Econometrica Supplementary Material

SUPPLEMENT TO "SEQUENTIAL VETO BARGAINING WITH INCOMPLETE INFORMATION"

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S. NAGEEB ALI
Department of Economics, Pennsylvania State University

NAVIN KARTIK
Department of Economics, Columbia University

ANDREAS KLEINER
Department of Economics, Arizona State University

APPENDIX C: SUPPLEMENTARY APPENDIX (FOR ONLINE PUBLICATION)

THIS SUPPLEMENTARY APPENDIX PROVIDES PROOFS for the lemmas stated in Appendix B.2. To reduce notation, we denote $S(v) := \overline{P}(t(v))$.

LEMMA 8: *For any v and $z < y \in T(v)$, we have $\overline{P}(z) < \overline{P}(y)$.*

PROOF: Suppose that there are v and $z < y$ such that $\overline{P}(z) \geq \overline{P}(y)$. We prove that $y \notin T(v)$. Since $\overline{P}$ is increasing, it is constant on $[z, y]$; call that value $\bar{p}$. It follows that

$$\begin{aligned}
&u(\bar{p})\big[F(v) - F(y)\big] + \delta R(y) \\
&\quad \leq u(\bar{p})\big[F(v) - F(y)\big] + \delta\big\{u(\bar{p})\big[F(y) - F(z)\big] + R(z)\big\} \\
&\quad < u(\bar{p})\big[F(v) - F(z)\big] + \delta R(z),
\end{aligned}$$

where the first inequality is because the payoff from any type in $[z, y]$ is at most $u(\bar{p})$ (and hence $R(y) - R(z) \leq u(\bar{p})[F(y) - F(z)]$). Thus, $y \notin T(v)$. *Q.E.D.*

Below, we will use the fact that T is upper hemicontinuous. This follows from the generalized theorem of the maximum in Ausubel and Deneckere (1989, p. 527). The theorem is applicable because: (i) the maximand function $u(\overline{P}(y))[F(v) - F(y)] + \delta R(y)$ is upper semicontinuous as a function of y for every v, which in turn is because $\overline{P}$ is upper semicontinuous, and u and F are continuous and increasing on the relevant range $\{y : y \leq v \text{ and } \overline{P}(y) \leq 1\}$;[1] and (ii) for any sequence $v_n \to v$, the maximand function converges uniformly.

PROOF OF LEMMA 5: Step 1: We begin by specifying beliefs and strategies:

- μ is derived from Bayes' rule whenever possible; if at history $h = (h', a)$ a probability 0 rejection occurs, $\mu(h)$ puts probability 1 on $\overline{v}$ if $\overline{v} \leq 1/2$ and probability 1 on 0 if $\overline{v} > 1/2$ (in the latter case, $\underline{v} \leq 0$ by assumption);

S. Nageeb Ali: nageeb@psu.edu
Navin Kartik: nkartik@columbia.edu
Andreas Kleiner: andreas.kleiner@asu.edu

[1]There is no loss in restricting attention to this range by a similar argument to that in the proof of Lemma 8.

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- At any history $h = (h', a)$, any Vetoer type not in the support of Proposer's current belief plays an arbitrary best response; type $v \geq 0$ in the support accepts a if and only if $a \in [0, \overline{P}(v)]$; type $v < 0$ in the support accepts if and only if $u_V(a, v) \geq u_V(0, v)$;
- Proposer's first offer is $S(\overline{v})$. To describe the rest of Proposer's strategy, consider any history $h = (h', a)$. Given Vetoer's strategy and the belief updating specified above, if Proposer holds a nondegenerate belief upon rejection of a then this belief equals $F_{[\underline{v},d]}$ for some d. We stipulate that if $a = \overline{P}(d) = P(d)$, then Proposer offers $S(d)$; if $a = \overline{P}(d) > P(d)$, then Proposer offers $\lim_{d'\uparrow d} S(d')$; if $a \in [\lim_{d'\uparrow d} \overline{P}(d'), \overline{P}(d))$, then Proposer randomizes between $\lim_{d'\uparrow d} S(d')$ and $S(d)$ so that type d is indifferent between a in the current period and the lottery in the next period; and for any $a \notin [\overline{P}(\underline{v}), \overline{P}(\overline{v})]$, Proposer offers $S(d)$. Finally, whenever Proposer's belief is degenerate on $x \geq 0$ ($x \in \{0, \overline{v}\}$), Proposer offers $\min\{2x, 1\}$ in all future periods.

Observe that at any history, Proposer's subsequent on path offers are decreasing, either trivially if the current belief is degenerate, or for any nondegenerate belief because the belief cutoffs are decreasing by definition and $\overline{P}$ and t are increasing.

Step 2: We verify that Proposer is playing a best response to Vetoer's strategy given beliefs μ. As this is obvious whenever he has a degenerate belief, assume he has a non-degenerate belief. As noted above, any such belief is of the form $F_{[\underline{v},d]}$ for some d. Proposer's strategy prescribes some randomization (possibly degenerate) between $S(d)$ and $\lim_{d'\uparrow d} S(d')$.

We first claim that $S(d)$ is an optimal proposal. Given Vetoer's strategy, $R(d)$ is an upper bound on Proposer's payoff. Furthermore, it follows from Lemma 8 that Vetoer's strategy has all types above $t(d)$ accepting $S(d)$ and all types strictly below rejecting. The claim follows.

We next claim that $\lim_{d'\uparrow d} S(d')$ is also an optimal proposal. Since T is upper hemicontinuous, $\lim_{d'\uparrow d} t(d') \in T(d)$. Hence, given Vetoer's strategy, $\overline{P}(\lim_{d'\uparrow d} t(d'))$ is an optimal proposal. It therefore suffices to show that $\lim_{d'\uparrow d} S(d') = \overline{P}(\lim_{d'\uparrow d} t(d'))$, or equivalently, $\lim_{d'\uparrow d} \overline{P}(t(d')) = \overline{P}(\lim_{d'\uparrow d} t(d'))$. Note that $\lim_{d'\uparrow d} \overline{P}(t(d')) \leq \overline{P}(\lim_{d'\uparrow d} t(d'))$ because t and $\overline{P}$ are increasing. But if $\lim_{d'\uparrow d} \overline{P}(t(d')) < \overline{P}(\lim_{d'\uparrow d} t(d'))$ then continuity of R and u and strict monotonicity of u in the relevant range imply the contradiction

$$
\begin{aligned}
R(d) &= u\Big(\lim_{d'\uparrow d} \overline{P}\big(t(d')\big)\Big)\Big[F(d) - F\Big(\lim_{d'\uparrow d} t(d')\Big)\Big] + \delta R\Big(\lim_{d'\uparrow d} t(d')\Big) \\
&< u\Big(\overline{P}\Big(\lim_{d'\uparrow d} t(d')\Big)\Big)\Big[F(d) - F\Big(\lim_{d'\uparrow d} t(d')\Big)\Big] + \delta R\Big(\lim_{d'\uparrow d} t(d')\Big) = R(d).
\end{aligned}
$$

All that remains is to verify that at a history $h = (h', a)$ with $a \in [\lim_{d'\uparrow d} \overline{P}(d'), \overline{P}(d))$, there is a randomization between $S(d)$ and $\lim_{d'\uparrow d} S(d')$ that makes type d indifferent between a in the current period and the lottery in the next period. To confirm this, note that since P is right-continuous and $P(v) \geq v$ for any v, we have

$$
u_V\Big(\lim_{d'\uparrow d} P(d'), d\Big) \geq u_V(a, d) \geq u_V\big(P(d), d\big).
$$

The existence of a suitable randomization now follows from continuity of $u_V(\cdot, d)$ and equation (7).

Step 3: We verify that Vetoer is playing a best response at each history. Consider any history (h, a) with $\mu(h) = F_{[\underline{v},q]}$. Since types outside of the support of Proposer's belief play a best response by assumption, we only consider types in $[\underline{v}, q]$.

- If $a > \overline{P}(q)$, Vetoer's strategy prescribes that no type below q accepts, and Proposer will propose $S(q)$ next period. Since type q is indifferent between $P(q)$ in the current period and $S(q)$ next period, and $S(q) \leq P(q) \leq \overline{P}(q) < a$, type q prefers $S(q)$ next period to a in the current period. The same holds for all lower types, and hence Vetoer is playing a best response.
- If $a < 0$, then: (i) it is clearly a best response for all types $v \geq 0$ to reject; and (ii) types $v < 0$ accept if and only if they prefer a to 0, which is a best response because Proposer will never make a strictly negative offer in the continuation equilibrium.
- If a is positive but below the range of $\overline{P}$, all types $v \geq 0$ accept. After a rejection, Proposer will either perpetually offer 0 or $2\overline{v}$, yielding a continuation payoff of 0 to all types, and so it is a best response for any type $v \geq 0$ to accept a.
- Otherwise, a is between $\overline{P}(\underline{v})$ and $\overline{P}(q)$.

  If $a = \overline{P}(d) = P(d)$ for some $d \leq q$, Vetoer's strategy prescribes that all and only those types above d accept.[2] On path, Proposer will propose $S(d)$ next period followed by lower offers; since type d is indifferent between a in the current period and $S(d)$ next period, and all future offers are below a, SCED implies that it is a best response for all higher types to accept and for all lower types to reject. Hence, Vetoer is playing a best response.

  If there is $d \leq q$ such that $a = \overline{P}(d) > P(d)$, Vetoer's strategy prescribes that all and only those types above d accept. Proposer will propose $\lim_{d' \uparrow d} S(d')$ next period, followed by lower offers. Since type d' is indifferent between $P(d')$ in the current period and $S(d')$ next period, continuity of u implies that type d is indifferent between $\lim_{d' \uparrow d} P(d') = \overline{P}(d) = a$ in the current period and $\lim_{d' \uparrow d} S(d')$ next period. Hence, Vetoer is playing a best response.

  If there is $d \leq q$ such that $a \in [\lim_{d' \uparrow d} \overline{P}(d'), \overline{P}(d))$, Vetoer's strategy again prescribes that all and only those types above d accept. Proposer will randomize next period between $\lim_{d' \uparrow d} S(d')$ and $S(d)$ to make type d indifferent between accepting a or getting the lottery next period. Therefore, Vetoer is playing a best response. *Q.E.D.*

PROOF OF LEMMA 6: Step 1: Suppose $\underline{v} > 0$. We claim that there is $\varepsilon > 0$ such that (R, P) given by

$$R(v) := u(2\underline{v})F(v),$$

$$P(v) := v + \sqrt{v^2 - 4\delta\underline{v}(v - \underline{v})}$$

supports a skimming equilibrium on $[\underline{v}, \underline{v} + \varepsilon]$. Plainly, R and P are continuous, given that F is continuous. Also, P is increasing, and hence $\overline{P} = P$. Some algebra confirms that $R(v)$ is the value from securing acceptance from all types below v on action $2\underline{v}$, while $P(v)$ is the action that makes type v indifferent between accepting that action now and getting action $2\underline{v}$ in the next period. Therefore, it is sufficient for us to show that there is $\varepsilon > 0$ such that for all $v \in [\underline{v}, \underline{v} + \varepsilon]$ the unique maximizer of the RHS of equation (6) is $\underline{v}$, which implies $t(v) = \underline{v}$.

To that end, observe that the derivative of the objective function in equation (6) with respect to y is

$$u'\big(\overline{P}(y)\big)\overline{P}'(y)\big[F(v) - F(y)\big] - u\big(\overline{P}(y)\big)f(y) + \delta u(2\underline{v})f(y). \tag{11}$$

[2]If there are multiple values of d satisfying $a = \overline{P}(d)$, all types above the lowest one accept.

Since $0 < u(2\underline{v}) \le u(\overline{P}(y))$ and f is bounded away from 0, the sum of the last two terms in expression (11) is strictly negative and bounded away from 0. Since $u'(\overline{P}(y))$ is bounded (by concavity), $\overline{P}'(y)$ is bounded (as $v^2 - 4\delta\underline{v}(v-\underline{v}) > 0$ for all v), F is continuous, and $v, y \in [\underline{v}, \underline{v}+\varepsilon]$, the first term in expression (11) goes to 0 as $\varepsilon \to 0$. It follows that there is $\varepsilon > 0$ such that expression (11) is strictly negative for all $y \in [\underline{v}, \underline{v}+\varepsilon]$, and hence the maximum of the RHS of equation (6) is attained uniquely at $t(v) = \underline{v}$ whenever $v \le \underline{v}+\varepsilon$.

Step 2: Suppose (R_{v^*}, P_{v^*}) supports a skimming equilibrium on $[\underline{v}, v^*]$, where $0 < \underline{v} < v^* < \overline{v}$. We will show that there is (R, P) that supports a skimming equilibrium on $[\underline{v}, \overline{v}]$ with the property that $P(v) = P_{v^*}(v)$ and $R(v) = R_{v^*}(v)$ for all $v \in [\underline{v}, v^*]$.

Pick $v' \in (v^*, \overline{v}]$ as large as possible such that

$$u(1)\big[F(v') - F(v^*)\big] \le (1/2)(1-\delta)R_{v^*}(v^*). \tag{12}$$

Note that v' is well-defined because F is continuous and $R_{v^*}(v^*) > 0$ (this inequality holds because of $v^* > \underline{v}$ and the property noted at the end of the paragraph following Definition 1). Moreover, letting $\overline{f}$ denote an upper bound for f, it holds that

$$v' - v^* \ge \frac{(1/2)(1-\delta)R_{v^*}(v^*)}{u(1)\overline{f}} > 0. \tag{13}$$

We extend R_{v^*} to $R_{v'}$ defined on $[\underline{v}, v']$ by setting $R_{v'}(v) := R_{v^*}(v)$ for $v \in [\underline{v}, v^*]$, and for $v \in (v^*, v']$,

$$R_{v'}(v) := \max_{y\in[\underline{v},v^*]} \big\{u(\overline{P}_{v^*}(y))\big[F(v) - F(y)\big] + \delta R_{v^*}(y)\big\}$$

and define $t_{v'}(v)$ to be the largest value in the argmax correspondence. Observe that $\overline{P}_{v^*}$ is upper semicontinuous (since P_{v^*} is right-continuous by assumption, and hence $\overline{P}_{v^*}$ is right-continuous) and R_{v^*} is continuous; hence, $R_{v'}(v)$ and $t_{v'}(v)$ are well-defined. We extend P_{v^*} to $P_{v'}$ defined on $[\underline{v}, v']$ by setting $P_{v'}(v) := P_{v^*}(v)$ for $v \in [\underline{v}, v^*]$, and for $v \in (v^*, v']$ by letting $P_{v'}(v)$ be the largest value satisfying

$$u_V\big(P_{v'}(v), v\big) = \delta u_V\big(\overline{P}_{v^*}\big(t_{v'}(v)\big), v\big).$$

So $(R_{v'}, P_{v'})$ satisfies equation (7). We can apply the generalized theorem of the maximum in Ausubel and Deneckere (1989, p. 527) analogously to the discussion after Lemma 8 and conclude that $R_{v'}$ is continuous and $T_{v'}$ is nonempty and upper hemicontinuous. Therefore, $t_{v'}$ is upper semicontinuous, and since it is increasing, right-continuous. These properties of $t_{v'}$ and the hypothesis that P_{v^*} is right-continuous imply that $P_{v'}$ is right-continuous. $(R_{v'}, P_{v'})$ also satisfies equation (6), that is,

$$R_{v'}(v) = \max_{y\in[\underline{v},v]} \big\{u(\overline{P}_{v'}(y))\big[F(v) - F(y)\big] + \delta R_{v'}(y)\big\}$$

for all $v \in [\underline{v}, v']$, because for all $y \in [v^*, v]$,

$$\begin{aligned} &u(\overline{P}_{v'}(y))\big[F(v) - F(y)\big] + \delta R_{v'}(y) \\ &\quad \le u(1)\big[F(v) - F(y)\big] + \delta R_{v'}(y) \\ &\quad \le (1/2)(1-\delta)R_{v^*}(v^*) + \delta R_{v'}(y) \end{aligned}$$

$$\le (1/2)(1-\delta)R_{v'}(y) + \delta R_{v'}(y)$$
$$< R_{v'}(v).$$

Here, the second inequality is because the choice of v' satisfies inequality (12) and the second inequality is because $R_{v^*}(v^*) = R_{v'}(v^*)$ and $R_{v'}$ is increasing. Therefore, the maximum is attained for $y \in [\underline{v}, v^*)$ and the claim follows since $R_{v'}(y) = R_{v^*}(y)$ for any such y.

We have established that $(R_{v'}, P_{v'})$ supports a skimming equilibrium on $[\underline{v}, v']$. Since $R_{v'}$ is increasing, it follows from inequality (13) that a finite number of repetitions of this argument extends (R_{v^*}, P_{v^*}) to the entire $[\underline{v}, \overline{v}]$ interval.

Step 3: By an approximation argument analogous to that in Ausubel and Deneckere (1989, Theorem 4.2), there exists (R, P) that supports a skimming equilibrium on $[\underline{v}, \overline{v}]$ if $\underline{v} = 0$; we omit details. The case of $\underline{v} < 0$ is handled by setting $R(v) = 0$ and $P(v) = 0$ for all $v < 0$, and pasting that to a solution when we take $\underline{v} = 0$ and set the distribution on $[0, \overline{v}]$ to be the conditional distribution $F_{[0,\overline{v}]}$. *Q.E.D.*

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