ for each $\lambda > 1$. Thus,

$$\psi_*(0) = \sup_{[0,1]}(-\psi) > 0.$$

*Q.E.D.*

LEMMA S2: *If $F(q) = q$ for each $q$, one has $\psi_*(0) = -\ln \frac{\pi}{4}$.*

PROOF: Recall from the proof of Lemma S1 that $\psi_*(0) = -\min_{[0,1]} \psi$. Since $F(q) = q$, one has $f_H(q) = f_L(1-q) = 2q$ for each $q \in [0, 1]$, hence $\psi(\lambda) = \psi(1-\lambda)$ for each $\lambda \in [0, 1]$. Since $\psi$ is convex on $[0, 1]$, this implies that

$$\min_{[0,1]} \psi = \psi\left(\frac{1}{2}\right),$$

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and therefore,

$$\psi_*(0) = -\psi\left(\frac{1}{2}\right) = -\ln \int_0^1 2\sqrt{q(1-q)}\, dq = -\ln \frac{\pi}{4},$$

where the last equality follows using routine computations. *Q.E.D.*

## S2. THE LAST-OBSERVED SETUP: PROOF OF THEOREM 3

In this section, we prove Theorem 3, following closely the outline in Section 3.1. We thus assume that $\phi_n(a_1, \dots, a_{n-1}) = a_n$. The social belief at time $n$ is given by $\pi_n = \mathbf{P}(\tilde{\theta} = H \mid a_n)$.

For $n \geq 1$, we denote by $x_n := \mathbf{P}_L(a_n = h)$ the probability that agent $n$ makes the wrong choice (given $L$). Under Assumptions A1 and A2, we show that

$$\sum_{n\geq 1} x_n < +\infty \quad \Leftrightarrow \quad \int_0^1 \frac{q}{\displaystyle\int_0^q F(x)\, dx}\, dx < \infty. \tag{S2.1}$$

Since $\mathbf{E}_L[W] = \sum_{n=1}^{+\infty} x_n$, Theorem 3 will follow.

We start with some simple properties of the sequence $(x_n)$. The core of the argument is in Section S2.2.

### S2.1. *The Sequence* $(x_n)$

LEMMA S3: *For all $n \geq 1$, one has $x_{n+1} - x_n = -2\int_0^{x_n} F(q)\, dq$.*

PROOF: Fix $n \geq 1$. By Bayes's rule and for each $a \in \{l, h\}$, one has

$$\frac{\mathbf{P}(\tilde{\theta} = H \mid a_n = a)}{\mathbf{P}(\tilde{\theta} = L \mid a_n = a)} = \frac{\mathbf{P}_H(a_n = a)}{\mathbf{P}_L(a_n = a)}. \tag{S2.2}$$

Since $a_{n+1} = h$ if and only if $q_{n+1} + \pi_{n+1} \geq 1$, it follows from (S2.2) that

$$\begin{aligned}\mathbf{P}_\theta(a_{n+1} = h) &= \sum_{a\in\{l,h\}} \mathbf{P}_\theta(a_n = h)\mathbf{P}_\theta\big(q_{n+1} + \mathbf{P}_\theta(H \mid a_n = a) \geq 1\big) \\ &= \sum_{a\in\{l,h\}} \mathbf{P}_\theta(a_n = a)\mathbf{P}_\theta\left(q_{n+1} \geq 1 - \frac{\mathbf{P}_H(a_n = a)}{\mathbf{P}_L(a_n = a) + \mathbf{P}_H(a_n = a)}\right).\end{aligned} \tag{S2.3}$$

The symmetry Assumption A1 implies inductively that $\mathbf{P}_H(a_n = l) = \mathbf{P}_L(a_n = h)$ for each $n$ or, equivalently, $\mathbf{P}_L(a_n = a) + \mathbf{P}_H(a_n = a) = 1$ for each $a$ and $n \in \mathbf{N}$. Equation (S2.3) thus yields

$$\begin{aligned}x_{n+1} &= \sum_{a\in\{l,h\}} \mathbf{P}_L(a_n = a) \times \mathbf{P}_L\big(q_{n+1} \geq \mathbf{P}_L(a_n = a)\big) \\ &= x_n\big(1 - F_L(x_n)\big) + (1 - x_n)\big(1 - F_L(1 - x_n)\big).\end{aligned}$$

Substituting $F_L(q) = 2(1-q)F(q) + 2\int_0^q F(x)\, dx$ (see Section A), elementary manipulations lead to the desired result. *Q.E.D.*

Since $F_L$ and $F_H$ have the same support, the set of guesses that agent $n$ makes with positive probability is independent of $\theta$. By symmetry, both guesses are made with positive probability: $\mathbf{P}_\theta(a_n = a) > 0$, for each $a$, $\theta$, and $n$. Thus, $x_n > 0$.

LEMMA S4: *The sequence* $(x_n)$ *is nonincreasing, with* $\lim x_n = 0$ *if and only if* $q_{\min} = 0$.

PROOF: Denote by $l \geq 0$ the limit of the nonnegative, nonincreasing sequence $(x_n)$, and observe that $l$ solves $\int_0^l F(t)\,dt = 0$. If $q_{\min} = 0$, one has $F(q) > 0$ for all $q > 0$; therefore, $l = 0$. If $q_{\min} > 0$, either $x_n > q_{\min}$ for all $n$ and $l \geq q_{\min}$ or $x_{\bar{n}} \leq q_{\min}$ for some $\bar{n}$, in which case $x_n = x_{\bar{n}}$ for all $n \geq \bar{n}$ and, thus, $l = x_{\bar{n}} > 0$. *Q.E.D.*

Lemma S4 allows us to dispose of the case where $q_{\min} > 0$. In that case, the sequence $(x_n)$ is bounded away from zero; therefore, $\sum_n x_n = +\infty$: learning is inefficient and the equivalence (S2.1) holds.

In the rest of the proof, we assume that $F(q) > 0$ for all $q > 0$, and we set $G(x) := 2\int_0^x F(t)\,dt$.

## S2.2. *The Continuous-Time Approximation*

We use a time-change technique to assess the convergence of $\sum x_n$. Fix $a > 0$ such that $a\alpha > 1$, where $\alpha > 0$ is given by Assumption A2, and for $k \geq 1$, set

$$\omega_k := \inf\left\{n : x_n < \frac{1}{k^a}\right\} \quad (\text{with } \inf\emptyset = +\infty).$$

Note that $\omega_{k+1} \geq \omega_k$ and that $\omega_k < +\infty$ for each $k$ since $(x_n) \to 0$.

Heuristically, the derivation of the continuous-time approximation is sufficiently simple. For $\omega_k \leq n < \omega_{k+1}$, $x_n$ is of the order of $1/k^a$ and $x_n - x_{n+1}$ is of the order of $G(1/k^a)$. Therefore, $\omega_{k+1} - \omega_k$ is approximately equal to $\frac{\frac{1}{k^a} - \frac{1}{(k+1)^a}}{G(1/k^a)}$, which is of the order of $\frac{1}{k^{a+1}G(1/k^a)}$. Thus, $\sum_{n=1}^{+\infty} x_n = \sum_{k=1}^{+\infty}\sum_{\omega_k}^{\omega_{k+1}-1} x_n$ is of the order of $\sum_{k=1}^{+\infty} \frac{1}{k^{2a+1}} \frac{1}{G(\frac{1}{k^a})}$ (Lemmas S6, S7, and S8). We then conclude with a simple series-integral comparison argument. The details are, however, somewhat cumbersome.

Lemma S5 is the only place in the proof where Assumption A2 is used.

LEMMA S5: *The sequence* $(\omega_k)$ *is eventually strictly increasing.*

PROOF: When integrating Assumption A2, one obtains $G(x) \leq \frac{2C}{\alpha+1}x^{\alpha+1}$ for $x$ sufficiently close to 0. In particular, $G(\frac{1}{k^a}) \leq \frac{2C}{\alpha+1}(\frac{1}{k})^{a+a\alpha}$ for $k$ large; thus,

$$G\left(\frac{1}{k^a}\right) = o\left(\frac{1}{k^{a+1}}\right) \quad \text{as } k \to +\infty,$$

since $a\alpha > 1$.

Since $\frac{1}{k^a} - \frac{1}{(k+1)^a} \sim \frac{a}{k^{a+1}}$ as $k \to +\infty$, this implies the existence of $K_0 \in \mathbf{N}$ such that

$$G\left(\frac{1}{k^a}\right) < \frac{1}{k^a} - \frac{1}{(k+1)^a} \quad \text{for each } k \geq K_0.$$

On the other hand, let $\bar{q}$ be the median of $F$: $F(\bar{q}) = \frac{1}{2}$. Since $1 - G'(x) = 1 - 2F(x)$, the map $x \mapsto x - G(x)$ is nondecreasing on $[0, \bar{q}]$. Let $N$ be such that $x_n < \bar{q}$ for each $n \geq N$ and $K_1$ be such that $\omega_{K_1} > N + 1$. Finally, set $K_* := \max(K_0, K_1)$. We will prove that $\omega_{k+1} > \omega_k$ for each $k \geq K_*$.

Let $k \geq K_*$ be arbitrary and set $n := \omega_k - 1$. Since $k \geq K_1$, one has $n > N$, so

$$\frac{1}{k^a} \leq x_{\omega_k - 1} = x_n < \bar{q}.$$

Thus,

$$x_{n+1} = x_n - G(x_n) \geq \frac{1}{k^a} - G\left(\frac{1}{k^a}\right) > \frac{1}{(k+1)^a},$$

where the first inequality holds since $G$ is nondecreasing on $[0, \bar{q}]$ and the second holds since $k \geq K_0$. Since $n + 1 = \omega_k$, we have thus proven that $x_{\omega_k} > \frac{1}{(k+1)^a}$, which implies $\omega_{k+1} > \omega_k$. *Q.E.D.*

LEMMA S6: *One has*

$$x_{\omega_k} - x_{\omega_{k+1}} \sim \frac{a}{k^{a+1}}, \quad \textit{as } k \to +\infty.$$

PROOF: We let $K_*$ be defined as in the proof of Lemma S5. For $k \geq K_*$, one has

$$\frac{1}{k^a} \geq x_{\omega_k} = x_{\omega_k - 1} - G(x_{\omega_k - 1}) \geq \frac{1}{k^a} - G\left(\frac{1}{k^a}\right), \tag{S2.4}$$

where the first inequality holds by definition of $\omega_k$ and the second holds since $x_{\omega_k - 1} \in [\frac{1}{k^a}, \bar{q}]$ and since $x \mapsto x - G(x)$ is nonincreasing on $[0, \bar{q}]$.

For the same reason,

$$\frac{1}{(k+1)^a} \geq x_{\omega_{k+1}} \geq \frac{1}{(k+1)^a} - G\left(\frac{1}{(k+1)^a}\right). \tag{S2.5}$$

By combining (S2.4) and (S2.5), one obtains

$$\frac{1}{k^a} - \frac{1}{(k+1)^a} - G\left(\frac{1}{k^a}\right) \leq x_{\omega_k} - x_{\omega_{k+1}} \leq \frac{1}{k^a} - \frac{1}{(k+1)^a} + G\left(\frac{1}{(k+1)^a}\right).$$

Since $\frac{1}{k^a} - \frac{1}{(k+1)^a} \sim \frac{a}{k^{a+1}}$ and $G(\frac{1}{k^a}) = o(\frac{1}{k^{a+1}})$ as $k \to +\infty$ (see the proof of Lemma S5), the result follows. *Q.E.D.*

LEMMA S7: *One has* $\sum_{n=1}^{+\infty} x_n < +\infty \Leftrightarrow \sum_{k=1}^{+\infty} \frac{\omega_{k+1} - \omega_k}{k^a} < +\infty$.

PROOF: Since $\frac{1}{(k+1)^a} < x_n \leq \frac{1}{k^a}$ when $\omega_k \leq n < \omega_{k+1}$, one has

$$\sum_{k=1}^{+\infty} \frac{\omega_{k+1} - \omega_k}{(k+1)^a} < \sum_{n=\omega_1}^{+\infty} x_n \leq \sum_{k=K_*}^{+\infty} \frac{\omega_{k+1} - \omega_k}{k^a}.$$

Since $\frac{1}{k^a} \sim \frac{1}{(k+1)^a}$ as $k \to +\infty$, the result follows. *Q.E.D.*

LEMMA S8: *One has* $\sum_{n=1}^{+\infty} x_n < +\infty \Leftrightarrow \sum_{k=1}^{+\infty} \frac{1}{k^{2a+1}} \frac{1}{G(\frac{1}{k^a})} < +\infty$.

PROOF: For each $k$ and $n$ such that $\omega_k \le n < \omega_{k+1}$,

$$G\left(\frac{1}{(k+1)^a}\right) \le x_n - x_{n+1} \le G\left(\frac{1}{k^a}\right);$$

hence, by summing over $n$,

$$(\omega_{k+1} - \omega_k) G\left(\frac{1}{(k+1)^a}\right) \le x_{\omega_k} - x_{\omega_{k+1}} \le (\omega_{k+1} - \omega_k) G\left(\frac{1}{k^a}\right). \tag{S2.6}$$

We note that without further information about $F$, it is unclear whether $G(\frac{1}{(k+1)^a}) \sim G(\frac{1}{k^a})$ as $k \to +\infty$. Hence, it is not possible to derive from (S2.6) an asymptotic equivalent for $\omega_{k+1} - \omega_k$; more work is needed.

If $\sum_n x_n < +\infty$, then $\sum_k \frac{\omega_{k+1}-\omega_k}{k^a} < +\infty$ by Lemma S7; hence, $\sum_k \frac{x_{\omega_k} - x_{\omega_{k+1}}}{k^a G(\frac{1}{k^a})} < +\infty$ by (S2.6), which by Lemma S6 implies $\sum_k \frac{1}{k^{2a+1} G(1/k^a)} < +\infty$.

Conversely, if $\sum_k \frac{1}{k^{2a+1} G(1/k^a)} < +\infty$, then $\sum_k \frac{1}{(k-1)^{2a+1} G(1/k^a)} < +\infty$ since $\frac{1}{(k-1)^{2a+1}} \sim \frac{1}{k^a}$ as $k \to +\infty$, hence $\sum_k \frac{1}{k^{2a+1} G(1/(k+1)^a)} < +\infty$, which by Lemma S6 implies $\sum_k \frac{x_{\omega_k} - x_{\omega_{k+1}}}{k^a G(1/(k+1)^a)} < +\infty$ and, therefore, $\sum_k \frac{\omega_{k+1}-\omega_k}{k^a} < +\infty$ by (S2.6), which yields $\sum_n x_n < +\infty$ by Lemma S7. *Q.E.D.*

To simplify the following statement, we introduce

$$a(t) := \frac{1}{t^{2a+1}} \quad \text{and} \quad b(t) := G\left(\frac{1}{k^a}\right) \quad (t > 0).$$

LEMMA S9: *One has* $\sum_{k=1}^{+\infty} \frac{a(k)}{b(k)} < +\infty \Leftrightarrow \int_1^{+\infty} \frac{a(t)}{b(t)} < +\infty$.

PROOF: Since $a(\cdot)$ and $b(\cdot)$ are decreasing on $[1, +\infty)$,

$$\frac{a(k+1)}{b(k)} \le \int_k^{k+1} \frac{a(t)}{b(t)}\, dt \le \frac{a(k)}{b(k+1)}$$

for each $k$ and, therefore,

$$\sum_{k=1}^{+\infty} \frac{a(k+1)}{b(k)} \le \int_1^{+\infty} \frac{a(t)}{b(t)}\, dt \le \sum_{k=1}^{+\infty} \frac{a(k)}{b(k+1)}. \tag{S2.7}$$

Since $a(k) \sim a(k+1)$ as $k \to +\infty$, the three series $\sum \frac{a(k+1)}{b(k)}$, $\sum \frac{a(k)}{b(k+1)}$, and $\sum \frac{a(k)}{b(k)}$ are simultaneously convergent or divergent; hence, the result follows from (S2.7). *Q.E.D.*

Observe now that $\int_1^{+\infty} \frac{a(t)}{b(t)}\, dt = \int_0^1 \frac{q}{G(q)}\, dq$, using the change of variables $q = 1/t^a$. We have thus proven that $\sum x_n < +\infty$ if and only if $\int_0^1 \frac{q}{G(q)} < +\infty$. This concludes the proof of Theorem 3.

## S3. ILLUSTRATIONS: PROOFS OF PROPOSITIONS 1 AND 2

Denote by $\Phi$ the cdf of the standard normal distribution. We start with the proof of Proposition 1. Assume w.l.o.g. that $\Delta\mu := \mu_H - \mu_L > 0$, and denote by $g_\theta$ the conditional density of $s_n$ given $\tilde{\theta} = \theta$. Following a signal realization $\tilde{s}$ and by Bayes's rule, the private belief $\tilde{q}$ is given by

$$\ln \frac{\tilde{q}}{1-\tilde{q}} = \ln \frac{g_H(\tilde{s})}{g_L(\tilde{s})} = \frac{\Delta\mu}{\sigma^2}\left(\tilde{s} - \frac{\mu_H + \mu_L}{2}\right)$$

and is therefore increasing in $\tilde{s}$. Thus, for $q \in (0,1)$, one has

$$F_L(q) = \mathbf{P}_L(\tilde{q} \le q) = \mathbf{P}_L\left(\ln \frac{\tilde{q}}{1-\tilde{q}} \le \ln \frac{q}{1-q}\right) = \mathbf{P}_L\left(\tilde{s} \le \frac{\mu_H + \mu_L}{2} + \frac{\sigma^2}{\Delta\mu} \ln \frac{q}{1-q}\right).$$

Since the r.v. $\frac{\tilde{s}-\mu_L}{\sigma}$ follows a standard normal distribution conditional on $\tilde{\theta} = L$, this yields $F_L(q) = \Phi(x(q))$ for each $q$, where

$$x(q) := \frac{\Delta\mu}{2\sigma} + \frac{\sigma}{\Delta\mu} \ln \frac{q}{1-q}.$$

We will use the inequality $\Phi(x) \le e^{-x^2/2}$, which holds for all $x < 0$ such that $|x|$ is large enough. Since

$$x(q)^2 = \left(\frac{\Delta\mu}{2\sigma}\right)^2 + \ln \frac{q}{1-q} + \left(\frac{\sigma}{\Delta\mu}\right)^2 \left(\ln \frac{q}{1-q}\right)^2 \ge \ln q + \frac{\sigma^2}{(\Delta\mu)^2}\left(\ln \frac{q}{1-q}\right)^2,$$

one obtains

$$F_L(q) \le e^{-(x(q))^2/2} \le \frac{1}{\sqrt{q}} \exp\left\{-\frac{\sigma^2}{2(\Delta\mu)^2}\left(\ln \frac{q}{1-q}\right)^2\right\} \tag{S3.1}$$

for all $q$ close enough to zero. The right-hand side of (S3.1) is equivalent to $\frac{1}{\sqrt{q}} \exp(-\frac{\sigma^2}{2(\Delta\mu)^2}(\ln q)^2)$ in the neighborhood of zero,[1] which around zero is negligible relative to any polynomial function of $q$. Proposition 1 follows.

We turn to the proof of Proposition 2, which is similar. We assume w.l.o.g. that $\mu_H = \mu_L = 0$. Following a signal realization $\tilde{s}$ and by Bayes's rule, the private belief $\tilde{q}$ is given by

$$\frac{\tilde{q}}{1-\tilde{q}} = \frac{\sigma_L}{\sigma_H} e^{-\frac{\tilde{s}^2}{2\delta}},$$

[1]Indeed, the ratio of these two quantities is given by $\exp(\frac{1}{2a^2}\ln(1-q)\ln\frac{2q}{1-q})$. Around zero, the expression within the exponential is equivalent to $-\frac{1}{2a^2} \times q \ln q$, which converges to zero as $q \to 0$.

with $\frac{1}{\delta} = \frac{1}{\sigma_H^2} - \frac{1}{\sigma_L^2} > 0$. Since the likelihood ratio $\frac{\tilde{q}}{1-\tilde{q}}$ does not exceed $\frac{\sigma_L}{\sigma_H}$, the private belief $\tilde{q}$ cannot possibly exceed $q_{\max} := \frac{\sigma_L}{\sigma_L+\sigma_H} < 1$. For $q \in (0, q_{\max}]$, one has

$$\begin{aligned} F_L(q) &= \mathbf{P}_L\left(\frac{\sigma_L}{\sigma_H} e^{-\frac{\tilde{s}^2}{2\delta}} \leq \frac{q}{1-q}\right) \\ &= \mathbf{P}_L\left(\tilde{s}^2 \geq 2\delta \ln \frac{1-q}{q} + 2\delta \ln \frac{\sigma_L}{\sigma_H}\right) \\ &= 2\mathbf{P}_L\left(\frac{\tilde{s}}{\sigma_L} \geq \frac{1}{\sigma_L}\sqrt{2\delta \ln\left(\frac{1-q}{q} \times \frac{\sigma_L}{\sigma_H}\right)}\right). \end{aligned}$$

Since the r.v. $\frac{\tilde{s}}{\sigma_L}$ follows a standard normal distribution, one has

$$F_L(q) = 2\big(1 - \Phi\big(z(q)\big)\big),$$

with $z(q) := \frac{1}{\sigma_L}\sqrt{2\delta \ln(\frac{1-q}{q} \times \frac{\sigma_L}{\sigma_H})}$. Recall from Section A that $F_L(q) \sim_0 2F(q)$ as $q \to 0$. Using the inequalities

$$\frac{z}{z^2+1} e^{-z^2/2} \leq 1 - \Phi(z) \leq \frac{1}{z} e^{-z^2/2} \quad \text{for } z > 0,$$

see, for example, Revuz and Yor (1999, p. 30), it follows that $F(q) \sim \frac{1}{z(q)} e^{-z(q)^2/2}$ as $q \to 0$, and thus, that $\int_0^1 \frac{1}{F(q)}\, dq < +\infty$ is equivalent to $\int_0^1 z(q) e^{z(q)^2/2}\, dq < +\infty$.

Next, observe that $z(q) \sim \frac{\sqrt{2\delta}}{\sigma_L}\sqrt{|\ln q|}$ as $q \to 0$, and that

$$e^{z(q)^2/2} = \exp\left(\frac{\delta}{\sigma_L^2} \ln\left(\frac{1-q}{q} \times \frac{\sigma_L}{\sigma_H}\right)\right) = \left(\frac{1-q}{q}\right)^{\delta/\sigma_L^2} \left(\frac{\sigma_L}{\sigma_H}\right)^{\delta/\sigma_L^2},$$

hence

$$z(q) e^{z(q)^2/2} \sim C_2 \frac{\sqrt{|\ln q|}}{q^{\delta/\sigma_L^2}}$$

as $q \to 0$, for some constant $C_2 > 0$. It follows that the integral $\int_0^1 z(q) e^{z(q)^2/2}\, dp$ is finite if and only if $\delta/\sigma_L^2 < 1$, or equivalently, $\sigma_L^2 > 2\sigma_H^2$, as desired.

## S4. RATES OF CONVERGENCE: PROOF OF THEOREM 4

The proof of Lemma 1 relies on Lemma S10 below, which is a classical result on asymptotic expansions of sequences. An equivalent statement appears in Francinou, Gianella, and Nicolas (2013, in French). Related analysis may be found in De Bruijn (1961).

LEMMA S10: *Let $g : \mathbf{R}_+ \to \mathbf{R}_+$, and $(u_n)$ a sequence be given, such that $u_{n+1} = g(u_n)$ for each $n$. Assume that $\lim u_n = 0$ and that $g(x) = x - ax^\beta + o(x^\beta)$ in the neighborhood of zero, with $a > 0$ and $\beta > 1$. Then*

$$u_n \sim \left(\frac{1}{a(\beta-1)} \frac{1}{n}\right)^{1/(\beta-1)} \quad \text{as } n \to +\infty.$$

PROOF: We follow the proof in Francinou, Gianella, and Nicolas (2013). For $x > 0$,

$$\begin{aligned} g(x)^{1-\beta} - x^{1-\beta} &= \left(x - ax^\beta + o\left(x^\beta\right)\right)^{1-\beta} - x^{1-\beta} \\ &= x^{1-\beta}\left(\left(1 - ax^{\beta-1} + o\left(x^{\beta-1}\right)\right)^{1-\beta} - 1\right) \\ &= x^{1-\beta}\left(-a(1-\beta)x^{\beta-1} + o\left(x^{\beta-1}\right)\right) = a(\beta-1) + o(1), \end{aligned}$$

hence $\lim_{x\to 0}(g(x)^{1-\beta} - x^{1-\beta}) = a(\beta - 1)$. Since $\lim u_n = 0$ and $u_{n+1} = g(u_n)$, this implies $\lim(u_{n+1}^{1-\beta} - u_n^{1-\beta}) = a(\beta - 1)$. By Cesaro Theorem, one has therefore $\lim \frac{u_n^{1-\beta}}{n} = a(\beta - 1)$ as well, hence $u_n \sim (a(\beta - 1)n)^{1/(1-\beta)}$ as $n \to +\infty$, as desired. *Q.E.D.*

PROOF OF LEMMA 1: We assume first that all choices are public, and recall that $\mathbf{P}_L(\tau > n) \sim (1 - \pi_n^*)\mathbf{P}_H(a_m = h \text{ for all } m)$ as $n \to +\infty$, using the notations of Section B.2. Set $u_n := 1 - \pi_n^*$. From (B.6), one has

$$\frac{u_{n+1}}{1 - u_{n+1}} = \frac{u_n}{1 - u_n} \times \frac{1 - F_L(u_n)}{1 - F_H(u_n)},$$

or equivalently,

$$u_{n+1} = g(u_n) := \frac{u_n\left(1 - F_L(u_n)\right)}{u_n\left(1 - F_L(u_n)\right) + (1 - u_n)\left(1 - F_H(u_n)\right)}. \tag{S4.1}$$

Under the assumption that $F(q) = aq^\alpha + o(q^\alpha)$ as $q \to 0$, Section A yields $F_L(q) = 2aq^\alpha + o(q^\alpha)$ and $F_H(q) = o(q^\alpha)$ as $q \to 0$. Plugging into (S4.1), we obtain

$$g(x) = x - 2ax^{\alpha+1} + o\left(x^{\alpha+1}\right) \quad \text{as } x \to 0.$$

The result then follows from Lemma S10.

Assume now that only the previous choice is observed. From Lemma 14, and the assumption on $F$, one has

$$x_{n+1} = x_n - \int_0^{x_n} F(q)\,dq = x_n - \frac{2a}{\alpha+1}x_n^{\alpha+1} + o\left(x_n^{\alpha+1}\right) \quad \text{as } n \to +\infty.$$

The result again follows from Lemma S10. *Q.E.D.*

PROOF OF THEOREM 4: We rely on the following elementary observation on divergent series. Let $(x_n)$ and $(u_n)$ be two bounded sequences such that $x_n \sim u_n$ as $n \to +\infty$. Assume that $u_n > 0$ for each $n$ and that the series $\sum u_n$ is divergent. Then $\sum_{k=1}^n x_k \sim \sum_{k=1}^n u_k$ as $n \to +\infty$, and $\sum_{k=1}^{+\infty} \delta^{k-1} x_k \sim \sum_{k=1}^{+\infty} \delta^{k-1} u_k$ as $\delta \to 1$.

Assume as stated that $F(q) \sim aq^\alpha$ as $q \to 0$, with $\alpha \geq 1$. In the all-observed setup, let $x_n := \mathbf{P}_L(\tau > n)$ and $u_n := c_1 1/n^{1/\alpha}$. Since $\sum u_n$ is divergent, and since $\mathbf{E}_L[\min(\tau, n)] = 1 + \sum_{k=1}^{n-1} x_k$, one has

$$\mathbf{E}_L\left[\min(\tau, n)\right] \sim c_1 \sum_{k=1}^n \frac{1}{k^{1/\alpha}} \quad \text{as } n \to +\infty.$$

Since $\alpha \geq 1$ and since $\sum 1/k^{1/\alpha}$ is divergent, a usual series-integral comparison argument yields $\sum_{k=1}^{n} \frac{1}{k^{1/\alpha}} \sim \int_1^n \frac{1}{x^{1/\alpha}}\,dx$ as $\to +\infty$, and the first claim follows.

In the last-observed setup, we let $x_n := \mathbf{P}_L(a_n = h)$ and $u_n := c_2 1/n^{1/\alpha}$. Since $\mathbf{E}_L[W_n] = \sum_{k=1}^{n} x_k$, it follows as in the previous paragraph that

$$\mathbf{E}_L[W_n] \sim c_2 \int_1^n \frac{1}{x^{1/\alpha}}\,dx \quad \text{as } n \to +\infty.$$

We are left with the estimate of $\mathbf{E}_L[W_\delta]$. Using the notations of the previous paragraph, one has

$$\mathbf{E}_L[W_\delta] = \sum_{k=1}^{+\infty} \delta^{k-1} x_k \sim c_2 \sum_{k=1}^{+\infty} \frac{\delta^{k-1}}{k^{1/\alpha}},$$

which in turn is equivalent to $c_2 \sum_{k=1}^{+\infty} \frac{\delta^k}{k^{1/\alpha}}$ as $\delta \to 1$.

Since

$$\frac{\delta^{k+1}}{(k+1)^{1/\alpha}} \leq \int_k^{k+1} \frac{\delta^x}{x^{1/\alpha}}\,dx \leq \frac{\delta^k}{k^{1/\alpha}} \quad \text{for each } k \geq 1,$$

one gets, by summation over $k$,

$$\sum_{k=2}^{+\infty} \frac{\delta^k}{k^{1/\alpha}} \leq \int_1^{+\infty} \frac{\delta^x}{x^{1/\alpha}}\,dx \leq \sum_{k=1}^{+\infty} \frac{\delta^k}{k^{1/\alpha}},$$

and therefore, $\sum_{k=1}^{+\infty} \frac{\delta^k}{k^{1/\alpha}} \sim \int_1^{+\infty} \frac{\delta^x}{x^{1/\alpha}}\,dx$ as $\delta \to 1$, since $\lim_{\delta\to 1} \int_1^{+\infty} \frac{\delta^x}{x^{1/\alpha}}\,dx = +\infty$.

Using the change of variable $y = -x \ln \delta$, the latter integral is equal to

$$\int_1^{+\infty} \frac{\delta^x}{x^{1/\alpha}}\,dx = (-\ln \delta)^{1/\alpha - 1} \times \int_{-\ln \delta}^{+\infty} e^{-y} y^{-1/\alpha}\,dy. \tag{S4.2}$$

If $\alpha > 1$, the desired estimate follows from equation (S4.2) since $-\ln(\delta) \sim (1-\delta)$ and since $\int_{-\ln\delta}^{+\infty} e^{-y} y^{-1/\alpha}\,dy$ converges to $\int_0^{+\infty} e^{-y} y^{-1/\alpha}\,dy = \Gamma(1 - 1/\alpha)$ as $\delta \to 1$.

If $\alpha = 1$, the integral $\int_0^{+\infty} e^{-y} y^{-1/\alpha}\,dy$ is infinite. Since $e^{-y}/y \sim 1/y$ as $y \to 0$, routine arguments show that

$$\int_{-\ln\delta}^{+\infty} \frac{e^{-y}}{y}\,dy \sim \int_{-\ln\delta}^{1} \frac{1}{y}\,dy = -\ln\ln\frac{1}{\delta},$$

and the result also follows from equation (S4.2). *Q.E.D.*

For completeness, we give a quick proof that the constants $c_1$ and $c_2$ in Lemma 1 are equal to $\frac{1}{\pi}$ and to 1, when private beliefs are uniformly distributed.

When all guesses are public, one has $u_n := \mathbf{P}_L(\tau > n) = \prod_{k=1}^{n}(1 - F_L(1 - \pi_k))$. With $F(p) = p$, one has $F_L(p) = p(2-p)$ and $F_H(p) = p^2$, hence $u_n = \prod_{k=1}^{n} \pi_k^2$ and the belief updating equation (3.6) reduces to $\frac{\pi_{n+1}}{1-\pi_{n+1}} = \frac{2-\pi_n}{1-\pi_n}$, from which it follows that $(\frac{1}{1-\pi_n})_n$ is an arithmetic sequence, and $\pi_n = 1 - \frac{1}{2n}$ for each $n \geq 1$.

Consequently,

$$u_n = \left(\prod_{k=1}^{n}\left(1-\frac{1}{2k}\right)\right)^2 = \left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2.$$

Using the Stirling formula, it follows that $u_n \sim \frac{1}{\pi n}$ as $n \to +\infty$.

When only the previous guess is observed, the probability $x_n := \mathbf{P}_L(a_n = h)$ of a wrong guess is given by $x_{n+1} - x_n = -2\int_0^{x_n} F(p)\,dp$, which reduces to a discrete-time logistic equation

$$x_{n+1} = x_n(1-x_n). \tag{S4.3}$$

Since $x_1 \in (0,1)$, it is obvious from (S4.3) that $(x_n)$ is decreasing and must converge to zero. An easy induction shows that $x_n < \frac{1}{n+1}$ for all $n \geq 2$. Set now $y_n := nx_n$, and observe that

$$y_{n+1} - y_n = x_n\big(1-(n+1)x_n\big) \geq 0. \tag{S4.4}$$

The sequence $(y_n)$ being nondecreasing with $y_n \leq 1$, it has a positive limit, which we denote by $l > 0$.

Equation (S4.4) also yields

$$y_{n+1} - y_n = \frac{y_n(1-y_n)}{n} - \frac{y_n^2}{n^2}.$$

Since the sequence $(y_n)$ converges, the series $\sum(y_{n+1} - y_n)$ converges as well, hence $l = 1$.[2] We have thus shown that $x_n \sim \frac{1}{n}$ as $n \to +\infty$.

The latter estimate implies that $\mathbf{E}_L[\tau] < +\infty$, and therefore, that the two efficiency criteria $\mathbf{E}_L[W] < +\infty$ and $\mathbf{E}_L[\tau] < +\infty$ are not equivalent when only the previous action is observed. One indeed has, for each $n$, $\mathbf{P}(\tilde{\theta} = H \mid a_n = h) = \mathbf{P}_H(a_n = h) = 1 - x_n$, which implies

$$\begin{aligned}\mathbf{P}_L(\tau > n+1 \mid \tau > n) &= \mathbf{P}_L(a_{n+1} = h \mid a_n = h)\\ &= 1 - F_L(x_n) = (1-x_n)^2.\end{aligned}$$

The sequence $(\mathbf{P}_L(\tau > n))_n$ satisfies

$$\frac{\mathbf{P}_L(\tau > n+1)}{\mathbf{P}_L(\tau > n)} = (1-x_n)^2 = 1 - \frac{2}{n} + o\left(\frac{1}{n}\right).$$

This implies that the series $\sum \mathbf{P}_L(\tau > n)$ is convergent, using the Raabe–Duhamel rule, and therefore, $\mathbf{E}_L[\tau] < +\infty$.

## S5. INEFFICIENCY OF RANDOM SAMPLING: PROOF OF THEOREM 5

The proof of Theorem 5 follows closely the proof of Theorem 3 and we refer to Section C for notations. In addition, we will denote by $\bar{x}_n := \frac{1}{n}\sum_{k=1}^{n} x_n$ the expected proportion of wrong choices among the first $n$ agents, and by $\alpha_n$ the random action observed by agent $n+1$. Thus, the social belief is here equal to $\pi_n = \mathbf{P}(\tilde{\theta} = H \mid \alpha_{n-1})$.

[2]Otherwise, $y_{n+1} - y_n$ would be equivalent to $l(1-l)/n$.

LEMMA S11: *For each* $n \geq 1$, *one has* $\bar{x}_{n+1} - \bar{x}_n = -\frac{2}{n+1}\int_0^{\bar{x}_n} F(q)\,dq$.

PROOF: Since agent $n+1$ samples uniformly among all previous agents, one has

$$\mathbf{P}_\theta(\alpha_n = a) = \frac{1}{n}\sum_{k=1}^{n} \mathbf{P}_\theta(a_k = a) \quad \text{for each } \theta \text{ and } a.$$

On the event $\alpha_n = a$, Bayes's rule leads to $\frac{\pi_{n+1}}{1-\pi_{n+1}} = \frac{\pi_n}{1-\pi_n} \times \frac{\mathbf{P}_H(\alpha_n=a)}{\mathbf{P}_L(\alpha_n=a)}$. Using $\mathbf{P}_\theta(a_{n+1} = h) = \mathbf{P}_\theta(q_{n+1} \geq 1 - \pi_{n+1})$ and the symmetry Assumption A1, elementary manipulations similar to those in the proof of Lemma 14 lead to

$$\begin{aligned} x_{n+1} &= \bar{x}_n\big(1 - F_L(\bar{x}_n)\big) + (1 - \bar{x}_n)\big(1 - F_L(1 - \bar{x}_n)\big) \\ &= \bar{x}_n - 2\int_0^{\bar{x}_n} F(q)\,dq. \end{aligned}$$

Since $\bar{x}_{n+1} = \frac{n}{n+1}\bar{x}_n + \frac{1}{n+1}x_{n+1}$, the result follows. *Q.E.D.*

LEMMA S12: *One has* $\sum_{n=1}^{+\infty} x_n < +\infty \Leftrightarrow \sum_{n=1}^{+\infty} \bar{x}_n < +\infty$.

PROOF: The argument that $x_n > 0$ applies without change, and yields $\bar{x}_n > 0$ for each $n$. The proof of Lemma 15 requires minor changes. Set $l := \lim \bar{x}_n$. Since $(x_n)$ is nonincreasing, one has $\lim x_n = l$ as well. As in the proof of Lemma 15, and if $q_{\min} > 0$, either $\bar{x}_n > q_{\min}$ for all $n$, and then $l \geq q_{\min}$, or $\bar{x}_{n_1} \leq q_{\min}$ for some $n_1$, in which case $\bar{x}_n = \bar{x}_{n_1}$ for all $n \geq n_1$, and thus $l = \bar{x}_{n_1} > 0$. In that case, both $\sum x_n$ and $\sum \bar{x}_n$ are divergent.

In the rest of the proof, we may thus assume that $F(q) > 0$ for each $q > 0$. We claim that $l = 0$. Otherwise, indeed, one would have $\bar{x}_{n+1} - \bar{x}_n \sim -\frac{1}{n} \times \int_0^l F(q)\,dq$ as $n \to +\infty$. Since the series $\sum \frac{1}{n}$ is divergent, this would imply $\lim \bar{x}_n = -\infty$, a contradiction. Hence $l = 0$, as claimed.

Using again Lemma S11, $|x_{n+1} - \bar{x}_n| \leq 2\bar{x}_n F(\bar{x}_n)$, hence $x_{n+1} \sim \bar{x}_n$ as $n \to +\infty$ since $\lim F(\bar{x}_n) = 0$. Hence, the convergence of the series $\sum x_n$ is equivalent to that of $\sum \bar{x}_n$. *Q.E.D.*

By Assumption A2 (and when possibly lowering $\alpha$), one has $F(q) \leq \frac{1}{2}(\alpha+1)q^\alpha$ in a neighborhood of zero. Using Lemma S11, there is $N_0 \in \mathbf{N}$ s.t.

$$\bar{x}_{n+1} \geq \bar{x}_n - \frac{1}{n+1}\bar{x}_n^{1+\alpha} \quad \text{for all } n \geq N_0. \tag{S5.1}$$

On the other hand, the map $y \mapsto y - y^{1+\alpha}$ is increasing over the interval $[0, \frac{1}{(\alpha+1)^{1/\alpha}}]$. Choose $N_1$ s.t. $\bar{x}_n \in [0, \frac{1}{(\alpha+1)^{1/\alpha}}]$ for each $n \geq N_1$, and set $N := \max(N_0, N_1)$.

Introduce now a sequence $(y_n)$ s.t. $y_N = x_N$ and $y_{n+1} - y_n = -\frac{1}{n+1}y_n^{1+\alpha}$ for each $n \geq N$. From the choice of $N$, it follows by induction that $\bar{x}_n \geq y_n$ for all $n \geq N$. It is thus sufficient to prove that the series $\sum y_n$ is divergent.

Obviously, the sequence $(y_n)$ is positive, decreasing, with $\lim y_n = 0$.[3] Hence

$$\frac{y_{n+1}}{y_n} = 1 - \frac{1}{n} y_n^\alpha = 1 + o\left(\frac{1}{n}\right).$$

It follows from the Raabe–Duhamel criterion that $\sum y_n$ is divergent.

## S6. ALTERNATIVE SETUP: PROOF OF THEOREM 5

Since $F(q) = q$ satisfies Assumption A1, $\mathbf{E}_\theta[\tau]$ is independent of $\theta$. We choose $\theta = L$ for concreteness.

Let $C_2$ be an upper bound for the sequence $(d_{k+1}/d_k)$. For $k \geq 1$, denote by $\Delta_k := d_1 + \cdots + d_k$ the cumulative size of the first $k$ generations, with $\Delta_0 = 1$. We will prove that $\sum_{k=1}^{+\infty} d_k \mathbf{P}_L(\tau > \Delta_k) = +\infty$. Since

$$\mathbf{E}_L[\tau] = \sum_{k=1}^{+\infty} \sum_{n=\Delta_{k-1}+1}^{\Delta_k} \mathbf{P}_L(\tau \geq n) \geq \sum_{k=1}^{+\infty} d_k \mathbf{P}_L(\tau > \Delta_k),$$

the result will follow.

Since $F(q) = q$, one has $F_H(q) = q^2$ and $F_L(q) = q(2-q)$ for each $q$ (see Section A), and thus $1 - F_L(1-\rho) = \rho^2$ for each $\rho$. For $k \geq 1$, we denote by

$$\rho_k := \mathbf{P}_L(\tilde{\theta} = H \mid a_1 = \cdots = a_{\Delta_{k-1}} = h)$$

the (common) social belief of agents from the $k$th generation, in the event $\tau > \Delta_{k-1}$ where all agents from all previous generations have chosen $h$.

Conditional on $\tau > \Delta_{k-1}$, agent $n$ from the $k$th generation chooses $a_n = h$ if and only if $q_n \geq 1 - \rho_k$, which occurs with probability $1 - F_L(1-\rho_k) = \rho_k^2$ in state $L$. Since there are $d_k$ such agents, $\mathbf{P}_L(\tau > \Delta_k \mid \tau > \Delta_{k-1}) = \rho_k^{2d_k}$ and thus,

$$\mathbf{P}_L(\tau > \Delta_k) = \prod_{i=1}^{k} \rho_i^{2d_i}. \tag{S6.1}$$

On the other hand, Bayes's rule leads to the belief updating formula

$$\frac{\rho_{k+1}}{1-\rho_{k+1}} = \frac{\rho_k}{1-\rho_k} \times \left(\frac{1 - F_H(1-\rho_k)}{1 - F_L(1-\rho_k)}\right)^{d_k} = \frac{\rho_k}{1-\rho_k} \times \left(\frac{2-\rho_k}{\rho_k}\right)^{d_k}. \tag{S6.2}$$

Setting $u_k := \frac{1}{2}\frac{\rho_k}{1-\rho_k}$, we have $\rho_k = 1 - \frac{1}{1+2u_k}$, and (S6.2) rewrites

$$u_{k+1} = u_k \left(1 + \frac{1}{u_k}\right)^{d_k}. \tag{S6.3}$$

We proceed with a series of claims.

[3] If $(y_n)$ instead had a positive limit $l$, we would have $y_{n+1} - y_n \leq -\frac{l^\alpha}{n}$ for each $n$, which by summation would imply $\lim y_n = -\infty$.

CLAIM S1: *One has* $u_{k+1} \geq \Delta_k + 1$ *for all* $k$.

PROOF: The inequality $(1+x)^\alpha \geq 1 + \alpha x$ (valid for $\alpha > 1$, $x > 0$) yields $u_{n+1} \geq u_n + d_n$. The result then follows by induction. *Q.E.D.*

CLAIM S2: *The series* $\sum \frac{d_k}{(u_k)^2}$ *is convergent*.

PROOF: Thanks to Claim S1, since $u_1 = \frac{1}{2}$ and since $\Delta_k = \Delta_{k-1} + d_k \leq \Delta_{k-1}(1 + C_2)$, one has

$$\sum_{k=1}^{\infty} \frac{d_k}{(u_k)^2} \leq 4d_1 + \sum_{k=2}^{\infty} \frac{d_k}{(\Delta_{k-1})^2} \leq 4d_1 + (1 + C_2)^2 \sum_{k=1}^{+\infty} \frac{d_k}{(\Delta_k)^2}.$$

Observe finally that the series $\sum \frac{d_k}{(\Delta_k)^2}$ is convergent, since

$$\sum_{k=2}^{+\infty} \frac{d_k}{(\Delta_k)^2} = \sum_{k=2}^{+\infty} \frac{\Delta_k - \Delta_{k-1}}{(\Delta_k)^2} \leq \sum_{k=2}^{+\infty} \int_{\Delta_{k-1}}^{\Delta_k} \frac{1}{x^2}\, dx = \int_{d_1}^{+\infty} \frac{1}{x^2}\, dx.$$

*Q.E.D.*

CLAIM S3: *The series* $\sum \frac{d_k}{u_k}$ *is divergent*.

PROOF: Observe that $\frac{u_{k+1}}{u_k} = (1 + \frac{1}{u_k})^{d_k} \leq e^{d_k/u_k}$ (since $\ln(1+x) \leq x$ for $x > 0$). Taking products over $k$, this implies

$$\frac{1}{2} u_{n+1} \leq \exp\left(\sum_{k=1}^{n} \frac{d_k}{u_k}\right).$$

The result follows, since $\lim u_n = \infty$ by Claim S1. *Q.E.D.*

CLAIM S4: *The series* $\sum d_n e^{-\sum_{k=1}^{n} d_k/u_k}$ *is divergent*.

PROOF: Since $\ln(1+x) \geq x - x^2$ for $x \geq 0$, one has $\frac{u_{k+1}}{u_k} = (1 + \frac{1}{u_k})^{d_k} \geq \exp(\frac{d_k}{u_k} - \frac{d_k}{u_k^2})$, or equivalently,

$$\exp\left(-\frac{d_k}{u_k}\right) \geq \frac{u_k}{u_{k+1}} \times \exp\left(-\frac{d_k}{u_k^2}\right).$$

Taking products over $k$, and multiplying by $d_n$, one obtains

$$d_n \exp\left\{-\sum_{k=1}^{n} \frac{d_k}{u_k}\right\} \geq \frac{d_n}{2u_n} \exp\left\{-\sum_{k=1}^{+\infty} \frac{d_k}{(u_k)^2}\right\}. \tag{S6.4}$$

The result follows from Claims S2 and S3. *Q.E.D.*

We now conclude. Since $\lim \rho_k = 1$ and $\ln(1+x) \geq x - x^2$ for $x > -\frac{1}{2}$, one has

$$\ln \rho_k \geq \rho_k - 1 - (\rho_k - 1)^2 = -\frac{1}{1 + 2u_k} - \left(\frac{1}{1 + 2u_k}\right)^2 \geq -\frac{1}{2u_k} - \frac{1}{(2u_k)^2} \tag{S6.5}$$

for all $i$ large enough. Plugging into (S6.1), one gets

$$\mathbf{P}_L(\tau > \Delta_k) = \prod_{i=1}^{k} \rho^i 2d_i = \exp\left(\sum_{i=1}^{k} 2d_i \ln \rho_i\right) \geq \exp\left\{-\sum_{i=1}^{k-1} \frac{d_i}{u_i}\right\} \times \exp\left\{-\frac{1}{2}\sum_{i=1}^{k-1} \frac{d_i}{(u_i)^2}\right\}$$

for some $C_3 > 0$ and all $k \geq 1$.[4] From Claims S2 and S4, it follows that the series $\sum d_k \mathbf{P}_L(\tau > \Delta_k)$ is divergent, as desired.

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[4]The additional $C_3$ accounts for the first values of $i$ where (S6.5) need not hold.