SUPPLEMENT TO "AMBIGUITY, LEARNING, AND ASSET RETURNS": TECHNICAL APPENDIX

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THIS TECHNICAL APPENDIX describes the numerical method used to solve the model in the main paper. We follow the same notation closely and we refer the reader to the main paper for detailed interpretations.

S1. NUMERICAL METHOD FOR MODELS WITH LEARNING

Our numerical method consists of two steps.

Step 1—Solve for the value function. Conjecture $V_t(C) = G(\mu_t)C_t$, where G is a function to be determined. By definition,

$$(S1) \qquad G(\mu_t) = \left[(1-\beta) + \beta \left\{ \mathbb{E}_{\mu_t} \left(\mathbb{E}_{z_{t+1},t} \left[G(\mu_{t+1})^{1-\gamma} \right. \right. \right. \right.$$
$$\left. \left. \left. \left. \times \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \right] \right)^{(1-\eta)/(1-\gamma)} \right\}^{(1-\rho)/(1-\eta)} \right]^{1/(1-\rho)}.$$

We use the projection method (Judd (1998)) to solve for G. The detailed implementation is similar to that described in Ju and Miao (2007). The case with $\rho = 1$ (log exponential) is the limiting case.

To ensure there is a bounded solution $G(\mu_t)$, we need a transversality condition. We find a finite upper bound by supposing consumption growth is in the high growth state forever. Then there is no ambiguity or learning about the growth state. In this case, we solve (S1) to obtain

$$\bar{G}^{1-\rho} = \frac{1-\beta}{1 - \beta e^{(\kappa_1 + 0.5(1-\gamma)\sigma^2)(1-\rho)}}.$$

We need $\bar{G} > 0$. We thus assume

$$(S2) \qquad \beta e^{(\kappa_1 + 0.5(1-\gamma)\sigma^2)(1-\rho)} < 1.$$

Since $G(\mu_t) < \bar{G}$, the above equation gives a sufficient condition for a finite $G(\mu_t)$.

Step 2—Solve for the price–dividend ratio. Conjecture

$$\frac{P_{e,t}}{D_t} = \varphi(\mu_t),$$

where φ is a function to be determined. We substitute the return equation

$$R_{e,t+1} = \frac{P_{e,t+1} + D_{t+1}}{P_{e,t}} = \frac{\varphi(\mu_{t+1}) + 1}{\varphi(\mu_t)} \frac{D_{t+1}}{D_t}$$

into the Euler equation to derive

(S3) $$1 = \mathbb{E}_t \left\{ \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{V_{t+1}}{\mathcal{R}_t(V_{t+1})} \right)^{\rho - \gamma} \right.$$

$$\left. \times \frac{(\mathbb{E}_{z_{t+1},t}[V_{t+1}^{1-\gamma}])^{-(\eta-\gamma)/(1-\gamma)}}{[\mathcal{R}_t(V_{t+1})]^{-(\eta-\gamma)}} \frac{\varphi(\mu_{t+1}) + 1}{\varphi(\mu_t)} \frac{D_{t+1}}{D_t} \right\}.$$

We use $V_t = G(\mu_t) C_t$ to derive

$$\left(\frac{V_{t+1}}{\mathcal{R}_t(V_{t+1})} \right)^{\rho - \gamma} \frac{(\mathbb{E}_{z_{t+1},t}[V_{t+1}^{1-\gamma}])^{-(\eta-\gamma)/(1-\gamma)}}{[\mathcal{R}_t(V_{t+1})]^{-(\eta-\gamma)}}$$

$$= \left(\frac{G(\mu_{t+1}) \dfrac{C_{t+1}}{C_t}}{\mathcal{R}_t \left(G(\mu_{t+1}) \dfrac{C_{t+1}}{C_t} \right)} \right)^{\rho - \gamma}$$

$$\times \frac{\left(\mathbb{E}_{z_{t+1},t} \left[\left(G(\mu_{t+1}) \dfrac{C_{t+1}}{C_t} \right)^{1-\gamma} \right] \right)^{-(\eta-\gamma)/(1-\gamma)}}{\left[\mathcal{R}_t \left(G(\mu_{t+1}) \dfrac{C_{t+1}}{C_t} \right) \right]^{-(\eta-\gamma)}}.$$

We substitute this equation into (S3) and then use the projection method to solve for φ after we obtain the approximate solution for G in Step 1. We can similarly solve for the price–consumption ratio.

S2. NUMERICAL METHOD FOR BENCHMARK MODEL I

In the paper, we also solve a benchmark model with Epstein–Zin preferences under full information. We write the utility function as

(S4) $$V_t(C) = \left[(1 - \beta) C_t^{1-\rho} + \beta \{ \mathcal{R}_t(V_{t+1}(C)) \}^{1-\rho} \right]^{1/(1-\rho)},$$

(S5) $$\mathcal{R}_t(V_{t+1}(C)) = \left(\mathbb{E}_t [V_{t+1}^{1-\gamma}(C)] \right)^{1/(1-\gamma)}.$$

Step 1. We conjecture the value function takes the form $V(z) = G(z) C_t$ for $z = 1, 2$. The function G satisfies

(S6) $$G(z) = \left[(1 - \beta) + \beta \left(\mathbb{E}_z \left[G(z')^{1-\gamma} \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \right] \right)^{(1-\rho)/(1-\gamma)} \right]^{1/(1-\rho)}.$$

That is,

$$(S7) \quad G(1)^{1-\rho} = (1-\beta) + \beta\big[\lambda_{11}G(1)^{1-\gamma}\exp((1-\gamma)\kappa_1 + 0.5(1-\gamma)^2\sigma^2)$$
$$+ (1-\lambda_{11})G(2)^{1-\gamma}$$
$$\times \exp((1-\gamma)\kappa_2 + 0.5(1-\gamma)^2\sigma^2)\big]^{(1-\rho)/(1-\gamma)},$$

$$(S8) \quad G(2)^{1-\rho} = (1-\beta)$$
$$+ \beta\big[(1-\lambda_{22})G(1)^{1-\gamma}\exp((1-\gamma)\kappa_1 + 0.5(1-\gamma)^2\sigma^2)$$
$$+ \lambda_{22}G(2)^{1-\gamma}\exp((1-\gamma)\kappa_2 + 0.5(1-\gamma)^2\sigma^2)\big]^{(1-\rho)/(1-\gamma)}.$$

We can solve this system of nonlinear equations for two unknowns $G(1)$ and $G(2)$. We also need a transversality condition (S2).

Step 2. Given the solution for $G(z)$ in Step 1, solve the price–dividend ratio. Conjecture

$$\frac{P_{e,t}}{D_t} = \varphi(z_t),$$

where φ is a function to be determined. We substitute the return equation

$$R_{e,t+1} = \frac{P_{e,t+1} + D_{t+1}}{P_{e,t}} = \frac{\varphi(z_{t+1}) + 1}{\varphi(z_t)}\frac{D_{t+1}}{D_t}$$

into the Euler equation to derive

$$(S9) \quad 1 = \mathbb{E}_t\bigg\{\beta\bigg(\frac{C_{t+1}}{C_t}\bigg)^{-\rho}\bigg(\frac{V_{t+1}}{\mathcal{R}_t(V_{t+1})}\bigg)^{\rho-\gamma}\frac{\varphi(z_{t+1}) + 1}{\varphi(z_t)}\frac{D_{t+1}}{D_t}\bigg\}.$$

The above equation gives a system of two linear equations for two unknowns $\varphi(1)$ and $\varphi(2)$. We check solutions such that $\varphi(1) > 0$ and $\varphi(2) > 0$.

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