Econometrica Supplementary Material

SUPPLEMENT TO "RANDOM EXPECTED UTILITY"

(*Econometrica*, Vol. 74, No. 1, January 2006, 121–146)

BY FARUK GUL AND WOLFGANG PESENDORFER[1]

In this supplement, we extend the analysis of our paper to nonregular random utility functions. We also provide examples that demonstrate the independence of the assumptions in our paper and provide a detailed discussion of the related literature. Specifically, we relate our results to the work of McFadden and Richter (1990), Clark (1995), and Falmagne (1978).

KEYWORDS: Random utility, random choice, expected utility, tie-breakers.

THIS SUPPLEMENT USES the notation and definitions established in the published paper. Theorems, lemmas, and examples are numbered S1, S2, etc. in this supplement. Numbers without the prefix S refer to the published paper.

1. NONREGULAR RANDOM UTILITY

For a nonregular random utility function (RUF) we cannot identify a unique maximizing random choice rule (RCR) since there is a positive probability of a "tie" in some decision problems. More precisely, for some decision problem D there is a positive probability of choosing a utility function that does not have a unique maximizer in D.

To deal with nonregular RUFs we introduce tie-breakers. Suppose that the agent with RUF μ faces the decision problem D. Assume that to eliminate ties, the decision-maker chooses two utility functions (u, v) according to some measure η. If the set of maximizers of u in D (denoted $M(D, u)$) is a singleton, then the agent chooses the unique element of $M(D, u)$. Otherwise, the agent chooses an element of $M(D, u)$ that maximizes v; that is, an element of $M(M(D, u), v)$. If η is a product measure $\eta = \mu \times \hat{\mu}$ and $\hat{\mu}$ is regular, then it is clear that this procedure will lead to a unique choice with probability 1. In this case, the choice of v is independent of the choice of u and the regularity of $\hat{\mu}$ ensures that $M(M(D, u), v)$ is a singleton with probability 1. It turns out that independence is not necessary for a tie-breaker to generate a unique choice as long as the marginal on the second coordinate is a regular RUF. Therefore, we do not require η to be a product measure.

To describe the lexicographic procedure above formally, we need to describe a measure on the set $U \times U$. Let $\mathcal{F}^2$ denote the smallest algebra that contains

[1]This research was supported by Grants SES0236882 and SES0214050 from the National Science Foundation. We thank the editor and three anonymous referees for their suggestions and comments.

$\mathcal{F} \times \mathcal{F}$. The marginals η_i of η are defined by

$$\eta_1(F) = \eta(F, U),$$
$$\eta_2(F) = \eta(U, F)$$

for all $F \in \mathcal{F}$.

DEFINITION: (i) The measure η on $\mathcal{F}^2$ is a *tie-breaker* if η_2 is regular. (ii) The measure η is a tie-breaker for μ if $\eta_1 = \mu$ and η_2 is regular.

Let $N_l(D, x) = \{(u, v) | x \in M(M(D, u), v)\}$. Hence, $(u, v) \in N_l(D, x)$ if and only if x is a lexicographic maximizer of (u, v) in D. We show in Lemma 8 that $N_l(D, x) \in \mathcal{F}^2$ for all D, x. A random choice rule ρ maximizes the tie-breaker η if the probability of choosing x in D is equal to the probability of choosing some (u, v) in $N_l(D, x)$. The random choice rule maximizes the (not necessarily regular) RUF μ if ρ maximizes a tie-breaker for μ.

DEFINITION: (i) The RCR ρ maximizes the tie-breaker η if $\rho^D(x) = \eta(N_l(D, x))$ for all D, x. (ii) The RCR ρ maximizes the RUF μ if ρ maximizes a tie-breaker for μ.

Part (ii) of the definition above applies to regular and nonregular RUFs. To see this, note that

$$\mu(N^+(D, x)) \leq \eta(N_l(D, x)) \leq \mu(N(D, x))$$

for all D, x. The first inequality follows from the fact that if x is the unique maximizer of u in D, then x is the lexicographic maximizer of (u, v) for all $v \in U$. The second inequality follows from the fact that any lexicographic maximizer of (u, v) is a maximizer of u. Hence, if η is the tie-breaker for the regular RUF μ and $\rho^D(x) = \eta(N_l(D, x))$ for all D, x, then $\rho^D(x) = \mu(N(D, x))$ for all D, x. Therefore, this definition of maximizing a RUF agrees with the definition of maximizing a regular RUF presented in the published paper.

Theorem S1 demonstrates that tie-breakers have a unique maximizing RCR. Moreover, this RCR is monotone, mixture continuous, linear, and extreme.

THEOREM S1: *Every tie-breaker is maximized by a unique RCR. If the RCR ρ maximizes a tie-breaker, then ρ is monotone, mixture continuous, linear, and extreme.*

Theorem S1 follows from Lemmas S1–S3. For $D \in \mathcal{D}$ and $x \in D$, let $\mathcal{P}_x(D)$ denote the collection subset of D that contains x. For $X \subset \mathbb{R}^n$, let $\neg X = \mathbb{R}^n \setminus X$. Note that

$$N_l(D, x) = \bigcup_{B \in \mathcal{P}_x(D)} \left(\left(\bigcap_{y \in B} N(D, y) \cap \bigcap_{y \in \neg B \cap D} \neg N(D, y) \right) \times N(B, x) \right), \tag{S1}$$

where we let the intersection over an empty index set (i.e., for $B = D$) equal $\mathbb{R}^n$. Define

$$N_l^+(D, x) := \bigcup_{B \in \mathcal{P}_x(D)} \left(\left(\bigcap_{y \in B} N(D, y) \cap \bigcap_{y \in \neg B \cap D} \neg N(D, y) \right) \times N^+(B, x) \right). \tag{S2}$$

LEMMA S1: *We have* $N_l(D, x) \in \mathcal{F}^2$.

PROOF: The collection $\mathcal{F}$ is an algebra that contains $N(D', y)$ for all D', y. Since $\mathcal{F}^2$ contains $\mathcal{F} \times \mathcal{F}$, equation (11) implies that $\mathcal{F}^2$ contains $N_l(D, x)$. *Q.E.D.*

Let η be a tie-breaker and let $\rho : \mathcal{B} \to \Pi$ be defined as

$$\rho^D(B) = \sum_{x \in D \cap B} \eta(N_l(D, x)) \tag{S3}$$

for all $D \in \mathcal{D}$, $B \in \mathcal{B}$. Clearly, this ρ is the only candidate for a maximizer of the tie-breaker η. Lemma S2 shows that the ρ defined in (S3) is a well-defined RCR. This proves that every tie-breaker has a unique maximizing RCR.

LEMMA S2: *The function ρ defined in* (S3) *is a RCR.*

PROOF: To prove that ρ is a RCR it suffices to show that $\sum_{x \in D} \rho^D(x) = 1$ for all D, x. First, we show that $\eta(N_l(D, x)) = \eta(N_l^+(D, x))$ for all D, x. Clearly, $\eta(N_l(D, x)) \geq \eta(N_l^+(D, x))$. If $\eta(N_l(D, x)) > \eta(N_l^+(D, x))$, then by (S1) and (S2) there is $F \in \mathcal{F}$ and $B \in \mathcal{P}_x(D)$ such that $\eta(F \times N(B, x)) > \eta(F \times N^+(B, x))$. Since $\eta(U \setminus F \times N(B, x)) \geq \eta(U \setminus F \times N^+(B, x))$, this implies that $\eta(U \times N(B, x)) > \eta(U \times N^+(B, x))$, contradicting the regularity of η_2.

For $x \neq y$, $N_l^+(D, x) \cap N_l^+(D, y) = \emptyset$. Also, $\bigcup_{x \in D} N_l(D, x) = \mathbb{R}^n \times \mathbb{R}^n$. Therefore,

$$\rho^D(x) = \sum_{x \in D} \eta(N_l(D, x)) \geq \eta\left(\bigcup_{x \in D} N_l(D, x) \right) = \eta(\mathbb{R}^n \times \mathbb{R}^n) = 1,$$

$$\rho^D(x) = \sum_{x \in D} \eta(N_l^+(D, x)) = \eta\left(\bigcup_{x \in D} N_l^+(D, x) \right) \leq \eta(\mathbb{R}^n \times \mathbb{R}^n) = 1.$$

Hence ρ^D is a RCR. *Q.E.D.*

LEMMA S3: *Let the RCR ρ maximize the tie-breaker η. Then ρ is monotone, mixture continuous, linear, and extreme.*

PROOF: Note that for all D, $x \in D$, y and $\lambda \in (0,1)$, $N_l(D \cup \{y\}, x) \subset N_l(D, x)$ and $N_l(\lambda D + (1-\lambda)\{y\}, x + (1-\lambda)y) = N_l(D, x)$. Hence, monotonicity and linearity of ρ follow immediately from its definition.

Next, we prove that ρ is extreme. For any $B \subset D$, let $F_B(D)$ denote the intersection of all faces of $F(\operatorname{conv} D)$ that contain B. Obviously, $B \subset F_B(D) \cap D$. Suppose there exists $z \in F_B(D) \cap D$, $z \notin B$. Then $u \in \bigcap_{y \in B} N(D, y)$ implies $u \in N(D, z)$ and, therefore, $\bigcap_{y \in B} N(D, y) \cap \bigcap_{y \in \neg B \cap D} \neg N(D, y) = \emptyset$. Hence, in (S1) it suffices to consider B such that $B = F \cap D$ for some face $F \in F(\operatorname{conv} D)$. However, if $B = F \cap D$ for some $F \in F(\operatorname{conv} D)$ and $x \in B$ is not an extreme point of D, then it is not an extreme point of B, but then the regularity of η_2 ensures $\eta(\mathbb{R}^n, N(B, x)) = 0$, proving the extremeness of ρ.

To prove mixture continuity, it suffices to show that $\rho^{tD+t'D'}$ is continuous in t, t'. By an analogue of Proposition 3, $\rho^{tD+t'D'}(tx + t'x') = \nu(N_l(D, x) \cap N_l(D', x'))$, which implies that $\rho^{tD+t'D'}$ is continuous in (t, t') for $t, t' > 0$. Continuity at $(t, t') = (0, 0)$ is obvious. Hence, it remains to show that $\rho^{tD+D'} \to \rho^{D'}$ as $t \to 0$. Choose $\epsilon > 0$ small enough so that $B_\epsilon(x') \cap D' = \{x'\}$ and choose t small enough so that $x' + tx \in B_\epsilon(x')$ for all $x \in D$ and $x'' + tx \notin B_\epsilon(x')$ for all $x \in D$, $x'' \in D \setminus \{x'\}$. Proposition 3 and the fact that $\bigcup_{x \in D} N_l(D, x) = \mathbb{R}^n$ imply that

$$\begin{aligned}\rho^{tD+D'}(B_\epsilon(x')) &= \nu\left(\bigcup_{x \in D}(N_l(D, x) \cap N_l(D', x'))\right) \\ &= \nu(N_l(D', x')) = \rho^{D'}(x'),\end{aligned}$$

which establishes mixture continuity and completes the proof of the lemma. *Q.E.D.*

Example S1 gives a tie-breaker for the RUF that corresponds to a deterministic utility function $\bar{u}$.

EXAMPLE S1: There are three prizes ($n + 1 = 3$). Consider the RUF $\mu_{\bar{u}}$, which assigns probability 1 to the utility function $\bar{u} \neq (0, 0, 0)$. An example of a tie-breaker for $\mu_{\bar{u}}$ is the measure $\eta = \mu_{\bar{u}} \times \mu$, where μ is the uniform RUF defined in Example 2. The tie-breaker η is maximized by the following RCR ρ. If $M(D, \bar{u}) = \{x\}$ and hence $\bar{u}$ has a unique maximizer in D, then $\rho^D(x) = 1$. If $M(D, \bar{u})$ is not a singleton, then the convex hull of $M(D, \bar{u})$ is a line segment. In that case, ρ^D assigns probability $1/2$ to each endpoint of this line segment.

Let $\hat{\mu}$ be any regular random utility.[2] Then the product measure $\eta := \mu \times \hat{\mu}$ is a tie-breaker for μ. By Theorem S1, every tie-breaker has a maximizer and therefore it follows that every nonregular RUF has a maximizer. For a nonregular RUF, the choice of a tie-breaker affects behavior and therefore there are

[2]Lemma 3 proves the existence of a regular RUF.

multiple maximizing random choice rules. In contrast, regular random utilities have a unique maximizer. Theorem S2 summarizes these facts.

THEOREM S2: (i) *Every RUF μ has a maximizer*. (ii) *A RUF has a unique maximizer if and only if it is regular*.

PROOF: In Lemma 3 we construct a regular RUF μ_V. Obviously, $\mu \times \mu_V$ is a tie-breaker for μ. Then Theorem S1 proves part (i) of the theorem.

Let ρ be such that $\rho^D(x) = \eta(N_l(D, x))$ for all D, x and $\eta = \mu \times \mu_V$, where μ_V is the regular RUF constructed in Lemma 3. By Theorem S1, this identifies a unique RCR ρ that is a maximizer of μ. To construct a second maximizer, note that since μ is not full dimensional, there exists some polyhedral cone K_* such that $\dim K_* < n$ and $\mu(K_*) > 0$. By the argument given in the proof of Lemma 2, there is $x_* \neq 0$ such that $K_* \subset N(D_*, x_*) \cap N(D_*, -x_*)$ for $D_* = \{-x_*, x_*\}$. Define μ_* as

$$\mu_*(K) = \frac{V(B_1(o) \cap K \cap N(D_*, x_*))}{V(B_1(o) \cap N(D_*, x_*))}.$$

Repeating the arguments made for μ_V establishes that μ_* is a regular RUF.[3] Then let ρ_* be defined by $\rho_*^D(x) = \eta_*(N(D, x))$, where $\eta_* = \mu \times \mu_*$. Again by Theorem S1, ρ_* is a maximizer of μ. Note that $1 = \rho_*^{D_*}(x_*) \neq \rho^{D_*}(x_*) = .5$. Hence, $\rho_* \neq \rho$ and we have shown that there are multiple maximizers of μ. *Q.E.D.*

Theorem S1 shows that the generalization of RUF maximization to nonregular RUFs preserves the properties identified in Section 4. If ρ is a maximizer of some (not necessarily regular) RUF, then it satisfies monotonicity, linearity, mixture continuity, and extremeness. Therefore, we can apply Theorem 2 to conclude that ρ must also maximize some *regular* RUF μ'.

THEOREM S3: *If the RCR ρ maximizes some RUF, then ρ maximizes a regular RUF*.

The proof follows from Theorem S1 and Theorem 2.

Consider a nonregular RUF μ. Let η be a tie-breaker for μ and let ρ be the maximizer of η. By Theorem S3, the RCR ρ also maximizes a regular RUF μ'. Hence,

$$\mu'(N(D, x)) = \eta(N_l(D, x)) = \rho^D(x)$$

for all $D \in \mathcal{D}$ and $x \in D$. We call this μ' a dilation of μ.

[3]A similar construction is used in Regenwetter and Marley ((2001), p. 880).

DEFINITION: A RUF μ' is a dilation of the RUF μ if there exists a tie-breaker η for μ such that $\mu'(N(D,x)) = \eta(N_l(D,x))$ for all D, x.

A dilation μ' of μ satisfies

$$\mu(N^+(D,x)) \le \mu'(N(D,x)) \le \mu(N(D,x)).$$

Intuitively, a dilation of μ takes probability mass from lower dimensional subsets of U and (with the aid of the tie-breaker) spreads it over adjacent n-dimensional sets. Below, we illustrate a dilation of the RUF in Example S1.

EXAMPLE S1—CONTINUED: There are three prizes ($n+1=3$). Consider the RUF $\mu_{\bar{u}}$, which assigns probability 1 to the utility function $\bar{u} \neq (0,0,0)$. The following regular random utility μ' is a dilation of $\mu_{\bar{u}}$. Recall that for any $u, v \in U$, $F_{uv} := \{\alpha u + \beta v | \alpha, \beta > 0\}$. Let $\mu'(F_{uv}) = 1$ if $u \neq \lambda v$ (and hence F_{uv} is two dimensional) and $\bar{u}$ is in the relative interior of F_{uv}. Let $\mu'(F_{uv}) = 1/2$ if $u \neq \lambda v$ for $\lambda \in \mathbb{R}$ and $\bar{u}$ is on the boundary of F_{uv}. That is, $\bar{u} = \lambda u$ or $\bar{u} = \lambda v$ for some $\lambda > 0$. In all other cases, $\mu'(F_{uv}) = 0$. In particular, every one-dimensional subset of U has μ'-measure 0 and therefore, μ' is regular. The RUF μ' is maximized by the same RCR as the uniform tie-breaker described above: If $M(D,\bar{u}) = \{x\}$, then $\bar{u}$ is in the interior of $N(D,x)$. Therefore, $\rho^D(x) = \mu'(N(D,x)) = 1$ in this case. If $M(D,\bar{u})$ is not a singleton, then $\rho^D = \mu'(N(D,x)) = 1/2$ for any x that is an extreme point of $M(D,\bar{u})$. (Note that $M(D,\bar{u})$ has at most two extreme points.)

Theorem S4 shows that, except for the case of complete indifference, a dilation of a nonregular random utility is not countably additive. In other words, ties cannot be broken in a manner that preserves countable additivity. Let $o = (0,\dots,0)$ denote the utility function that is indifferent between all prizes.

THEOREM S4: *If μ' is a dilation of some nonregular μ such that $\mu(o) = 0$, then μ' is not countably additive.*

Theorem S4 is closely related to Theorem 3. Theorem 3 implies that a maximizer of a regular, countably additive RUF is continuous. In Lemma S4, we show that a maximizer of a nonregular RUF μ with $\mu(o) = 0$ must fail continuity and, therefore, Theorem 3 implies Theorem S4.

LEMMA S4: *Let ρ maximize some RUF μ such that $\mu(o) = 0$. If ρ is continuous, then μ is regular.*

PROOF: If ρ maximizes some μ, then

$$\mu(N^+(D,x)) \le \rho^D(x) \le \mu(N(D,x)). \tag{S4}$$

Suppose μ is not regular. By Lemma 2, this implies that μ is not full dimensional. By Proposition 6, $\mathcal{H} := \{\operatorname{ri} K | K \in \mathcal{K}\}$ is a semiring and every element of $\mathcal{F}$ can be written as a finite union of elements in $\mathcal{H}$. Therefore, $\mu(K) > 0$ for some $K \in \mathcal{K}$ with $\dim K < n$. By Proposition 1(i), $\dim K < n$ implies there is $x \neq 0$ such that $x, -x \in N(K, o)$. Since K is a pointed cone, o is an extreme point of K and, therefore, Proposition 1(iii) implies that $N^+(K, o)$ is nonempty. Hence there is z such that $u \cdot z < 0$ for all $u \in K$, $u \neq o$. Let $D_k := \{x, 1/k(-z), -x\}$ and note that $(D_k)_{k\geq 1}$ converges to $D = \{x, o, -x\}$ in the Hausdorff topology. Let O be an open ball that contains o but does not contain $x, -x$. Since $\mu(K) = \mu(K \setminus \{o\})$, for all k sufficiently large, (S4) implies $\rho^{D_k}(O) \geq \mu(K \setminus \{o\}) = \mu(K) > 0$. However, $\rho^D(O) = 0$ since ρ is extreme. *Q.E.D.*

PROOF OF THEOREM S4: Let μ' be a dilation of μ for some nonregular μ such that $\mu(o) = 0$. Let ρ maximize μ' (and hence maximize μ). Since ρ maximizes μ, Lemma S5 implies that ρ is not continuous. Since μ' is regular, Theorem 3 implies that μ' is not countably additive. *Q.E.D.*

We illustrate Theorem S4 by demonstrating that the dilation in Example S1 is not countably additive.

EXAMPLE S1—CONTINUED: In Example S1, we defined a dilation μ' of the random utility $\mu_{\bar{u}}$. To see that μ' is not countably additive, let $v \neq \lambda \bar{u}$ and let v_n be in the relative interior of the line segment that connects $\bar{u}$ and v. Choose the sequence v_n so that it converges to $\bar{u}$. Note that $\mu'(F_{vv_n}) = 0$ for all n, yet $\mu'(\bigcup_n F_{vv_n}) = 1/2$. Hence, the dilation μ' is not countably additive. Note that the original random utility $\mu_{\bar{u}}$ is countably additive.

We can interpret the results in this section as a justification for restricting attention to regular RUFs. When tie-breakers are used to resolve the ambiguity associated with nonregular RUFs, the resulting behavior maximizes some regular random utility. In this sense, the restriction to regular RUFs is without loss of generality. However, applying a tie-breaker to a nonregular μ typically results in a regular RUF (i.e., dilation of μ) that fails countable additivity.

2. COUNTEREXAMPLES

In this section, we provide examples that show that none of the assumptions in Theorems 2 and 3 in our main paper is redundant. Example S2 provides a RCR that is continuous (hence mixture continuous), linear, and extreme, but not monotone. This shows that monotonicity cannot be dispensed with in Theorems 2 and 3.

EXAMPLE S2: Let $n+1=2$. Hence, P can be identified with the unit interval and $x \in P$ is the probability of getting prize 2. For $D \in \mathcal{D}$, let $\underline{m}(D)$ denote the smallest element in D, let $\overline{m}(D)$ denote the largest element in D, and define

$$a(D) := \sup\{x - y | \underline{m}(D) \leq y \leq x \leq \overline{m}(D), (y, x) \cap D = \emptyset\}.$$

Hence, $a(D)$ is the length of the largest open interval that does not intersect D, but is contained in the convex hull of D. If $D = \{x\}$, then $\rho^D(x) = 1$. If D is not a singleton, let

$$\rho^D(\underline{m}(D)) = \frac{a(D)}{\overline{m}(D) - \underline{m}(D)},$$
$$\rho^D(\overline{m}(D)) = 1 - \rho^D(\underline{m}(D)),$$

and $\rho^D(x) = 0$ for $x \notin \{\underline{m}(D), \overline{m}(D)\}$. Then ρ is continuous (hence mixture continuous), linear, and extreme, but not monotone.

Example S3 provides a RCR that is continuous (hence mixture continuous), monotone, and linear, but not extreme. This shows that the requirement that the choice rule is extreme cannot be dropped in Theorems 2 and 3.

EXAMPLE S3: Let $n+1=2$ and let $x \in [0, 1]$ denote the probability of getting prize 2. For any $D = \{x_1, \dots, x_m\}$, where $x_1 < x_2 < \cdots < x_m$, let

$$\rho^D(x_1) = \begin{cases} 1, & \text{if } m = 1, \\ 1/2, & \text{otherwise.} \end{cases}$$

For $k > 1$, let

$$\rho^D(x_k) = \frac{x_k - x_{k-1}}{2(x_m - x_1)}.$$

Then ρ is continuous, monotone, and linear, but not extreme.

Example S4 provides a RCR that is continuous (hence mixture continuous), extreme, and monotone, but not linear. This shows that linearity cannot be dropped in Theorems 2 and 3.

EXAMPLE S4: Let $n+1=2$ and let $x \in [0, 1]$ denote the probability of getting prize 2. As in Example 4, let $\underline{m}(D)$ and $\overline{m}(D)$ be the smallest and largest elements in D. Let $\rho^D(x) = 1$ for $D = \{x\}$. If D is not a singleton, then

$$\rho^D(\overline{m}(D)) = \overline{m}(D),$$
$$\rho^D(\underline{m}(D)) = 1 - \overline{m}(D),$$

and $\rho^D(x) = 0$ for $x \notin \{\underline{m}(D), \overline{m}(D)\}$. Then ρ is continuous, monotone, and extreme, but not linear.

Example S5 provides a RCR that is monotone, linear, and extreme, but not mixture continuous (and hence is not continuous). This shows that mixture continuity cannot be dispensed with in Theorem 2 and continuity cannot be dispensed with in Theorem 3.

EXAMPLE S5: Let $n+1=3$. The RCR ρ takes on the values 0, $\frac{1}{2}$, and 1. If $N(D,x)=U$ and hence the decision problem is a singleton, then $\rho^D(x)=1$. There are three cases in which ρ takes on the value $\frac{1}{2}$:

$$\rho^D(x)=\frac{1}{2} \text{ if } N(D,x) \text{ is a half-space, or}$$

$$\text{if there is } \epsilon>0 \text{ such that}$$

$$(1+\epsilon,-1,0),(1,-1,0)\in N(D,x), \text{ or}$$

$$\text{if there is } \epsilon>0 \text{ such that}$$

$$(-1,1+\epsilon,0),(-1,1,0)\in N(D,x).$$

In all other cases, $\rho^D(x)=0$.

To see that this ρ is a well-defined RCR, note that $N(D,x)$ is a half-space if and only if D is one dimensional and x is an extreme point of D. Clearly, a one-dimensional decision problem has two extreme points. If D is two-dimensional, then $\rho^D(x)=1/2$ if x is the maximizer of $(1,-1,0)$ in D with the largest first coordinate or if x is the maximizer of $(-1,1,0)$ in D with the largest second coordinate.

This RCR is extreme by definition. It is linear because the probability of choosing x from D depends only on the set $N(D,x)$, which is invariant to linear translations of D. To see that the choice rule is monotone, note that the construction ensures that the probability of choosing x from D is monotone in $N(D,x)$. That is, $N(D,x)\subset N(D',y)$ implies $\rho^D(x)\le\rho^{D'}(y)$. Since $N(D\cup\{y\},x)\subset N(D,x)$, it follows that ρ is monotone. It remains to show that ρ is not mixture continuous.

Let $D=\{(\frac{1}{4},\frac{1}{2},\frac{1}{4}),(\frac{1}{2},\frac{1}{4},\frac{1}{4})\}$ and let $D'=\{(\frac{3}{8},\frac{3}{8},\frac{1}{4}),(\frac{1}{8},\frac{1}{8},\frac{3}{4})\}$. For $\lambda>0$, the agent chooses from $\lambda D+(1-\lambda)D'$ either $\lambda(\frac{1}{4},\frac{1}{2},\frac{1}{4})+(1-\lambda)(\frac{3}{8},\frac{3}{8},\frac{1}{4})$ or $\lambda(\frac{1}{2},\frac{1}{4},\frac{1}{4})+(1-\lambda)(\frac{3}{8},\frac{3}{8},\frac{1}{4})$, each with probability $\frac{1}{2}$. For $\lambda=0$, the agent chooses $(\frac{3}{8},\frac{3}{8},\frac{1}{4})$ or $(\frac{1}{8},\frac{1}{8},\frac{3}{4})$, each with probability $\frac{1}{2}$. Clearly, this violates mixture continuity at $\lambda=0$.

3. RELATIONSHIP TO THE LITERATURE

McFadden and Richter (1991) and Clark (1995) provide necessary and sufficient conditions for a RCR to maximize a RUF. They do not require the choice objects to be lotteries and, hence, do not analyze von Neumann–Morgenstern utility functions. Both Clark's condition and McFadden and Richter's condition can be thought of as a joint restriction on random choice rules and the space of utility functions. To facilitate the comparisons, we adopt their assumptions to the setting of our main paper: The notation below is taken from Section 2.

Clark (1995) introduces an axiom termed coherency. Coherency is closely related to a theorem of de Finetti that provides a necessary and sufficient condition for a function defined on a collection of subsets to have an extension to a finitely additive probability measure on the smallest algebra that contains those subsets. Clark (1995) shows that a random choice rule is coherent if and only if it maximizes some regular random utility function.

The definition below adapts Clark's axiom to our setting. For $A \subset U$, let I_A denote the indicator function on the set A. Hence $I_A(u) = 1$ if $u \in A$ and $I_A(u) = 0$ otherwise. For any $F : U \to \mathbb{R}$, we write $f \geq 0$ as a shorthand for $f(u) \geq 0\ \forall u \in U$.

DEFINITION: The RCR ρ is *coherent* if for every finite sequence $\{D_i, x_i\}_{i=1}^m$ with $D_i \in \mathcal{D}$, $x_i \in D_i$ and every finite sequence of real numbers $\{\lambda_i\}_{i=1}^m$,

$$\sum_{i=1}^{n} \lambda_i I_{N(D_i, x_i)} \geq 0 \quad \text{implies} \quad \sum_{i=1}^{n} \lambda_i \rho^{D_i}(x_i) \geq 0.$$

Clark (1995) shows that coherency is necessary and sufficient for the existence of a regular RUF μ such that for all D and $x \in D$,

$$\rho^D(x) = \mu(N(D, x)).$$

We can show that coherency implies all four assumptions of Theorem 2.

FACT S1: *A coherent RCR ρ is monotone*, *mixture continuous*, *linear*, *and extreme*.

PROOF: To show extremeness, let $y \in D$ with $y \notin \operatorname{ext} D$ and let $D' = \operatorname{ext} D$. Then $I_{N(D,x)} = I_{N(D',x)}$ for all $x \in D$ and, therefore, coherency implies

$$\sum_{x \in D} \rho^{D'}(x) = \sum_{x \in D} \rho^D(x) = 1,$$

which in turn implies that $\rho^D(y) = 0$ and establishes extremeness.

To show monotonicity, let $D' = D \cup \{y\}$. Then $N(D, x) \supset N(D', x)$ for all $x \in D$ and, therefore, $I_{N(D,x)} - I_{N(D',x)} \geq 0$, which implies $\rho^D(x) \geq \rho^{D'}(x)$.

To show linearity and mixture continuity, note that for any coherent RCR ρ,

$$\text{(S5)} \qquad N(D, x) = N(D', x') \quad \text{implies} \quad \rho^D(x) = \rho^{D'}(x')$$

and

$$\text{(S6)} \qquad N(D, x) = N(D', x') \cup N(D'', x'') \quad \text{implies}$$
$$\rho^D(x) \le \rho^{D'}(x') + \rho^{D''}(x'').$$

Since $N(D, x) = N(\lambda D + (1-\lambda)\{y\}, \lambda x + (1-\lambda)y)$, linearity follows from (S5). Using (S5) and (S6), it is straightforward to adapt the argument given in the proof of Theorem 2 to demonstrate that ρ is mixture continuous. *Q.E.D.*

Clark's theorem implies that coherency is necessary for a random choice rule to maximize a regular random utility function. We can use Fact S1 together with Theorem 2 to provide an alternative proof of Clark's theorem. Suppose $\sum_{i=1}^n \lambda_i I_{N(D_i,x_i)} \ge 0$ and that the RCR ρ maximizes some regular RUF μ. Then note that $0 \le \int_U \sum_{i=1}^n \lambda_i I_{N(D_i,x_i)}\, d\mu(u) = \sum_{i=1}^n \lambda_i \int_U I_{N(D_i,x_i)}\, d\mu(u) = \sum_{i=1}^n \lambda_i \mu(N(D_i, x_i)) = \sum_{i=1}^n \lambda_i \rho^{D_i}(x_i)$. Hence, coherency is necessary for ρ to maximize some regular μ. The sufficiency of coherency follows from Fact S1 and Theorem 2. Hence, a RCR is monotone, mixture continuous, linear, and extreme if and only if it is coherent.

Coherency can also be applied in settings where we observe choice behavior only in a subset of the possible decision problems. In that case, coherency is necessary and sufficient for the implied RUF to have an extension that is a probability measure. Thus whenever the observed choice probabilities satisfy coherency, one can construct a RUF μ such that the observed behavior is consistent with μ-maximization.

Coherency is hard to interpret behaviorally. Moreover, it seems difficult to construct experiments that "test" for coherency. By contrast, it is quite straightforward to construct tests of extremeness, linearity, and monotonicity. In fact, the experimental literature on expected utility has focused on the linearity axiom to point out violations of the expected utility framework and develop alternatives. This process of searching for violations of a theory and generalizing the theory to incorporate the documented violations requires interpretable axioms.

McFadden and Richter (1990) introduce a stochastic version of the strong axiom of revealed preference, an axiom they term *axiom of revealed stochastic preference* (ARSP). McFadden and Richter (1990) study a case where each utility function under consideration has a unique maximizer and show that ARSP is necessary and sufficient for (regular) random utility maximization.

In the definition below, we adapt ARSP to the framework in our main paper.

DEFINITION: The RCR satisfies ARSP if and only if for all $(D_i, x_i)_{i=1}^m$ with $D_i \in \mathcal{D}$, $x_i \in D_i$,

$$\text{(S7)} \qquad \sum_{i=1}^m \rho^{D_i}(x_i) \le \max_{u \in U} \sum_{i=1}^m I_{N^+(D_i,x_i)}(u).$$

To see that ARSP is necessary for regular random utility maximization, note that if ρ maximizes a regular RUF μ, then

$$\sum_{i=1}^{m} \rho^{D_i}(x_i) = \int_U \sum_{i=1}^{m} I_{N^+(D_i,x_i)}(u)\mu(du).$$

Obviously, the right-hand side of the equation above is less than or equal to the right-hand side of (S7).

Fact S2 shows that ARSP implies monotonicity, linearity, extremeness, and mixture continuity. Hence, Theorem 2 yields an alternative proof of the McFadden–Richter theorem and implies that a random choice rule satisfies ARSP if and only if it is monotone, mixture continuous, linear, and extreme.

FACT S2: *If the RCR ρ satisfies ARSP, then it is monotone, mixture continuous, linear, and extreme.*

PROOF: Extremeness is trivial since $N^+(D, x)$ is empty unless x is an extreme point of D.

For monotonicity, apply ARSP to $\{(D, x), (D \setminus \{y\}, z)_{z\neq x,y}\}$. This yields $\rho^D(x) \leq \rho^{D\setminus\{y\}}(x)$ and hence monotonicity.

Next, we show that

$$\text{(S8)} \qquad \rho^D(x) = \rho^{D'}(x') \quad \text{if} \quad N^+(D, x) = N^+(D', x').$$

Apply ARSP to $\{(D, x), (D', y)_{y\neq x'}\}$ to get $\rho^D(x) \leq \rho^{D'}(x')$. Reversing the roles of D and D' yields the reverse inequality and hence the result. Linearity now follows because $N^+(D, x) = N^+(\lambda D + (1 - \lambda)\{y\}, \lambda x + (1 - \lambda)y)$. To prove mixture continuity, we proceed as above. First, we show that (S5) and (S6) in the proof of Fact S1 above hold. To prove (S6), apply ARSP to $\{(D, x), (D', y)_{y\neq x'}, (D'', y)_{y\neq x''}\}$. Since $N(D, x) = N(D', x')$ implies $N^+(D, x) = N^+(D', x')$, (S5) follows from (S8). Using (S5) and (S6) it is again straightforward to adapt the argument given in the proof of Theorem 2 to demonstrate that ρ is mixture continuous. *Q.E.D.*

Falmagne (1978) studies a model with finitely many alternatives. Let Y be a finite set. A decision problem is a nonempty subset D of Y. Let $\mathcal{D}^*$ be the corresponding collection of decision problems. Let U^* be the set of all one-to-one utility functions on Y and let $\mathcal{F}^*$ be the algebra generated by the equivalence relation that identifies all ordinally equivalent utility functions (i.e., $u \in F$ implies $v \in F$ if and only if $[v(x) \geq v(y)$ if and only if $u(x) \geq u(y)]$ for all $x, y \in Y$). Let Π^* denote the set of all probability measures on $\mathcal{F}^*$. Falmagne identifies a finite number (depending on the number of available alternatives) of nonnegativity conditions as necessary and sufficient for random utility maximization.[4]

DEFINITION: For any RCR ρ, define the difference function Δ of ρ inductively as $\Delta_x(\emptyset, D) = \rho^D(x)$ for all $x \in D$ and $D \subset Y^*$. Let $\Delta_x(A \cup \{y\}, D) = \Delta_x(A, D) - \Delta_x(A, D \cup \{y\})$ for any $A, D \subset Y^*$ such that $x \in D$, $A \cap D = \emptyset$, and $y \in Y^* \setminus (A \cup D)$.

Falmagne (1978) shows that the RCR ρ maximizes some $\mu \in \Pi^*$ if and only if $\Delta_x(A, Y \setminus A) \geq 0$ for all A and $x \in Y \setminus A$. This condition turns out to be equivalent to $\Delta_x(A, D) \geq 0$ for all x, A, D such that $A \cap D = \emptyset$ and $x \in D$.

Note that for $A = \{y\}$, the condition $\Delta_x(A, D) \geq 0$ for all $x \in D$, $y \notin D$ corresponds to our monotonicity assumption and says that the probability of choosing x from D is at least as high as the probability of choosing x from $D \cup \{y\}$. These conditions also require that the difference in the probabilities between choosing x from D and $D \cup \{y\}$ does not increase as alternative z is added to D and that analogous higher order differences are nonincreasing as well. While monotonicity is a straightforward (necessary) condition, the higher order conditions are more difficult to interpret.

We can relate our theorem to Falmagne's if we interpret Y to be the set of extreme points of our simplex of lotteries P. Suppose Falmagne's conditions are satisfied and hence a RCR (on $\mathcal{D}^*$) maximizes some RUF μ. We can extend this RUF μ to a RUF $\hat{\mu}$ on our algebra $\mathcal{F}$ (i.e., the algebra generated by the normal cones $N(D, x)$) by choosing a single u from each $[u]$ and setting $\hat{\mu}(\{\lambda u | \lambda \geq 0\}) = \mu([u])$, where $[u]$ is the (equivalence) class of utility functions ordinally equivalent to u. Hence, $\hat{\mu}$ is a RUF on $\mathcal{F}$ that assigns positive probability to a finite number of rays and zero probability to all cones that do not contain one of those rays. By utilizing our Theorems S2(i) and S1, we can construct some monotone, mixture continuous, linear, and extreme $\hat{\rho}$ that maximize $\hat{\mu}$. This $\hat{\rho}$ must agree with ρ whenever $D \subset P$ consists of degenerate lotteries. Hence, any RCR that satisfies Falmagne's conditions can be extended to a RCR over lotteries that satisfy our conditions. Conversely, if a Falmagne RCR can be extended to a RCR on $\mathcal{F}$ that satisfies our conditions, then by Theorem 2, this RCR maximizes a regular RUF. It follows that the restriction of this RCR to sets of degenerate lotteries maximizes a Falmagne RUF and satisfies the conditions above. Thus, Falmagne's conditions are necessary and sufficient for a random choice rule over a finite set to have a monotone, mixture continuous, linear, and extreme extension to the set of all lotteries over that set.

Dept. of Economics, Princeton University, Princeton, NJ 08544, U.S.A.; pesendor@princeton.edu.

[4]Regenwetter and Marley (2001) show that Falmagne's definition of a random utility is equivalent to the definition used in this paper.