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SUPPLEMENT TO "TWO NEW CONDITIONS SUPPORTING THE FIRST-ORDER APPROACH TO MULTISIGNAL PRINCIPAL–AGENT PROBLEMS"

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This note contains some supplementary results. First, Section S1 gives Jewitt's (1988) original (unpublished) proof of his one-signal result, based on his condition (2.10a), and Section S2 shows why it is hard to generalize this proof to the multisignal case. Section S3 shows how to check for concavity of the signal technology in the agent's effort level. Section S4 gives an alternative derivation of the results in Section 8, which avoids the vector calculus machinery from Section 7. Section S5 presents a candidate canonical vector flow for a signal technology, but argues that this canonical flow is of limited usefulness. Section S6 shows how to derive the state-space representation in Proposition 3, using the vector flow in Lemma 3. Finally, Section S7 gives the details for the generalization of Jewitt's Theorem 2, mentioned in Section 9.

S1. JEWITT'S FULL ONE-SIGNAL CONDITIONS

THIS AND THE FOLLOWING SECTION show why it is difficult to generalize Jewitt's (1988) key one-signal condition (2.10a) to the multisignal case. The basic framework, including Jewitt's key condition and his original unpublished proof of his one-signal result, are presented in this section. Section S2 then shows why it is difficult to generalize this proof and condition to the multisignal case. Thus, the main paper must use the stronger condition, that signals are concave in the agent's effort, in its multisignal generalization of Jewitt's conditions.

Note that Jewitt also dropped his condition (2.10a) in his treatment of the multisignal case. In his main multisignal result (his Theorem 3), he also replaced his (2.10a) with concavity of the signal technology, as do I (though Jewitt also assumed independent signals). In his other multisignal result (his Theorem 2) he replaced (2.10a) with the much stronger convexity of the distribution function condition (CDFC).

Finally, it should be noted that these sections do not show that it is *impossible* to generalize Jewitt's condition (2.10a) to the multisignal case, but only argues that the most obvious generalizations do not work. Thus, a definitive resolution of this issue, either way, remains an interesting topic for future research.

There is one principal and one agent, with the agent choosing an effort level $a \geq 0$. Suppose initially that there is a one-dimensional signal $\tilde{x}$, with density and cumulative distribution functions

$$f(x|a) \quad \text{and} \quad F(x|a), \tag{S1}$$

respectively. Assume as usual that the support of $f(x|a)$ in (S1) is independent of a and that the density is bounded between two positive constants on its support.

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The principal is risk neutral, while the agent has a von Neumann–Morgenstern utility function $u(s) - a$, with s the agent's income. Also, the principal pays the agent using payment schedule $s(x)$. Thus, the principal's expected payoff is

$$V(s(\cdot), a) = \int [x - s(x)] f(x|a)\, dx, \tag{S2}$$

and the agent's expected payoff is

$$U(s(\cdot), a) = \int u(s(x)) f(x|a)\, dx - a. \tag{S3}$$

The principal's problem is to choose a payment schedule, $s^*(\cdot)$, and target action, a^*, by the agent to maximize (S2) given two constraints:

(S4) for the agent, a^* maximizes $U(s^*(\cdot), a)$

and

(S5) the resulting expected payoff to the agent, $U(s^*(\cdot), a^*) \geq U_0$,

where U_0 is the agent's reservation utility. Here (S4) and (S5) are the usual incentive compatibility and participation constraints.

The first-order approach assumes that one can replace the constraint (S4) with a "relaxed" constraint

$$U_a(s^*(\cdot), a^*) = 0, \tag{S6}$$

where subscripts denote partial derivatives. To ensure that (S6) implies (S4), it is sufficient for the agent's utility, $U(s^*(\cdot), a)$, to be a concave function of her effort a, when $s^*(\cdot)$ solves the *relaxed* problem involving (S6). To get this, we first ensure that $u(s^*(x))$ is a concave function of x. For this, the following two conditions are sufficient:

CONDITION (a): $f_a(x|a)/f(x|a)$ is nondecreasing concave in x for each a.

CONDITION (b): Jewitt's (1988) function, $\omega(\cdot)$, is increasing concave, where

$$\omega(z) = u((u')^{-1}(1/z)). \tag{S7}$$

These conditions ensure that, if $s^*(\cdot)$ is a solution to the relaxed problem, then $u(s^*(x))$ is concave in x (see the main paper). Jewitt also makes one more assumption:

CONDITION (c)—Jewitt's Condition (2.10a):

(S8) $\int_{-\infty}^{b} F(x|a)\,dx$ is nonincreasing convex in a for every b.

Using integration by parts and letting $h(x;b) = \min(0, x - b)$, (S8) is equivalent to

(S9) $h^T(a;b) = \int h(x;b)f(x|a)\,dx$ is nondecreasing concave in a for every b.

Here $h(\cdot, b)$ is a kind of "test function," since (S9) implies that

(S10) if $\varphi(\cdot)$ is nondecreasing concave, $\varphi^T(a) = \int \varphi(x)f(x|a)\,dx$ is also.

Jewitt cited an unpublished working paper (Jewitt and Kanbur (1988)) which shows that (S8) is equivalent to (S10). The equivalence of (S9) to (S10) is shown in the following lemma:

LEMMA S1: *Conditions* (S9) *and* (S10) *are equivalent*.

PROOF: The following proof is essentially the unpublished proof in the Jewitt–Kanbur paper (personal communication from Ian Jewitt), and it has a structure similar to the proof of Lemma 1 in the main paper. The basic idea is that any nondecreasing concave function $\varphi(\cdot)$ can clearly be approximated, up to a constant, by a positive linear combination, over different values of b, of the functions $h(\cdot;b)$, say

$$\varphi(x) \approx \alpha_0 + \sum_{i=1}^{n} \alpha_i h(x;b_i), \quad \text{where} \quad \alpha_i > 0 \text{ for } i \geq 1. \tag{S11}$$

Thus, applying the transformation in (S10) gives

$$\varphi^T(a) \approx \alpha_0 + \sum_{i=1}^{n} \alpha_i h^T(a;b_i), \tag{S12}$$

with each term nondecreasing concave by (S9). Taking limits, $\varphi^T(x)$ is also nondecreasing concave. *Q.E.D.*

PROPOSITION S1: *Assume* (a) $f_a(x|a)/f(x|a)$ *is nondecreasing concave in* x, (b) *Jewitt's function* $\omega(\cdot)$ *is increasing concave*, *and* (c) *Jewitt's condition* (2.10a) *holds*. *Then any solution to the relaxed problem*, *maximizing* (S2) *subject to* (S5) *and* (S6), *is also a solution to the full problem of maximizing* (S2) *subject to* (S4) *and* (S5).

PROOF: Let $u(s^*(x)) = \varphi(x)$. Then (a) and (b) show that $\varphi(x)$ is nondecreasing concave, as in the main paper. Thus, since Jewitt's condition (2.10a) implies (S9), which implies (S10), it follows that $U(s^*(\cdot), a) = \varphi^T(a) - a$ is concave in a. *Q.E.D.*

Conditions (a) and (b) generalize easily to the multisignal case (see Jewitt (1988) or the main paper). However, condition (2.10a) is difficult to generalize, as shown next.

S2. THE DIFFICULTY IN GENERALIZING JEWITT'S CONDITION (2.10a) TO THE MULTI-SIGNAL CASE

This section shows why it is difficult to generalize Jewitt's condition (2.10a) to the multisignal case. For the multisignal version of the problem, replace the random signal $\tilde{x}$ by the random vector $\tilde{\mathbf{x}}$, with density $f(\mathbf{x}|a)$ and cumulative distribution function $F(\mathbf{x}|a)$. Again assume that the support of $f(\mathbf{x}|a)$ is independent of a and that the density is bounded between two positive constants on its support. As in Sinclair-Desgagné (1994), let the monetary payoff to the principal as a function of $\mathbf{x}$ be $\pi(\mathbf{x})$. Also, assume the principal pays the agent using payment function $s(\mathbf{x})$. Then, analogous to Section S1, the principal's expected payoff is

$$V(s(\cdot), a) = \int [\pi(\mathbf{x}) - s(\mathbf{x})] f(\mathbf{x}|a)\, d\mathbf{x}, \tag{S13}$$

the agent's expected payoff is

$$U(s(\cdot), a) = \int u(s(\mathbf{x})) f(\mathbf{x}|a)\, d\mathbf{x} - a, \tag{S14}$$

and the principal's problem is to choose a payment schedule, $s^*(\cdot)$, and target action, a^*, to maximize (S13), given the constraints (S4) and (S5), where now $U(s(\cdot), a)$ is defined by (S14), not (S3). Also, as before, the first-order approach assumes that one can replace the constraint (S4) with a relaxed constraint, (S6) (i.e., $U_a(s^*(\cdot), a^*) = 0$).

To generalize Jewitt's condition (2.10a), we must determine conditions such that, for any *multivariable* function $\varphi(\mathbf{x})$,

(S15) if $\varphi(\cdot)$ is nondecreasing concave, then $\varphi^T(a) = \int \varphi(\mathbf{x}) f(\mathbf{x}|a)\, d\mathbf{x}$ is also.

To see how difficult it is to generalize Jewitt's condition (2.10a), we show that the obvious multivariable analogues to the test functions $h(x; b) = \min(0, x - b)$, used in (S9), are insufficient to approximate, or "span," as positive linear combinations, all multivariable nondecreasing concave functions, so the argument in (S11) and (S12) does not go through. For related results, see Johansen

(1974) and Bronshtein (1978), who showed that, in the multivariable case, the set of such generalized spanning test functions h must, remarkably, be *dense* in the set of all concave functions.

Focus for definiteness on the two variable case. One natural generalization of the $h(x; b)$ function is then

$$h(x_1, x_2; \alpha_1, \alpha_2, \beta) = \min(0, \alpha_1 x_1 + \alpha_2 x_2 - \beta) \tag{S16}$$

with α_1, $\alpha_2 \geq 0$, so h is nondecreasing in (x_1, x_2), and $\alpha_1 + \alpha_2 = 1$, say, as a harmless normalization. One would hope that all nondecreasing concave functions could be approximated as constants plus positive linear combinations of these h functions. If this were possible, one could easily extend Lemma S1. However, this approach does not work, as shown by the following lemma.

LEMMA S2: *The nondecreasing concave function $k(x_1, x_2) = \min(x_1, x_2)$ cannot be approximated by a constant plus a positive linear combination of functions of the form $h(x_1, x_2; \alpha_1, \alpha_2, \beta)$ from* (S16), *with α_1, $\alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$.*

For the proof, see Appendix A.

Thus, if one wants to use the functions in (S16) to extend Lemma S1 to the multisignal case, then the method of proof from Lemma S1 does not work. Of course, it could be possible for Lemma S1 to extend in this way to the multivariable case, even if the method of proof does not. That is, one might hope that if the transformed functions, $h^T(a; \alpha_1, \alpha_2, \beta) = \int h(\mathbf{x}; \alpha_1, \alpha_2, \beta) f(\mathbf{x}|a)\, d\mathbf{x}$, are concave in a for all α_1, $\alpha_2 \geq 0$ and all β, then (S15) holds. The next lemma shows that this is not the case.

LEMMA S3: *Consider the nondecreasing concave function given by $\varphi(x_1, x_2) = \min(x_1 - 1, x_2)$. There is a distribution, $f(\mathbf{x}|a)$, such that $h^T(a; \alpha_1, \alpha_2, \beta)$ is nondecreasing concave in a for all α_1, α_2, and β, with α_1, $\alpha_2 \geq 0$, but $\varphi^T(a)$ is not concave in a. Also, this distribution can be made to satisfy the monotone likelihood ratio* (*MLR*) *property, $f_a(\mathbf{x}|a)/f(\mathbf{x}|a)$ nondecreasing in* $\mathbf{x}$.

For the proof, see Appendix B.

Thus, the most natural generalization of (S9) does not work. Of course, concave functions can always be approximated by *some* type of piecewise linear concave function, since the graph of a concave function forms the lower envelope of its tangent planes. Thus, we could generalize Lemma S1 by replacing the $h(x; b)$ functions used in (S9) by arbitrary piecewise linear concave functions. However, this class of functions is *too* flexible, since it would be just as hard to check (S15) for all piecewise linear concave functions as to check it for *all* concave functions.

A third possibility might be to replace the $h(\cdot; b)$ function in (S9) with some sort of generalization of the $h(\cdot; b)$ function, which is not too flexible, but which is more flexible than the $h(\cdot, \cdot; \alpha_1, \alpha_2, \beta)$ functions in (S16).

The next most flexible generalization might be a nondecreasing flat-top pyramid function. Define such a function to have a graph with a flat top (just as the graph of $h(x; b)$ is flat for $x \geq b$). Then require the faces surrounding this flat top to be separated by edges which do not intersect. This is a reasonable restriction, since otherwise we would come close to allowing all piecewise linear concave functions, which would be too broad a class to check, as argued above. As an example of such a nondecreasing flat-top pyramid function, consider

$$\text{(S17)} \qquad \min(0, 2x_1, 2x_2, x_1 + x_2 - 8).$$

The flat top in (S17) is bounded by the lines $x_1 = 0$, $x_1 + x_2 = 8$, and $x_2 = 0$. The surrounding faces are separated by the edges $x_2 = x_1 + 8$ and $x_2 = x_1 - 8$, which do not intersect.

Of course, checking (S15) for all such pyramid functions would also be impractical, even if this gave sufficient conditions for (S15) in general. However, it probably does not give such sufficient conditions. For example, positive linear combinations of such functions are not enough to approximate the nondecreasing concave function

$$\text{(S18)} \qquad G(x_1, x_2) = \min(0, x_1, x_2, x_1 + x_2 + 1).$$

LEMMA S4: *$G(x_1, x_2)$ in* (S18) *is nondecreasing concave, but it cannot be approximated by a positive linear combination of nondecreasing flat-top pyramid functions.*

See Appendix C for the proof.

Thus, the argument in (S11) and (S12) again does not go through. Also, the proof of Lemma S4 could be adapted to the case of component pyramids *without* flat tops.

It would be interesting to construct a counterexample, analogous to the one in Lemma S3, to show definitively that even if (S15) holds for nondecreasing flat-top pyramids, it does not necessarily hold for arbitrary nondecreasing concave functions. However, the example in Appendix B is already quite complicated, and this additional counterexample would presumably be even more complicated. In any case, it seems to be extremely difficult to generalize Lemma S1 to the multisignal case.

Thus, we have shown that it is difficult to generalize Jewitt's condition (2.10a) to the multisignal case. In the main paper, therefore, I simply assume that $\mathbf{x}(a, \vartheta)$ is concave in a. Note that Jewitt (1988) also made this simple concavity assumption for his main multisignal result, his Theorem 3. It would be interesting to see whether there are any classes of functions, perhaps more flexible than those considered here, which could be used to obtain a multisignal generalization of Jewitt's condition (2.10a).

S3. CHECKING CONCAVITY IN a OF THE COORDINATES OF THE STATE-SPACE REPRESENTATION $\mathbf{x}(a, \vartheta)$

The state-space representation, $\mathbf{x}(a, \vartheta)$, in Proposition 3 of the main paper can be used to impose conditions for concavity of the $\mathbf{x}(a, \vartheta)$ function. This is a simple mechanical application of the following rather technical result. In the following proposition, let $\mathbf{F}(\mathbf{x}, a) = (F^1(\mathbf{x}, a), F^2(\mathbf{x}, a))'$ be a sufficiently smooth 2×1 vector function, with primes denoting transposes and with $\mathbf{x} = (x_1, x_2)'$. Define $x_1(a, \theta_1, \theta_2)$ and $x_2(a, \theta_1, \theta_2)$ implicitly by $\mathbf{F}(\mathbf{x}, a) = (\theta_1, \theta_2)'$. Let $\mathbf{v}$ be the 3×1 vector $(x_1, x_2, a)'$, let $\partial\mathbf{F}/\partial\mathbf{v}$ be the 2×3 Jacobian matrix of partial derivatives of $\mathbf{F}(\mathbf{x}, a) = \mathbf{F}(\mathbf{v})$ with respect to $\mathbf{v}$, and let $\mathbf{H}^i$ be the 3×3 Hessian of $F^i(\mathbf{x}, a) = F^i(\mathbf{v})$ with respect to $\mathbf{v}$.

PROPOSITION S2: *Suppose* $F^1_{x_1}F^2_{x_2} - F^2_{x_1}F^1_{x_2} > 0$. *Then for each* $i, j = 1, 2, i \neq j$, *the function* $x_i(a, \theta_1, \theta_2)$ *is concave in* a *if and only if*

$$\text{(S19)} \qquad \mathbf{z}'[\mathbf{H}^j F^i_{x_j} - \mathbf{H}^i F^j_{x_j}]\mathbf{z} \leq 0 \text{ for all } 3 \times 1 \text{ vectors } \mathbf{z} \text{ satisfying } (\partial\mathbf{F}/\partial\mathbf{v})\mathbf{z} = \mathbf{0}.$$

PROOF: Differentiating $\mathbf{F}(\mathbf{v}) = (\theta_1, \theta_2)'$ implicitly, for fixed (θ_1, θ_2), shows that

$$\text{(S20)} \qquad d\mathbf{v}/da = (dx_1/da, dx_2/da, 1)' \text{ is determined by } (\partial\mathbf{F}/\partial\mathbf{v})(d\mathbf{v}/da) = \mathbf{0},$$

where $d\mathbf{v}/da$ is determined *uniquely* since $F^1_{x_1}F^2_{x_2} - F^2_{x_1}F^1_{x_2} \neq 0$. Implicitly differentiating both coordinates of $(\partial\mathbf{F}/\partial\mathbf{v})(d\mathbf{v}/da) = \mathbf{0}$ from (S20) gives

$$(d\mathbf{v}/da)'\mathbf{H}^1(d\mathbf{v}/da) + (\partial F^1/\partial\mathbf{v})(d^2\mathbf{v}/da^2) = 0,$$
$$(d\mathbf{v}/da)'\mathbf{H}^2(d\mathbf{v}/da) + (\partial F^2/\partial\mathbf{v})(d^2\mathbf{v}/da^2) = 0.$$

Solving these two equations for $d^2\mathbf{v}/da^2 = (d^2x_1/da^2, d^2x_2/da^2, 0)'$ gives

$$\text{(S21)} \qquad d^2x_i/da^2 = (d\mathbf{v}/da)'\big[\mathbf{H}^j F^i_{x_j} - \mathbf{H}^i F^j_{x_j}\big](d\mathbf{v}/da)/\big[F^1_{x_1}F^2_{x_2} - F^2_{x_1}F^1_{x_2}\big],$$

where $i, j = 1, 2, i \neq j$. The result follows from (S20) and (S21). *Q.E.D.*

To apply Proposition S2 to our case, replace $F^1(x_1, x_2, a)$ and $F^2(x_1, x_2, a)$ with $F^1(x_1|a)$ and $F^2(x_2|x_1, a)$. Then $F^1_{x_1}F^2_{x_2} - F^2_{x_1}F^1_{x_2} > 0$ reduces to $f^1(x_1|a) \times f^2(x_2|x_1, a) > 0$, with the f^i's the obvious densities. One can therefore use (S19) to check concavity in a of $\mathbf{x}(a, \vartheta)$, from (11) in the main paper, by using the analogue to bordered Hessians, for example (see Debreu (1952)).

This may, however, be computationally messy. It might therefore be easier to use a shortcut to check concavity of $\mathbf{x}(a, \vartheta)$. For example, one might begin with a state-space representation or solve (11) explicitly.

Thus, to find conditions for the concavity of $\mathbf{x}(a, \vartheta)$ in a, consider for example $x_2(a, \theta_1, \theta_2)$ from Proposition 3. For simplicity represent this as

$$\begin{aligned} x_2(a, \theta_1, \theta_2) &= (F^2)^{-1}\big(\theta_2 | (F^1)^{-1}(\theta_1|a), a\big) \\ &= g(x_1(a, \theta_1), a) = g(h(a), a), \end{aligned}$$

where

$$g(x_1, a) = (F^2)^{-1}(\theta_2|x_1, a) \quad \text{and} \quad h(a) = (F^1)^{-1}(\theta_1|a) = x_1(a, \theta_1).$$

Here the θ_i's have been suppressed in g and h for brevity. Concavity now requires

$$\begin{aligned} \partial^2 x_2/\partial a^2 &= g_1(h(a), a)h''(a) + g_{11}(h(a), a)h'(a)^2 \\ &\quad + g_{22}(h(a), a) + 2g_{12}(h(a), a)h'(a) \le 0. \end{aligned}$$

Suppose $\tilde{x}_1$ and $\tilde{x}_2$ are positively related, so an increase in $\tilde{x}_1$ increases $\tilde{x}_2$ in the sense of first-order stochastic dominance. Then $g_1(h(a), a)$ will be nonnegative. Thus if h is concave (so x_1 depends concavely on a), then the first term will be nonpositive. Next, if g is concave in each of its arguments (so x_2 depends concavely on x_1 and a), then the next two terms will be nonpositive. This leaves the last term. If the marginal distribution of $\tilde{x}_1$ is increasing in a in the sense of first-order stochastic dominance, then $h'(a)$ is nonnegative. The key issue therefore becomes whether $g_{12}(h(a), a)$ is nonpositive. This requires roughly that the correlation between $\tilde{x}_1$ and $\tilde{x}_2$ be nonincreasing in a.

A similar condition is important in the concave increasing-set probability (CISP) approach (see Proposition 7, Equation (29), in the main paper). Note also that if g is concave *jointly* in x_1 and a, then its Hessian is negative semidefinite, so the *sum* of the last three terms is again nonpositive.

S4. ALTERNATIVE DERIVATION OF THE LOCAL CONDITIONS IN SECTION 8

This section presents alternative derivations of the results in Section 8 of the main paper, which avoid the vector calculus machinery in Section 7 of that paper. However, I believe that these alternative derivations are less illuminating than the derivations in the paper. Proposition S3 treats the nondecreasing increasing-set probability (NISP) condition and is identical to Proposition 6 in the paper, while Proposition S4 handles CISP and is identical to Proposition 7. Proposition S5 below then presents conditions which are more general, but also more complicated, than those in Propositions S3 and S4, and is identical to Lemma 3 in the paper. Throughout we focus on two-dimensional signals, $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y})$, on the unit square $\mathbf{S} = [0, 1] \times [0, 1]$. The results generalize in a straightforward manner to n signals.

Let $g(x|a)$ and $G(x|a)$ be the marginal density and cumulative distribution functions of $\tilde{x}$, and let $h(y|x, a)$ and $H(y|x, a)$ be the conditional density and cumulative distribution functions of $\tilde{y}$ given $\tilde{x} = x$. Thus the joint density of $(\tilde{x}, \tilde{y})$ is $f(x, y|a) = g(x|a)h(y|x, a)$.

Let **E** be an increasing set and let $y = b(x)$, for $x \in [0, 1]$, be the downward sloping curve that describes the southwest boundary of $\mathbf{E} \cap \mathbf{S}$, so $b'(x) \leq 0$. If this southwest boundary hits the upper boundary of **S**, then let $b(x) = 1$ on the relevant interval $[0, x_a]$; if it hits the lower boundary, let $b(x) = 0$ on the relevant interval $[x_b, 1]$. Assume that $b(\cdot)$ is continuous and piecewise differentiable (it can always be approximated by such a function). We start with the following lemma.

LEMMA S5: *Let* $P = \text{Prob}(\tilde{\mathbf{x}} \in \mathbf{E})$ *be the probability that the signal* $\tilde{\mathbf{x}}$ *is in* **E**. *Then*

$$dP/da = \int_0^1 \big[G_a(x|a)H_x(b(x)|x, a) - g(x|a)H_a(b(x)|x, a)\big]\,dx + \int_0^1 G_a(x|a)h(b(x)|x, a)b'(x)\,dx = I_A + I_B. \tag{S22}$$

PROOF: First,

$$P = \int_{x=0}^1 \int_{y=b(x)}^1 g(x|a)h(y|x, a)\,dy\,dx = \int_0^1 g(x|a)[1 - H(b(x)|x, a)]\,dx.$$

Thus, assuming one can differentiate under the integral sign,

$$dP/da = \int_0^1 g_a(x|a)[1 - H(b(x)|x, a)]\,dx - \int_0^1 g(x|a)H_a(b(x)|x, a)\,dx = I_1 - I_2,$$

where I_1 and I_2 are the obvious integrals. Next, integrate I_1 by parts, yielding

$$I_1 = G_a(x|a)[1 - H(b(x)|x, a)]\big|_{x=0}^1 + \int_0^1 G_a(x|a)H_x(b(x)|x, a)\,dx + \int_0^1 G_a(x|a)h(b(x)|x, a)b'(x)\,dx = 0 + I_{1a} + I_{1b},$$

where the first term is zero since $G_a(0|a) = G_a(1|a) = 0$. Here $I_A = I_{1a} - I_2$ and $I_B = I_{1b}$. Thus, $dP/da = I_A + I_B$, as in (S22). *Q.E.D.*

Note that the integration by parts on I_1 is useful since it is more natural to impose conditions on $G_a(x|a)$ and $H_x(y|x,a)$ than on $g_a(x|a)$. This is illustrated in the following proposition.

PROPOSITION S3: *Suppose $G(x|a)$ is nonincreasing in a, and $H(y|x,a)$ is nonincreasing in x and a. Then NISP holds.*

PROOF: First, since $G_a(x|a) \leq 0$, $H_x(y|x,a) \leq 0$, and $H_a(y|x,a) \leq 0$, the first integral in (S22), $I_A \geq 0$. Next, since $G_a(x|a) \leq 0$ and $b'(x) \leq 0$, the second integral $I_B \geq 0$ as well. Thus, $dP/da \geq 0$, so NISP holds. *Q.E.D.*

Note that Proposition S3 is Proposition 6 of the main paper, where its meaning is discussed in greater detail.

PROPOSITION S4: *Suppose $G_a(x|a)$ and $H_a(y|x,a)$ are negative (this follows from strict versions of the corresponding MLR properties for $\tilde{x}$ and $\tilde{y}$). Assume also that $g(x|a)$ and $h(y|x,a)$ are strictly positive for $x, y \in [0,1]$. Finally assume $H_x(y|x,a) < 0$ (so $\tilde{x}$ and $\tilde{y}$ are positively related). Then the conditions*

(S23) $$h_a(y|x,a)/h(y|x,a) \leq -G_{aa}(x|a)/G_a(x|a),$$

(S24) $$g_a(x|a)/g(x|a) \leq -H_{aa}(y|x,a)/H_a(y|x,a),$$

(S25) $$H_{ax}(y|x,a)/H_x(y|x,a) \leq -G_{aa}(x|a)/G_a(x|a)$$

are sufficient to ensure CISP.

PROOF: First, differentiate the integrand of I_B in (S22) with respect to a, yielding $[G_{aa}(x|a)h(y|x,a) + G_a(x|a)h_a(y|x,a)]b'(x)$. Now, (S23) plus $G_a(x|a) < 0$ and $b'(x) \leq 0$ ensure that this is less than or equal to zero, so I_B is nonincreasing in a. Similarly, (S25) plus $H_x(y|x,a) < 0$ and $G_a(x|a) < 0$ ensure that the first term in the integrand of I_A is nonincreasing in a. Finally, (S24) and $H_a(y|x,a) < 0$ ensure that the second term in I_A, after the minus sign, is nondecreasing in a. Thus, I_A is nonincreasing in a, so dP/da is nonincreasing in a, and CISP holds. *Q.E.D.*

Again, Proposition S4 is Proposition 7 of the main paper, where its meaning is discussed in greater detail.

Next, (S23) can be weakened if we are willing to strengthen (S24) and visa versa, as explained in the main paper. We show how to do this in Proposition S5 below, which replicates Lemma 3. This proposition will build on the following lemma.

LEMMA S6: *Let $\phi(x,y,a)$ and $\psi(x,y,a)$ satisfy*

(S26) $$\phi_x(x,y,a) + \psi_y(x,y,a) = 0$$

and

(S27) $$\phi(0, y, a) = \phi(1, y, a) = \psi(x, 0, a) = \psi(x, 1, a) = 0.$$

Then

(S28) $$\int_0^1 \psi(x, b(x), a)\, dx - \int_0^1 \phi(x, b(x), a) b'(x)\, dx = I_\psi - I_\phi = 0.$$

PROOF: First, since $\psi(x, 1, a) = 0$,

(S29) $$\begin{aligned} I_\psi &= -\int_0^1 \big[\psi(x, 1, a) - \psi(x, b(x), a)\big]\, dx \\ &= -\int_{x=0}^1 \int_{y=b(x)}^1 \psi_y(x, y, a)\, dy\, dx, \end{aligned}$$

where the second step uses the fundamental theorem of calculus.

Next, suppose $b(x)$ is strictly decreasing, and so invertible, and represent the southwest boundary of **E** by $x = c(y)$. This inverts $b(x)$ on the downward sloping part of the curve. Also let $c(y)$ equal 0 for $y \geq b(0)$ and equal 1 for $y \leq b(1)$ if the curve hits the left or right boundary of the square **S**. Then, letting $b(x) = y$, so $x = c(y)$, gives

(S30) $$\begin{aligned} I_\phi &= \int_{b(0)}^{b(1)} \phi(c(y), y, a)\, dy = -\int_{b(1)}^{b(0)} \phi(c(y), y, a)\, dy \\ &= -\int_0^1 \phi(c(y), y, a)\, dy. \end{aligned}$$

Here the last step uses $b(0) > b(1)$; it also uses, for $y \leq b(1)$, that $\phi(c(y), y, a) = \phi(1, y, a) = 0$, and similarly for $y \geq b(0)$. Next, since $\phi(1, y, a) = 0$, (S30) equals

(S31) $$\begin{aligned} I_\phi &= \int_0^1 \big[\phi(1, y, a) - \phi(c(y), y, a)\big]\, dy \\ &= \int_{y=0}^1 \int_{x=c(y)}^1 \phi_x(x, y, a)\, dx\, dy. \end{aligned}$$

Also, an approximation argument can be used if $b(x)$ is not *strictly* decreasing. Now, both (S29) and (S31) are simply integrals over $\mathbf{E} \cap \mathbf{S}$, so using Fubini's theorem,

$$I_\psi - I_\phi = -\int\int_{\mathbf{E}\cap\mathbf{S}} [\phi_x(x, y, a) + \psi_y(x, y, a)]\, dx\, dy = 0,$$

by (S26). *Q.E.D.*

PROPOSITION S5: *Suppose there are functions* $\phi(x, y, a)$ *and* $\psi(x, y, a)$ *that satisfy* (S26) *and* (S27), *and such that*

$$G_a(x|a)h(y|x, a) + \phi(x, y, a) \tag{S32}$$

and

$$g(x|a)H_a(y|x, a) - G_a(x|a)H_x(y|x, a) + \psi(x, y, a) \tag{S33}$$

are less than or equal to zero for all $(x, y) \in \mathbf{S}$. *Then* $f(x, y|a)$ *satisfies NISP. If there are* $\phi(x, y, a)$ *and* $\psi(x, y, a)$ *that satisfy* (S26) *and* (S27), *and such that* (S32) *and* (S33) *are nondecreasing in* a, *then* $f(x, y|a)$ *satisfies CISP.*

PROOF: Subtract (S28) from (S22) and use $b'(x) \leq 0$. *Q.E.D.*

Proposition S5, which is Lemma 3 in the main paper, now allows us to choose the ϕ and ψ functions to weaken (S23) if we are willing to strengthen (S24) and visa versa. For example it can be used to show that the generalized CDFC (GCDFC) implies CISP, as in the main paper.

S5. IS THERE A CANONICAL VECTOR FLOW FOR SIGNAL TECHNOLOGIES $f(\mathbf{x}|a)$ IN PRINCIPAL–AGENT MODELS?

Sections 7 and 8 of the main paper show that it is possible to represent any principal–agent technology $f(\mathbf{x}|a)$ by a vector flow $\mathbf{v}(\mathbf{x}, a)$, such that $f_a(\mathbf{x}|a) = -\operatorname{div} \mathbf{v}(\mathbf{x}, a)$. This lets us confirm the *global* NISP and CISP conditions in that paper by checking certain easy-to-verify *local* conditions on the vector field $\mathbf{v}(\mathbf{x}, a)$.

Thus assume, for specificity, that the signal $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y})$ is two dimensional and that $\tilde{\mathbf{x}}$ is distributed in the unit square, $\mathbf{S} = [0, 1] \times [0, 1]$, as in the paper. Say that a set $\mathbf{E}$ is an *increasing set* if $\mathbf{x} \in \mathbf{E}$ and $\mathbf{y} \geq \mathbf{x}$ implies $\mathbf{y} \in \mathbf{E}$. Then NISP is the condition that $\text{Prob}(\tilde{\mathbf{x}} \in \mathbf{E}|a)$ is nondecreasing in a, and CISP is the condition that $\text{Prob}(\tilde{\mathbf{x}} \in \mathbf{E}|a)$ is concave in a, for all increasing sets $\mathbf{E}$.

Next assume that, as a increases, the density $f(\mathbf{x}|a)$ follows the density flux $\mathbf{v}(\mathbf{x}, a) = (u(x, y, a), v(x, y, a))$, with $u(0, y, a) = u(1, y, a) = v(x, 0, a) = v(x, 1, a) = 0$, so no density flows out of $\mathbf{S}$. Thus $f_a(\mathbf{x}|a) = -\operatorname{div} \mathbf{v}(\mathbf{x}, a)$, as in the main paper. The divergence theorem then shows that

$$\frac{d}{da} \text{Prob}(\tilde{\mathbf{x}} \in \mathbf{E}|a) = -\int_0^1 \mathbf{v}(\mathbf{x}(t), a) \cdot \mathbf{n}(t)\, dt, \tag{S34}$$

where $\mathbf{x}(t) = (x(t), y(t))$ traces the boundary of $\mathbf{E} \cap \mathbf{S}$ counterclockwise, $\mathbf{n}(t) = (y'(t), -x'(t))$ is the outward-pointing normal (perpendicular) to the boundary of $\mathbf{E} \cap \mathbf{S}$, and $\mathbf{v}(\mathbf{x}(t), a) \cdot \mathbf{n}(t)$ is the dot product of $\mathbf{v}(\mathbf{x}(t), a)$ and $\mathbf{n}(t)$, that is,

$$\mathbf{v}(\mathbf{x}(t), a) \cdot \mathbf{n}(t) = u(\mathbf{x}(t), a)y'(t) - v(\mathbf{x}(t), a)x'(t);$$

see the main paper for details. Lemma 2 in that paper uses (S34) to show that, if the coordinates of $\mathbf{v}(\mathbf{x}, a)$ are everywhere nonnegative, then NISP holds, while if the coordinates of $\mathbf{v}(\mathbf{x}, a)$ are everywhere nonincreasing in a, then CISP holds.

It is reasonable to conjecture that Lemma 2 in the paper has a converse. That is, if $f(\mathbf{x}|a)$ satisfies NISP, then there should exist a vector field $\mathbf{v}(\mathbf{x}, a)$, as above, with coordinates everywhere nonnegative, while if $f(\mathbf{x}|a)$ satisfies CISP, then there should be a vector field $\mathbf{v}(\mathbf{x}, a)$ as above with coordinates nonincreasing in a. However, while these conjectures seem intuitively obvious, I have not been able to prove them.

In addition, the vector field, $\mathbf{v}(\mathbf{x}, a)$, constructed in Section 8 of the main paper, is rather ad hoc. This section therefore considers a possible *canonical* vector flow corresponding to $f(\mathbf{x}|a)$, and argues that this canonical flow is not obviously superior to the ad hoc flow constructed in the main paper. Thus, at this point there seems to be no obvious alternative superior to the ad hoc flow in the main paper. On the other hand, if the conditions in the main paper turn out to be inadequate for some future applications, then these future applications may themselves suggest new vector flows.

In any case, the difficulty with representing $f(\mathbf{x}|a)$ by a vector flow is precisely that there are too many vector flows capable of representing any technology $f(\mathbf{x}|a)$. This suggests representing $f(\mathbf{x}|a)$ using a vector flow which is in some sense *minimal*. Specifically, we look for a vector flow $\mathbf{v}(\mathbf{x}, a) = (u(\mathbf{x}, a), v(\mathbf{x}, a))$ which solves

$$\min_{\mathbf{v}(\cdot,\cdot)} \int_0^1 \int_0^1 \|\mathbf{v}(\mathbf{x}, a)\|^2 \, d\mathbf{x} \quad \text{subject to} \quad f_a(\mathbf{x}|a) = -\operatorname{div} \mathbf{v}(\mathbf{x}, a)$$

and

$$u(0, y, a) = u(1, y, a) = v(x, 0, a) = v(x, 1, a) = 0, \tag{S35}$$

where $\|\mathbf{v}\|^2 = u^2 + v^2$ for $\mathbf{v} = (u, v)$. We can write the Lagrangian for this problem as

$$L = \int_0^1 \int_0^1 \|\mathbf{v}(\mathbf{x}, a)\|^2 \, d\mathbf{x} + \int_0^1 \int_0^1 \Phi(\mathbf{x}, a)[f_a(\mathbf{x}|a) + \operatorname{div} \mathbf{v}(\mathbf{x}, a)] \, d\mathbf{x} + \begin{pmatrix} \text{boundary} \\ \text{terms} \end{pmatrix},$$

where $\Phi(\mathbf{x}, a)$ is the Lagrange multiplier for the constraint $f_a(\mathbf{x}|a) = -\operatorname{div} \mathbf{v}(\mathbf{x}, a)$, and the "boundary terms" are the terms that correspond to the boundary constraints in (S35).

Performing an integration by parts on the $\operatorname{div} \mathbf{v}(\mathbf{x}, a)$ term shows that

$$L = \int_0^1 \int_0^1 \|\mathbf{v}(\mathbf{x}, a)\|^2 \, d\mathbf{x}$$

$$+\int_0^1\int_0^1 \left[\Phi(\mathbf{x},a)f_a(\mathbf{x}|a) - \mathbf{v}(\mathbf{x},a)\cdot \operatorname{grad}\Phi(\mathbf{x},a)\right] d\mathbf{x}$$
$$+\begin{pmatrix}\text{boundary}\\ \text{terms}\end{pmatrix}.$$

Thus, following the usual variational argument, we consider a small change, or variation, $\delta\mathbf{v}(\mathbf{x}, a)$, in the vector field $\mathbf{v}(\mathbf{x}, a)$, leaving the boundary conditions fixed. If we consider only first-order terms in $\delta\mathbf{v}$, then, at a minimum point, this should leave the Lagrangian L unchanged, so

$$0 = \delta L = \int_0^1\int_0^1 [2\mathbf{v}(\mathbf{x},a) - \operatorname{grad}\Phi(\mathbf{x},a)]\cdot\delta\mathbf{v}(\mathbf{x},a)\,d\mathbf{x}.$$

Now, this must hold for all variations, $\delta\mathbf{v}(\mathbf{x}, a)$, in the vector flow, satisfying the boundary conditions. In particular, it must hold for a "blip" around an arbitrary point $\mathbf{x}_0$, which equals zero outside of a neighborhood around $\mathbf{x}_0$. Choosing $\delta\mathbf{v}(\mathbf{x}, a)$ judiciously inside this small neighborhood, in the usual way, then shows that $2\mathbf{v}(\mathbf{x}, a) - \operatorname{grad}\Phi(\mathbf{x}, a) = 0$ for all $\mathbf{x} \in \mathbf{S}$, or

$$\mathbf{v}(\mathbf{x},a) = (1/2)\operatorname{grad}\Phi(\mathbf{x},a).$$

That is, the minimal flow is a "potential flow" (Chorin and Marsden (1990, pp. 47–68)), with the potential given by the Lagrange multiplier. Next let Δ be the "Laplacian"operator, so

$$\begin{aligned}\Delta\Phi(\mathbf{x},a) &= \Phi_{xx}(\mathbf{x},a) + \Phi_{yy}(\mathbf{x},a)\\ &= \operatorname{div}\operatorname{grad}\Phi(\mathbf{x},a) = 2\operatorname{div}\mathbf{v}(\mathbf{x},a) = -2f_a(\mathbf{x}|a).\end{aligned}$$

Then this Lagrange multiplier potential solves the "Neumann problem" (Berg and McGregor (1966), Folland (1976))

$$\text{(S36)}\qquad \Delta\Phi(\mathbf{x},a) = -2f_a(\mathbf{x}|a),$$
$$\text{subject to}\quad \Phi_x(0,y) = \Phi_x(1,y) = \Phi_y(x,0) = \Phi_y(x,1) = 0.$$

The fact that a minimal flow solves (S36) is essentially Dirichlet's principle (Folland (1976, pp. 112–117)). Thus, if one can find such a potential function $\Phi(\mathbf{x}, a)$, it becomes easy to check NISP and CISP. For example, NISP simply follows from $\Phi(\mathbf{x}, a)$ nondecreasing in $\mathbf{x}$, while CISP follows from $\operatorname{grad}\Phi(\mathbf{x}, a)$ nonincreasing in a.

However, while there are various methods for solving Neumann problems (see, e.g., Folland (1976)), none seems easy to implement in practice. For example, one method uses a Green's function, $G(\mathbf{x}, \mathbf{z})$, which gives

$$\Phi(\mathbf{x},a) = -2\int_0^1\int_0^1 G(\mathbf{x},\mathbf{z})f_a(\mathbf{z}|a)\,d\mathbf{z}.$$

The problem then becomes to find an appropriate Green's function. For a Neumann problem on a square, it does not seem to be possible to write this Green's function in closed form, though it can be represented as a double Fourier series,

$$G(x, y; z, w) = 4\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\gamma_{mn}\frac{\cos\pi mx\cos\pi mz\cos\pi ny\cos\pi nw}{\pi^2(m^2+n^2)},$$

where $\gamma_{00} = 0$, $\gamma_{m0} = \gamma_{0n} = 1/2$, and $\gamma_{mn} = 1$ otherwise (Roach (1982, p. 268)). It is not clear how to use such solutions to derive useful conditions for NISP or CISP.

In addition, while the solution above seems to be canonical in the sense that it minimizes the total flow necessary to represent the signal technology, it is not canonical in the sense of providing *necessary* as well as sufficient conditions for NISP and CISP. Thus, while $\Phi(\mathbf{x}, a)$ nondecreasing in $\mathbf{x}$ is sufficient to assure NISP, it is not necessary. For example, consider a function $\Phi(\mathbf{x}, a)$ which is increasing in $\mathbf{x}$ outside of some small neighborhood around the point $(0.5, 0.5)$, say, but decreasing slightly inside of that neighborhood. Thus the vector field $\mathbf{v}(\mathbf{x}, a) = (1/2)\,\mathrm{grad}\,\Phi(\mathbf{x}, a)$ has positive coordinates except in that small neighborhood. Clearly this can be done so that the integral (S34) will always be positive (recall that the normal vector $\mathbf{n}(t)$ in (S34) is pointing southwest along the southwest border). Thus NISP will hold, even though $\Phi(\mathbf{x}, a)$ is *not* always nondecreasing in $\mathbf{x}$.

This underscores the difficulty of finding easy-to-check *local* conditions to verify hard-to-check *global* conditions like NISP or CISP. Thus, it seems the best we can reasonably do at this point is to use ad hoc vector flows like those in Section 8 of the main paper. Also, those ad hoc vector flows yield quite useful conditions, as argued in that paper. Nevertheless, in the process of considering different applications, even more useful conditions may be discovered.

S6. DERIVING STATE-SPACE REPRESENTATIONS FROM VECTOR FLOWS

If we begin with the basic vector flow in Equation (26) of the main paper, the state-space representation it implies is the representation in Proposition 3, as mentioned in footnote 10 of the main paper. This basic vector flow is

$$\begin{aligned}\mathbf{v}^b(x, y, a) = \big(&-G_a(x|a)h(y|x, a),\\ &-g(x|a)H_a(y|x, a) + G_a(x|a)H_x(y|x, a)\big).\end{aligned}$$

We want to show that, for this vector flow, the solution, $\mathbf{x}(a, \mathbf{x}_0)$, to the differential system

$$\mathbf{x}_a(a, \mathbf{x}_0) = \frac{1}{f(\mathbf{x}(a, \mathbf{x}_0)|a)}\mathbf{v}^b(\mathbf{x}(a, \mathbf{x}_0), a),$$

subject to the initial condition, $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$, is essentially the state-space representation in Proposition 3. Consider one coordinate at a time. The first coordinate of the above equation becomes

$$\text{(S37)} \qquad \begin{aligned} x_a(a, \mathbf{x}_0) &= -\frac{G_a(x(a, \mathbf{x}_0)|a)h\big(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a\big)}{g(x(a, \mathbf{x}_0)|a)h\big(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a\big)} \\ &= -\frac{G_a(x(a, \mathbf{x}_0)|a)}{g(x(a, \mathbf{x}_0)|a)} \end{aligned}$$

or

$$g(x(a, \mathbf{x}_0)|a)x_a(a, \mathbf{x}_0) + G_a(x(a, \mathbf{x}_0)|a) = 0.$$

Integrating this with respect to a gives

$$G(x(a, \mathbf{x}_0)|a) = \text{constant in } a = G(x_0|a = 0).$$

Letting $G(x_0|a = 0) = \theta_1$ and noting that the marginal cumulative distribution function G, here, is the marginal cumulative distribution function F^1 in Proposition 3 of the paper, yields the first half of (11) in that proposition.

Similarly, the second coordinate of the above vector-differential equation gives

$$\begin{aligned} &y_a(a, \mathbf{x}_0) \\ &\quad = -\big[g(x(a, \mathbf{x}_0)|a)H_a(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a) \\ &\qquad\quad - G_a(x(a, \mathbf{x}_0)|a)H_x(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a)\big] \\ &\qquad /g(x(a, \mathbf{x}_0)|a)h(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a). \end{aligned}$$

Simplifying and using (S37) gives

$$\begin{aligned} y_a(a, \mathbf{x}_0) = &-\frac{H_a(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a)}{h(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a)} \\ &- \frac{H_x(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a)}{h(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a)} x_a(a, \mathbf{x}_0) \end{aligned}$$

or

$$\begin{aligned} &h(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a)y_a(a, \mathbf{x}_0) \\ &\quad + H_a(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a) + H_x(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a)x_a(a, \mathbf{x}_0) = 0. \end{aligned}$$

Again, integrating with respect to a gives

$$H(y(a, \mathbf{x}_0)|x(a, \mathbf{x}_0), a) = \text{constant in } a = H(y_0|x_0, a = 0).$$

Finally, again letting $H(y_0|x_0, a = 0) = \theta_2$ gives the second half of (11) in the paper and we are done.

S7. GENERALIZING JEWITT'S THEOREM 2

The main paper provides generalizations of two of the three major sets of conditions that justify the first-order approach to multisignal principal–agent problems: the Sinclair-Desgagné (1994) conditions and Jewitt's (1988) Theorem 3. It also briefly describes a generalization of the third major set of conditions, that in Jewitt's Theorem 2 (see Section 9 of the main paper). This note presents that generalization in detail. I would like to thank Ian Jewitt for suggesting this generalization to me.

Notation is as in the main paper. Suppose the principal is risk neutral and observes a two-signal random vector $(\tilde{x}, \tilde{y})$ (note that this result does *not* generalize easily beyond the two-signal case). Jewitt, Theorem 2, assumes that the signals are independent. Let their cumulative distribution functions be $F^1(x|a)$ and $F^2(y|a)$, with sufficiently smooth densities $f^1(x|a)$ and $f^2(y|a)$. Also, recall Jewitt's function $\omega(z) = u((u')^{-1}(1/z))$ from (S7). In the two-signal case, the principal's first-order condition for the cost minimizing schedule $s^*(\cdot)$ to induce action a, given the *agent's* first-order condition $U_a(s(\cdot), a) = 0$, yields

$$u(s^*(x, y)) = \omega\left(\lambda + \mu\left[\frac{f_a^1(x|a)}{f^1(x|a)} + \frac{f_a^2(x|a)}{f^2(x|a)}\right]\right), \tag{S38}$$

with λ and μ Lagrange multipliers, and $\mu \geq 0$, as argued in Jewitt (1988). With this setup, Jewitt's Theorem 2 is the following:

THEOREM—Jewitt's Theorem 2: *The first-order approach is valid if $\omega(z)$ is concave in z, and $F^1(x|a)$ and $F^2(y|a)$ satisfy the MLR property and the CDFC.*

To extend this theorem to the nonindependence case, consider the general density and cumulative distribution functions, $f(x, y|a)$ and $F(x, y|a)$, and impose the conditions in the following two definitions:

DEFINITION S1: The distribution $F(x, y|a)$ satisfies the *lower quadrant convexity condition* (LQCC) if, for every fixed (x, y), the probability, $\text{Prob}(\tilde{x} \geq x \text{ or } \tilde{y} \geq y|a)$, is concave in a, so the probability of the corresponding lower quadrant, $\text{Prob}(\tilde{x} \leq x \text{ and } \tilde{y} \leq y|a)$, is convex.

DEFINITION S2: The distribution $F(x, y|a)$ satisfies the *submodular likelihood ratio* (SLR) property if the likelihood ratio, $f_a(x, y|a)/f(x, y|a)$, is submodular, so $\partial^2[f_a(x, y|a)/f(x, y|a)]/\partial x\, \partial y \leq 0$ for all x, y, and a.

With these conditions, we can now state the generalization of Jewitt's Theorem 2.

PROPOSITION S6: *The first-order approach is valid if $\omega(z)$ is concave in z and $F(x, y|a)$ satisfies MLR, SLR, and LQCC.*

Note that LQCC follows if $\tilde{x}$ and $\tilde{y}$ are independent and each satisfies the CDFC and MLR, since $\text{Prob}(\tilde{x} \le x|a)\,\text{Prob}(\tilde{y} \le y|a)$ is convex in a if $\tilde{x}$ and $\tilde{y}$ satisfy CDFC and MLR. Also, if $\tilde{x}$ and $\tilde{y}$ are independent, then the SLR property is met, since in this case,

$$\text{(S39)} \qquad \frac{f_a(x, y|a)}{f(x, y|a)} = \frac{f_a^1(x|a)}{f^1(x|a)} + \frac{f_a^2(y|a)}{f^2(y|a)},$$

so the cross-partial of the likelihood ratio in (S39) is zero. Thus, Proposition S6 does generalize Jewitt's Theorem 2. To prove Proposition S6, we need a lemma. This lemma is similar to the $n = 2$ case of Theorem 3.3.15 in Müller and Stoyan (2002).

LEMMA S7: *Define the function $h(x, y; x_0, y_0) = 1$, if $x \ge x_0$ or $y \ge y_0$, $= 0$ otherwise. Suppose that $g(x, y)$ is nondecreasing in x and y and that $g_{xy}(x, y) \le 0$. Then $g(\cdot, \cdot)$ can be approximated as a constant plus a positive linear combination of the $h(\cdot, \cdot; x_0, y_0)$ functions*:

$$\text{(S40)} \qquad g(x, y) \approx \alpha_0 + \sum_{i=1}^{n} \alpha_i h(x, y; x_i, y_i), \quad \textit{with} \quad \alpha_i > 0 \textit{ for } i \ge 1.$$

PROOF: In point of fact we represent $g(\cdot, \cdot)$ as an integral of the $h(\cdot, \cdot; x_0, y_0)$ functions. For simplicity suppose that all functions are defined on the square $\mathbf{S} = [0, 1] \times [0, 1]$ (a limiting argument can extend the result to functions on any domain). Then

$$\text{(S41)} \qquad \begin{aligned} g(x, y) = g(0, 0) &+ \int_0^1 g_x(u, 1) h(x, y; u, 1)\, du \\ &+ \int_0^1 g_y(1, v) h(x, y; 1, v)\, dv \\ &- \int_0^1 \int_0^1 g_{xy}(u, v) h(x, y; u, v)\, du\, dv. \end{aligned}$$

To see this, note that, using the definition of $h(x, y; u, v)$, the integrals in (S41) can be rewritten, giving

$$\begin{aligned} &\text{right-hand side of (S41)} \\ &\quad = g(0, 0) + \int_0^x g_x(u, 1)\, du + \int_0^y g_y(1, v)\, dv \end{aligned}$$

$$- \int_{u=0}^{x} \int_{v=0}^{1} g_{xy}(u, v)\, dv\, du - \int_{u=x}^{1} \int_{v=0}^{y} g_{xy}(u, v)\, dv\, du.$$

Using the fundamental theorem of calculus, this equals

$$\begin{aligned} & g(0,0) + g(x,1) - g(0,1) + g(1,y) - g(1,0) \\ & \quad - \int_0^x [g_x(u,1) - g_x(u,0)]\, du - \int_x^1 [g_x(u,y) - g_x(u,0)]\, du \\ & = g(0,0) + g(x,1) - g(0,1) + g(1,y) - g(1,0) \\ & \quad - [g(x,1) - g(x,0) - g(0,1) + g(0,0)] \\ & \quad - [g(1,y) - g(1,0) - g(x,y) + g(x,0)] = g(x,y). \end{aligned}$$

Now the coefficients $g_x(u, 1)$ and $g_y(1, v)$ in the first two integrals in (S41) are nonnegative since $g(x, y)$ is nondecreasing in x and y. Also, the coefficient $g_{xy}(u, v)$ in the last integral in (S41) is nonpositive, since $g_{xy}(x, y) \le 0$. Since this last integral is being subtracted, it is clear that (S41) expresses $g(x, y)$ as a constant plus a sum of integrals of functions of the form $h(x, y; u, v)$, as u and v vary, with positive coefficients. Hence it can be approximated as a constant plus a linear combination of such functions with positive coefficients, as in (S40). *Q.E.D.*

PROOF OF PROPOSITION S6: First, the generalization of (S38) to the nonindependence case is $u(s^*(x, y)) = \omega(\lambda + \mu[f_a(x, y|a)/f(x, y|a)])$. Using MLR, SLR, and concavity of $\omega(z)$ in this proves that $u(s^*(x, y))$ is nondecreasing and submodular in x and y. The lemma then shows that $u(s^*(x, y))$ can be approximated as in (S40). Thus $U(s^*(\cdot), a)$ can be approximated by a linear combination, with positive coefficients, of the functions $h^T(a; x_i, y_i)$, and these are concave in a by the LQCC. Any solution to the relaxed problem therefore yields $U(s^*(\cdot), a)$ concave in a, so this solution satisfies the unrelaxed constraints and so is a solution to the unrelaxed problem. The first-order approach is therefore valid. *Q.E.D.*

APPENDIX A: PROOF OF LEMMA S2

Intuitively, if the positive linear combination yielding $k(x_1, x_2) = \min(x_1, x_2)$ includes the function $h(x_1, x_2; \alpha_1^0, \alpha_2^0, \beta^0)$, then $k(x_1, x_2)$ should have a kink along the negatively sloped line $\alpha_1^0 x_1 + \alpha_2^0 x_2 = \beta^0$. However, the only points where the $k(x_1, x_2)$ function is not perfectly flat are contained in the *positively* sloped line

$$\mathbf{K} = \{(x_1, x_2) : x_1 = x_2\}.$$

This yields a contradiction.

To make this argument precise, one must show that, if $k(x_1, x_2)$ can be *approximated* by a constant plus a positive linear combination of the $h(x_1, x_2; \alpha_1, \alpha_2, \beta)$ functions, then it can be expressed *exactly* as a constant plus an *integral* of these functions, where the integral is over $(\alpha_1, \alpha_2, \beta)$. This requires a compactness argument.

To facilitate this argument, consider approximations of $k(x_1, x_2)$ on the square

$$(x_1, x_2) \in \mathbf{S} = [-1, 1] \times [-1, 1].$$

It is enough to prove that there are no such approximations on **S**. However, for approximations on **S**, it is enough to consider values of $\beta \in [-1, 1]$. This is because the expression $\alpha_1 x_1 + \alpha_2 x_2$ is in $[-1, 1]$ for $(x_1, x_2) \in \mathbf{S}$, since α_1, $\alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$. Thus, for $\beta < -1$, $h(x_1, x_2; \alpha_1, \alpha_2, \beta) = 0$ on **S**, and for $\beta > 1$, $h(x_1, x_2; \alpha_1, \alpha_2, \beta) = \alpha_1 x_1 + \alpha_2 x_2 - \beta = h(x_1, x_2; \alpha_1, \alpha_2, 1) + 1 - \beta$ on **S**. It is therefore enough to use the $h(x_1, x_2; \alpha_1, \alpha_2, \beta)$ functions with $(\alpha_1, \alpha_2, \beta)$ in the compact set

$$\mathbf{T} = \{(\alpha_1, \alpha_2, \beta) : \alpha_1, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 1, -1 \leq \beta \leq 1\}.$$

Thus, consider a sequence of approximations to $k(x_1, x_2)$ given by

$$\text{(SA1)} \qquad k_n^A(x_1, x_2) = A_n^0 + B_n^0 \int_{\mathbf{T}} h(x_1, x_2; \alpha_1, \alpha_2, \beta)\, d\mu_n^0(\alpha_1, \alpha_2, \beta),$$

where μ_n^0 is a measure in the compact set, $\Delta\mathbf{T}$, of probability measures on **T** under weak convergence. Assume k_n^A converges uniformly to k on **S**. Of course, if the approximations in (SA1) are ordinary sums with finite numbers of terms, then each μ_n^0 has support which is a finite set of points.

Unfortunately, the sequence $\{B_n^0\}$ might be unbounded if the support of μ_n^0 becomes increasingly concentrated near $\beta = -1$. The following somewhat complicated argument, through equation (SA3) below, shows that this is not a problem.

First let $\mathbf{T}^* = \{(\alpha_1, \alpha_2, \beta) \in \mathbf{T} : -0.5 \leq \beta \leq 1\}$. Then $B_n^0 \mu_n^0(\mathbf{T}^*)$ is a bounded sequence. This can be seen since

$$\begin{aligned} B_n^0 \mu_n^0(\mathbf{T}^*) &= 2B_n^0 \int_{\mathbf{T}^*} [(-0.5 - \beta) - (-1 - \beta)]\, d\mu_n^0(\alpha_1, \alpha_2, \beta) \\ &= 2B_n^0 \int_{\mathbf{T}^*} [h(-0.5, -0.5; \alpha_1, \alpha_2, \beta) \\ &\qquad - h(-1, -1; \alpha_1, \alpha_2, \beta)]\, d\mu_n^0(\alpha_1, \alpha_2, \beta) \\ &\leq 2B_n^0 \int_{\mathbf{T}} [h(-0.5, -0.5; \alpha_1, \alpha_2, \beta) \end{aligned}$$

$$- h(-1,-1;\alpha_1,\alpha_2,\beta)]\,d\mu_n^0(\alpha_1,\alpha_2,\beta)$$
$$= 2[k_n^A(-0.5,-0.5) - k_n^A(-1,-1)].$$

Here the second step follows by $\alpha_1+\alpha_2=1$, $\beta\geq -0.5$ and the definition of the h function in (S16), while the third step follows since the integrand is nonnegative and $\mathbf{T}^*\subseteq\mathbf{T}$. Finally, the last sequence above is bounded, since $k_n^A(x_1,x_2)$ converges uniformly to $k(x_1,x_2)$ on $\mathbf{S}$. Thus $B_n^0\mu_n^0(\mathbf{T}^*)$ is a bounded sequence, as desired.

Next, $k(x_1,x_2)=2k(0.5x_1,0.5x_2)$, so $k(x_1,x_2)$ can be approximated on $\mathbf{S}$ by

$$\text{(SA2)}\quad 2k_n^A(0.5x_1,0.5x_2) = 2A_n^0 + 2B_n^0\int_{\mathbf{T}} h(0.5x_1,0.5x_2;\alpha_1,\alpha_2,\beta)\,d\mu_n^0(\alpha_1,\alpha_2,\beta) = 2A_n^0 + B_n^0\int_{\mathbf{T}} h(x_1,x_2;\alpha_1,\alpha_2,2\beta)\,d\mu_n^0(\alpha_1,\alpha_2,\beta),$$

where the last step follows since $h(x_1,x_2;\alpha_1,\alpha_2,\beta)$ is homogeneous of degree 1 in (x_1,x_2,β). Now, for $(x_1,x_2)\in\mathbf{S}$, the range of integration in (SA2) can be restricted to $\mathbf{T}^*$, since $h(x_1,x_2;\alpha_1,\alpha_2,2\beta)=0$ for $\beta\in[-1,-0.5)$. Let $\mu_n^1\in\Delta\mathbf{T}^*$ be the measure μ_n^0, restricted to $\mathbf{T}^*$ and divided by $\mu_n^0(\mathbf{T}^*)$ to obtain another probability measure. Also, let $A_n^1=2A_n^0$ and $B_n^1=B_n^0\mu_n^0(\mathbf{T}^*)$. Then $k(x_1,x_2)$ is uniformly approximated on $\mathbf{S}$ by the sequence

$$\text{(SA3)}\quad A_n^1 + B_n^1\int_{\mathbf{T}^*} h(x_1,x_2;\alpha_1,\alpha_2,2\beta)\,d\mu_n^1(\alpha_1,\alpha_2,\beta).$$

Also $B_n^1=B_n^0\mu_n^0(\mathbf{T}^*)$ is bounded, as shown above. In addition, since (SA3) and B_n^1 are bounded, A_n^1 must be bounded as well.

Now, (A_n^1,B_n^1,μ_n^1) is a sequence in a compact set (since $\Delta\mathbf{T}^*$, like $\Delta\mathbf{T}$, is compact). Thus it has a convergent subsequence. Let the limit of this subsequence be $(A_\infty^1,B_\infty^1,\mu_\infty^1)$. Then

$$\text{(SA4)}\quad k(x_1,x_2) = A_\infty^1 + B_\infty^1\int_{\mathbf{T}^*} h(x_1,x_2;\alpha_1,\alpha_2,2\beta)\,d\mu_\infty^1(\alpha_1,\alpha_2,\beta)\quad \text{on } \mathbf{S},$$

by continuity of the integral with respect to weak convergence of measures.

Now, the support of μ_∞^1 must contain a point $(\alpha_1^0,\alpha_2^0,\beta^0)$ such that the line $\alpha_1^0x_1+\alpha_2^0x_2=\beta^0$ passes through the interior of $\mathbf{S}$, since otherwise the integral in (SA4) would be linear (or more precisely, affine) in (x_1,x_2) on $\mathbf{S}$. There is then a small neighborhood around $(\alpha_1^0,\alpha_2^0,\beta^0)$ with positive measure under μ_∞^1.

However, the negatively sloped line $\alpha_1^0x_1+\alpha_2^0x_2=\beta^0$ is not contained in the set $\mathbf{K}=\{(x_1,x_2): x_1=x_2\}$ of edges of the graph of the function $k(x_1,x_2)=$

$\min(x_1, x_2)$. Thus, there must be a point, $(x_1^0, x_2^0) \in \mathbf{S}$, on the line $\alpha_1^0 x_1 + \alpha_2^0 x_2 = \beta^0$, but not in $\mathbf{K}$. But $k(x_1, x_2)$ should then have some curvature at (x_1^0, x_2^0), which is impossible since $(x_1^0, x_2^0) \notin \mathbf{K}$.

APPENDIX B: PROOF OF LEMMA S3

Let the density $f(x_1, x_2|a)$, on the rectangle $[0, 3] \times [0, 1]$, be given by

$$f(x_1, x_2|a) = \begin{cases} (1/3) + (2x_1 - 3)a\varepsilon_0 + (1 - x_1)(2x_2 - 1)(2a - a^2)\varepsilon_1, \\ \quad \text{for } 0 \le x_1 \le 1, \\ (1/3) + (2x_1 - 3)a\varepsilon_0, \\ \quad \text{for } 1 \le x_1 \le 2, \\ (1/3) + (2x_1 - 3)a\varepsilon_0 + (x_1 - 2)(2x_2 - 1)a^2\varepsilon_2, \\ \quad \text{for } 2 \le x_1 \le 3. \end{cases}$$

Note that this breaks up the rectangle $[0, 3] \times [0, 1]$ into three regions: Region I, the square $[0, 1] \times [0, 1]$, Region II, the square $[1, 2] \times [0, 1]$, and Region III, the square $[2, 3] \times [0, 1]$. As a rises, the mass of the density tends to move in a concave manner in Region I, linearly in Region II, and in a *convex* manner in Region III.

Now, $\varphi(x_1, x_2) = \min(x_1 - 1, x_2)$. This equals $x_1 - 1$ in Region I and it equals x_2 in Region III. In Region II, it equals $x_1 - 1$ above the diagonal $x_2 = x_1 - 1$ (i.e., for $x_2 \ge x_1 - 1$), and it equals x_2 below this diagonal. Call the part of Region II above the diagonal Region IIa, and the part below the diagonal, Region IIb. Then

$$\varphi^T(a) = E[\varphi(\tilde{\mathbf{x}})|a] = I_{\mathrm{I}} + I_{\mathrm{IIa}} + I_{\mathrm{IIb}} + I_{\mathrm{III}},$$

where $I_{\mathrm{I}} = \int_{\text{Region I}} \varphi(\mathbf{x}) f(\mathbf{x}|a)\, d\mathbf{x}$ and so forth. Now, the first three of these integrals are linear in a, while the fourth is strictly convex in a. As an illustration,

$$I_{\mathrm{I}} = \int_0^1 \int_0^1 (x_1 - 1)\big[(1/3) + (2x_1 - 3)a\varepsilon_0 + (1 - x_1)(2x_2 - 1)(2a - a^2)\varepsilon_1\big]\, dx_1\, dx_2,$$

and it is easy to check that the coefficient of a^2 in this expression integrates out to zero (since $\int_0^1 (2x_2 - 1)\, dx_2 = 0$), so only terms linear in a survive. Similarly,

$$I_{\mathrm{IIb}} = \int_1^2 \int_0^{x_1 - 1} x_2[(1/3) + (2x_1 - 3)a\varepsilon_0]\, dx_2\, dx_1,$$

which is also clearly linear in a. Since the first three integrals are linear and the fourth is strictly convex, it follows that $\varphi^T(a)$ is strictly convex, and so is not concave.

Next, suppose ε_2 is smaller than ε_1. Then we will show that

$$\int_0^3 \int_0^1 h(x_1, x_2; \alpha_1, \alpha_2, \beta) f(x_1, x_2 | a)\, dx_2\, dx_1$$

is concave in a for all α_1, α_2, and β with $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$. To see this, note that the line $\alpha_1 x_1 + \alpha_2 x_2 = \beta$, or $x_2 = (\beta - \alpha_1 x_1)/\alpha_2$, divides Regions I, II, and III into at most two subregions each, for a total of six subregions. Of these six subregions, only the three below the line $x_2 = (\beta - \alpha_1 x_1)/\alpha_2$ matter, since the integrand is zero above the line. Also, the only terms that ultimately matter are those that involve a^2, since the other terms are either constant or linear in a.

The terms that involve a^2 come from the parts of Regions I and III that are below the line $x_2 = (\beta - \alpha_1 x_1)/\alpha_2$. Suppose $(\beta - \alpha_1 x_1)/\alpha_2 \in [0, 1]$ for all $x_1 \in [0, 3]$ (the other cases can be handled similarly). Then the resulting coefficients of a^2 are

$$-\int_0^1 \int_0^{(\beta - \alpha_1 x_1)/\alpha_2} (\alpha_1 x_1 + \alpha_2 x_2 - \beta)(1 - x_1)(2x_2 - 1)\varepsilon_1\, dx_2\, dx_1$$

from Region I, and

$$\int_2^3 \int_0^{(\beta - \alpha_1 x_1)/\alpha_2} (\alpha_1 x_1 + \alpha_2 x_2 - \beta)(x_1 - 2)(2x_2 - 1)\varepsilon_2\, dx_2\, dx_1$$

from Region III. Now, it is easily checked that the first expression is negative and the second is positive. Also, if $\varepsilon_1 > \varepsilon_2$, then the first expression is bigger in absolute value than the second. Thus, the total coefficient on a^2 is negative and $h^T(a; \alpha_1, \alpha_2, \beta)$ is indeed concave.

It is also easy to check that, if ε_0 is sufficiently large compared to ε_1 and ε_2, then the distribution function satisfies the MLR property as well.

APPENDIX C: PROOF OF LEMMA S4

As was the case for Lemma S2, if the function can be approximated as a positive linear combination of nondecreasing flat-top pyramids, it can be represented exactly as a positive integral of flat-top pyramids.

Now, the edges of the graph of $G(x_1, x_2) = \min(0, x_1, x_2, x_1 + x_2 + 1)$ are given by (a) the ray $x_1 = 0$, $x_2 \geq 0$, (b) the ray $x_2 = 0$, $x_1 \geq 0$, (c) the ray $x_1 = -1$, $x_2 \leq -1$, (d) the ray $x_2 = -1$, $x_1 \leq -1$, and (e) the *finite* line segment from the point $(-1, -1)$ to the point $(0, 0)$. Key among these is the positively sloped finite line segment (e). Since the graphs of nondecreasing flat-top pyramids have no edges which are positively sloped finite line segments, it is impossible to get a function with such an edge from a positive linear combination of such flat-top pyramids.

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