SUPPLEMENTAL MATERIAL TO "LONG-TERM RISK: A MARTINGALE APPROACH"

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A. PROOFS FOR SECTION 3

WE FIRST RECALL SOME RESULTS about semimartingale topology originally introduced by Émery (1979) (see Czichowsky and Schweizer (2006), Kardaras (2013), and Cuchiero and Teichmann (2015) for recent applications in mathematical finance). The semimartingale topology is stronger than the topology of uniform convergence in probability on compacts (ucp). In the latter case, the supremum in Eq. (2.1) is only taken over integrands in the form $\eta_t = 1_{[0,s]}(t)$ for every $s > 0$:

$$d_{\mathrm{ucp}}(X, Y) = \sum_{n\geq 1} 2^{-n}\mathbb{E}^{\mathbb{P}}\Big[1 \wedge \sup_{s\leq n} |X_s - Y_s|\Big].$$

The following inequality due to Burkholder is useful for proving convergence in the semimartingale topology in Theorem 3.1 (see Meyer (1972, Theorem 47, p. 50) for discrete martingales and Cuchiero and Teichmann (2015) for continuous martingales, where a proof is provided inside the proof of their Lemma 4.7).

LEMMA A.1: *For every martingale X and every predictable process η bounded by* 1, $|\eta_t| \leq 1$, *it holds that*

$$a\mathbb{P}\bigg(\sup_{s\in[0,t]}\bigg|\int_0^s \eta_u\, dX_u\bigg| > a\bigg) \leq 18\mathbb{E}^{\mathbb{P}}\big[|X_t|\big]$$

for all $a \geq 0$ and $t > 0$.

We will also use the following result (see Kardaras (2013, Proposition 2.10)).

LEMMA A.2: *If $X^n \xrightarrow{\mathcal{S}} X$ and $Y^n \xrightarrow{\mathcal{S}} Y$, then $X^nY^n \xrightarrow{\mathcal{S}} XY$.*

We will also make use of the following lemma.

LEMMA A.3: *Let $(X_t^n)_{t\geq 0}$ be a sequence of martingales such that $X_t^n \xrightarrow{\mathrm{L}^1} X_t^\infty$ for each $t \geq 0$. Then $(X_t^\infty)_{t\geq 0}$ is a martingale.*

PROOF: It is immediate that $\mathbb{E}[|X_t^\infty|] < \infty$ for all t. We need to verify that $\mathbb{E}_s[X_t^\infty] = X_s^\infty$ for $t > s \geq 0$. First we show that, from $X_t^n \xrightarrow{\mathrm{L}^1} X_t^\infty$, it follows that

$$\mathbb{E}_s\big[X_t^n\big] \xrightarrow{\mathrm{L}^1} \mathbb{E}_s\big[X_t^\infty\big] \tag{A.1}$$

for each $s < t$. By Jensen's inequality, for each $0 \leq s < t$, we have $|\mathbb{E}_s[X_t^n - X_t^\infty]| \leq \mathbb{E}_s[|X_t^n - X_t^\infty|]$. Taking expectations on both sides, we have

$$\mathbb{E}\big|\mathbb{E}_s\big[X_t^n\big] - \mathbb{E}_s\big[X_t^\infty\big]\big| \leq \mathbb{E}_s\big[\big|X_t^n - X_t^\infty\big|\big].$$

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Thus, $X_t^n \xrightarrow{\mathrm{L}^1} X_t^\infty$ implies $\mathbb{E}_s[X_t^n] \xrightarrow{\mathrm{L}^1} \mathbb{E}_s[X_t^\infty]$ for each $s < t$.

Since X_t^n are martingale, $\mathbb{E}_s[X_t^n] = X_s^n$. By (A.1) for $t \geq s$, $X_s^n \xrightarrow{\mathrm{L}^1} \mathbb{E}_s[X_t^\infty]$. On the other hand, $X_s^n \xrightarrow{\mathrm{L}^1} X_s^\infty$. Thus, $\mathbb{E}_s[X_t^\infty] = X_s^\infty$ for $t > s$, hence X_t^∞ is a martingale. *Q.E.D.*

PROOF OF THEOREM 3.1: (i) It is easy to see that Eq. (3.1) implies M_t^T converges to M_t^∞ in L^1 under $\mathbb{P}$. Since $(M_t^T)_{t\geq 0}$ are positive $\mathbb{P}$-martingales with $M_0^T = 1$, and for each $t \geq 0$ random variables M_t^T converge to $M_t^\infty > 0$ in L^1, by Lemma A.3 $(M_t^\infty)_{t\geq 0}$ is also a positive $\mathbb{P}$-martingale with $M_0^\infty = 1$. Emery's distance between the martingale M^T for some $T > 0$ and M^∞ is

$$d_{\mathcal{S}}\big(M^T, M^\infty\big) = \sum_{n\geq 1} 2^{-n} \sup_{|\eta|\leq 1} \mathbb{E}^{\mathbb{P}}\left[1 \wedge \left|\int_0^n \eta_s \, d\big(M^T - M^\infty\big)_s\right|\right].$$

To prove $M^T \xrightarrow{\mathcal{S}} M^\infty$, it suffices to prove that for all n,

$$\lim_{T\to\infty} \sup_{|\eta|\leq 1} \mathbb{E}^{\mathbb{P}}\left[1 \wedge \left|\int_0^n \eta_s \, d\big(M^T - M^\infty\big)_s\right|\right] = 0. \tag{A.2}$$

We can write for an arbitrary $\varepsilon > 0$ (for any random variable X, it holds that $\mathbb{E}[1 \wedge |X|] \leq \mathbb{P}(|X| > \varepsilon) + \varepsilon$):

$$\mathbb{E}^{\mathbb{P}}\left[1 \wedge \left|\int_0^n \eta_s \, d\big(M^T - M^\infty\big)_s\right|\right] \leq \mathbb{P}\left(\left|\int_0^n \eta_s \, d\big(M^T - M^\infty\big)_s\right| > \varepsilon\right) + \varepsilon.$$

By Lemma A.1,

$$\sup_{|\eta|\leq 1} \mathbb{E}^{\mathbb{P}}\left[1 \wedge \left|\int_0^n \eta_s \, d\big(M^T - M^\infty\big)_s\right|\right] \leq \frac{18}{\varepsilon} \mathbb{E}^{\mathbb{P}}\big[\big|M_n^T - M_n^\infty\big|\big] + \varepsilon.$$

Since $\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}}[|M_n^T - M_n^\infty|] = 0$, and ε can be taken arbitrarily small, Eq. (A.2) is verified and, hence, $M^T \xrightarrow{\mathcal{S}} M^\infty$.

(ii) We have shown that $S_t B_t^T = M_t^T \xrightarrow{\mathcal{S}} M_t^\infty := S_t B_t^\infty$. By Lemma A.2, $B_t^T \xrightarrow{\mathcal{S}} B_t^\infty$, and B_t^∞ is the long bond according to Definition 3.1 (the semimartingale convergence is stronger than the ucp convergence).

Part (iii) is a direct consequence of (i) and (ii).

(iv) Define a new probability measure $\mathbb{Q}^\infty$ by $\mathbb{Q}^\infty|_{\mathcal{F}_t} = M_t^\infty \mathbb{P}|_{\mathcal{F}_t}$ for each $t \geq 0$. The distance in total variation between the measure $\mathbb{Q}^T$ for some $T > 0$ and $\mathbb{Q}^\infty$ on $\mathcal{F}_t$ is

$$2 \sup_{A\in\mathcal{F}_t} \big|\mathbb{Q}^T(A) - \mathbb{Q}^\infty(A)\big|.$$

For each $t \geq 0$, we can write

$$0 = \lim_{T\to\infty} \mathbb{E}^{\mathbb{P}}\big[\big|M_t^T - M_t^\infty\big|\big] = \lim_{T\to\infty} \mathbb{E}^{\mathbb{P}}\big[M_t^\infty \big|B_t^T/B_t^\infty - 1\big|\big] = \lim_{T\to\infty} \mathbb{E}^{\mathbb{Q}^\infty}\big[\big|B_t^T/B_t^\infty - 1\big|\big].$$

Thus,

$$\lim_{T\to\infty} \sup_{A\in\mathcal{F}_t} \big|\mathbb{E}^{\mathbb{Q}^\infty}\big[\big(B_t^T/B_t^\infty\big)\mathbf{1}_A\big] - \mathbb{E}^{\mathbb{Q}^\infty}[\mathbf{1}_A]\big| = 0.$$

Since $\frac{d\mathbb{Q}^T}{d\mathbb{Q}^\infty}|_{\mathcal{F}_t} = \frac{B_t^T}{B_t^\infty}$, it follows that

$$\lim_{T\to\infty} \sup_{A\in\mathcal{F}_t} \left| \mathbb{E}^{\mathbb{Q}^T}[\mathbf{1}_A] - \mathbb{E}^{\mathbb{Q}^\infty}[\mathbf{1}_A] \right| = 0.$$

Thus, $\mathbb{Q}^T$ converge to $\mathbb{Q}^\infty$ in total variation on $\mathcal{F}_t$ for each t. Since convergence in total variation implies strong convergence of measures, this shows that $\mathbb{Q}^\infty$ is the long forward measure according to Definition 3.2, $\mathbb{Q}^\infty = \mathbb{L}$. *Q.E.D.*

PROOF OF THEOREM 3.2: (i) Define functions $h(t) := P_0^{\log t}$ and $g(t) := \lim_{T\to\infty} P_0^{T-t}/sP_0^T$ (the latter is defined for each t due to our assumption). Then, for all $0 < a < 1$,

$$\lim_{t\to\infty} \frac{h(at)}{h(t)} = \lim_{t\to\infty} \frac{P_0^{\log at}}{P_0^{\log t}} = \lim_{t\to\infty} \frac{P_0^{\log t + \log a}}{P_0^{\log t}} = g(-\log a).$$

Thus, $h(t)$ is a regularly varying function (see Bingham, Goldie, and Teugels (1989)). By Karamata's characterization theorem (see Bingham, Goldie, and Teugels (1989, Theorem 1.4.1)), there exists a real number λ such that $\lim_{t\to\infty} \frac{h(at)}{h(t)} = g(-\log a) = a^{-\lambda}$ and a slowly varying function $L(t)$ such that $h(t) = t^{-\lambda}L(t)$. Rewriting it gives $g(t) = e^{\lambda t}$ and $P_0^t = e^{-\lambda t}L(e^t)$.

(ii) By Eq. (3.1), $S_t P_t^T/P_0^T$ converges to M_t^∞ in L^1 under $\mathbb{P}$ as $T \to \infty$. Thus, it also converges in probability under $\mathbb{P}$, as well as under any measure locally equivalent to $\mathbb{P}$. Hence, P_t^T/P_0^T, as well as $\log(P_t^T/P_0^T)$, converge in probability (from now on, as well as in the proofs of Theorems 3.3–3.5, we omit explicit dependency on the probability measure when we talk about convergence in probability, since it holds under all locally equivalent measures). Thus, $\log(P_t^T/P_0^T)/(T-t)$ converges to zero in probability. Since $P_0^T = e^{-\lambda T}L(e^T)$ and for any slowly varying function $\lim_{T\to\infty} \frac{1}{T-t}\log(L(e^T)) = 0$ (see Proposition 1.3.6 of Bingham, Goldie, and Teugels (1989)), we have

$$\lim_{T\to\infty} \log P_0^T/(T-t) = -\lambda.$$

Combining these two properties yields (ii).

(iii) and (iv) By Bingham, Goldie, and Teugels (1989, Theorem 1.2.1), P_0^T/P_0^{T-t} converges to $1/g(t)$ as $T \to \infty$ uniformly on compacts, and thus also in semimartingale topology. By Theorem 3.1, B_t^T (and thus P_t^T/P_0^T) converges to B_t^∞ in semimartingale toplogy. Thus, by Lemma A.2, the ratio P_t^T/P_0^{T-t} converges in semimartingale topology as $T \to \infty$, and we denote the limit π_t. The decomposition of B_t^∞ is then immediate.

(v) Since $M_t^\infty = S_t B_t^\infty = S_t \pi_t e^{\lambda t}$ is a martingale, we have $\mathbb{E}_t^{\mathbb{P}}[S_T \pi_T e^{\lambda T}] = S_t \pi_t e^{\lambda t}$. Rewriting it yields Eq. (3.5). Combining the fact that $P_t^T = \mathbb{E}_t^{\mathbb{L}}[B_t^\infty/B_T^\infty] = e^{-\lambda(T-t)}\mathbb{E}_t^{\mathbb{L}}[\pi_t/\pi_T]$ and Eq. (3.3) yields Eq. (3.6). *Q.E.D.*

PROOF OF THEOREM 3.3: (i) By assumption, we have, for $T > T'$,

$$\frac{c}{C\mathbb{E}_t^{\mathbb{L}}[1/\pi_T]} < \frac{\mathbb{E}_t^{\mathbb{L}}[C_T]}{\mathbb{E}_t^{\mathbb{L}}[C_T/\pi_T]} < \frac{C}{c\mathbb{E}_t^{\mathbb{L}}[1/\pi_T]}.$$

Combining it with Eq. (3.6) yields that $\log(\frac{\mathbb{E}_t^{\mathbb{L}}[C_T]}{\mathbb{E}_t^{\mathbb{L}}[C_T/\pi_T]})/(T-t)$ converges to zero in probability. Substituting it into Eq. (3.7), we arrive at part (i). Part (ii) is proved similarly. *Q.E.D.*

PROOF OF THEOREM 3.4: (i) Since $P_0^T = O(t^{-\gamma})$, $\lambda = 0$ and $B_t^\infty = \pi_t$. Similarly to the proof of Theorem 3.2, $(-\log P_t^T)/\log(T-t)$ converges to γ in probability as $T \to \infty$. Since $P_t^T = \mathbb{E}_t^{\mathbb{L}}[B_t^\infty / B_T^\infty] = \mathbb{E}_t^{\mathbb{L}}[\pi_t/\pi_T]$, we have $(-\log \mathbb{E}_t^{\mathbb{L}}[1/\pi_T])/\log(T-t)$ converges to γ in probability. By assumption, we have, for $T > T'$,

$$\frac{c}{C\mathbb{E}_t^{\mathbb{L}}[1/\pi_T]} < \frac{\mathbb{E}_t^{\mathbb{L}}[C_T]}{\mathbb{E}_t^{\mathbb{L}}[C_T/\pi_T]} < \frac{C}{c\mathbb{E}_t^{\mathbb{L}}[1/\pi_T]}.$$

Thus, $\log(\frac{\mathbb{E}_t^{\mathbb{L}}[C_T]}{\mathbb{E}_t^{\mathbb{L}}[C_T/\pi_T]})/\log(T-t)$ converges to γ in probability. Part (ii) can be proved similarly. *Q.E.D.*

PROOF OF THEOREM 3.5: By assumptions and Eq. (3.8), we can write, by changing the probability measure to $\mathbb{G}$,

$$\rho_{t,T}^{\mathbb{L}}(G_T) = \lambda + \frac{1}{T-t}\log\left(\frac{\mathbb{E}_t^{\mathbb{G}}[\pi_T/\pi_T^G]}{\pi_t \mathbb{E}_t^{\mathbb{G}}[1/\pi_T^G]}\right).$$

By assumption, we immediately have Eq. (3.9). *Q.E.D.*

B. DISCRETE-TIME ENVIRONMENT

We will show how the results of Alvarez and Jermann (2005) in discrete-time environments are naturally nested in our Theorems 3.1 and 3.2. Alvarez and Jermann (2005) worked in discrete time with the pricing kernel S_t, $t = 0, 1, \ldots$, and made the following assumptions (below, $P_t^{t+\tau}$ is the time-t price of a pure discount bond with maturity at time $t+\tau$ and unit face value, where $t, \tau = 0, 1, \ldots$).

ASSUMPTION B.1—Alvarez and Jermann (2005, Assumptions 1 and 2): (i) *There exists a constant* λ *such that* $0 < \lim_{\tau\to\infty} e^{\lambda\tau} P_t^{t+\tau} < \infty$ *almost surely for all* $t = 0, 1, \ldots$.

(ii) *For each* $t = 1, 2, \ldots$, *there exists a random variable* x_t *with* $\mathbb{E}_{t-1}^{\mathbb{P}}[x_t] < \infty$ *such that* $e^{\lambda(t+\tau)} S_t P_t^{t+\tau} \le x_t$ *almost surely for all* $\tau = 0, 1, \ldots$.

Any discrete-time adapted process can be embedded into a continuous-time semimartingale as follows. For a discrete-time process $(X_t, t = 0, 1, \ldots)$, define a continuous-time process $(\tilde{X}_t)_{t\ge 0}$ such that, at integer times, it takes the same values as the discrete-time process X, and is piece-wise constant between integer times, that is, $\tilde{X}_t = X_{[t]}$, where $[t]$ denotes the integer part (floor) of t. This process has RCLL paths and is a semimartingale (it is of finite variation). The following result shows that Proposition 1 in Alvarez and Jermann (2005) is nested in Theorems 3.1 and 3.2.

PROPOSITION B.1: *Consider a discrete-time positive pricing kernel* $(S_t, t = 0, 1, \ldots)$ *with* $\mathbb{E}^{\mathbb{P}}[S_{t+\tau}/S_t] < \infty$ *for all* t, τ. *Suppose pure discount bonds* $P_t^{t+\tau} = \mathbb{E}_t^{\mathbb{P}}[S_{t+\tau}/S_t]$ *satisfy Assumption* B.1. *Then the corresponding continuous-time positive semimartingale pricing kernel* $(\tilde{S}_t)_{t\ge 0}$ *satisfies the conditions in Theorem* 3.1 *and Theorem* 3.2; *hence, all results in Theorem* 3.1 *and Theorem* 3.2 *hold*.

PROOF: We first prove Eq. (3.1) is satisfied. We first consider integer values of t and τ. By assumption (ii), we have

$$S_t \frac{P_t^{t+\tau}}{P_0^{t+\tau}} e^{\lambda(t+\tau)} P_0^{t+\tau} \le x_t.$$

Recall that

$$M_t^{t+\tau} = \frac{\mathbb{E}_t^{\mathbb{P}}[S_{t+\tau}]}{\mathbb{E}^{\mathbb{P}}[S_{t+\tau}]} = S_t \frac{P_t^{t+\tau}}{P_0^{t+\tau}}$$

for $t, \tau \geq 0$. Thus, we have that

$$M_t^{t+\tau} e^{\lambda(t+\tau)} P_0^{t+\tau} \leq x_t$$

for $t, \tau \geq 0$. By assumption (i), $e^{\lambda T} P_0^T$ has a positive finite limit as $T \to \infty$. Thus, there exists a constant $c > 0$ such that $e^{\lambda T} P_0^T > c$ for all T. Hence, $M_t^{t+\tau} \leq c^{-1} x_t$ for $t, \tau \geq 0$.

Furthermore, for all $t, \tau \geq 0$, we can write

$$M_t^{t+\tau} = S_t \frac{P_t^{t+\tau}}{P_0^{t+\tau}} = e^{\lambda t} S_t \frac{e^{\lambda \tau} P_t^{t+\tau}}{e^{\lambda(t+\tau)} P_0^{t+\tau}}.$$

By assumption (i), $\lim_{\tau\to\infty} M_t^{t+\tau}$ exists almost surely and is positive. We denote it M_t^∞. Since $M_t^{t+\tau} \leq c^{-1} x_t$ and x_t is integrable, by the dominated convergence theorem we have that $M_t^{t+\tau} \to M_t^\infty$ in L^1 as $\tau \to \infty$ for each fixed integer $t = 0, 1, \ldots$.

We now consider real values of t and τ and recall our embedding of discrete-time adapted processes into continuous semimartingales with piece-wise constant paths. For each real t and τ, we have $M_t^{t+\tau} = M_n^N$, where n, N are two integers such that $t \in [n, n+1)$ and $t + \tau \in [N, N+1)$. Thus, $M_t^{t+\tau} \to M_t^\infty$ in L^1 as $\tau \to \infty$ for each fixed real $t \geq 0$. This prove Eq. (3.1).

P_0^{T-t}/P_0^T converges for all $t > 0$ is a simple consequence of the fact that $e^{\lambda t} P_0^t$ converges for all $t > 0$. *Q.E.D.*

C. PROOFS FOR SECTION 4

We start with proving the following measurability property of the bond pricing function $P(t, x)$ under Assumption 4.1.

LEMMA C.1: *If the pricing kernel S_t satisfies Assumption* 4.1, *then the bond pricing function $P(t, x) = \mathbb{E}_x^{\mathbb{P}}[S_t]$ is jointly measurable with respect to $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E}^*$, where $\mathcal{B}(\mathbb{R}_+)$ is the Borel σ-algebra on $\mathbb{R}_+$, $\mathcal{E}^*$ is the σ-algebra of universally measurable sets on E (see Sharpe* (1988, *p.* 1)).

PROOF: Let $P^n(t, x) = \mathbb{E}_x^{\mathbb{P}}[S_t \wedge n]$. By Chen and Fukushima (2011, Exercise A.1.20), for fixed t, $P^n(t, x)$ is $\mathcal{E}^*$-measurable. Since S_t is right-continuous, by the bounded convergence theorem for fixed x the function $P^n(t, x)$ is right-continuous in t. Thus, on $[0, 1) \times E$, we can write

$$P^n(t, x) = \lim_{m\to\infty} P_m^n(t, x),$$

where

$$P_m^n(t, x) := \sum_{i=1}^m 1_{[(i-1)/m, i/m)}(t) P^n\big((i-1)/m, x\big). \tag{C.1}$$

Thus, on $[0, 1) \times E$, the function $P_m^n(t, x)$ is jointly measurable with respect to $\mathcal{B}([0, 1)) \otimes \mathcal{E}^*$. Similarly, we can prove that $P_m^n(t, x)$ is jointly measurable with respect to $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E}^*$.

By Eq. (C.1), $P^n(t,x)$ is then also jointly measurable with respect to $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{E}^*$. Since S_t is integrable, by the dominated convergence theorem $\lim_{n\to\infty} P^n(t,x)=P(t,x)$. Thus, $P(t,x)$ is also jointly measurable with respect to $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{E}^*$. *Q.E.D.*

Next, we prove the following result.

LEMMA C.2: *Suppose the PK S satisfies Assumption* 4.1 *and Eq.* (3.1) *holds under $\mathbb{P}_x$ for each $x\in E$. Then, for each $t>0$ and $x\in E$, we can write for the long bond*

$$B_t^\infty(x)=b^\infty(t,x,X_t)>0 \tag{C.2}$$

$\mathbb{P}_x$-almost surely, where $b^\infty(t,x,y)$ is a universally measurable function of y for each fixed $t>0$ and $x\in E$.

PROOF: The long bond $B_t^\infty(x)$ is the ucp limit of the processes $B_t^T(x)$ defined in Section 3. Dependence on the initial state $X_0=x$ comes from dividing by the initial bond price $P(0,x)$ at time zero in the definition of B_t^T. For each $t>0$ and $x\in E$, the random variables $B_t^T(x)=P_t^T(x)/P_0^T(x)=P(T-t,X_t)/P(T,x)$ with $T\geq t$ converge to $B_t^\infty(x)$ as $T\to\infty$ in probability. By Lemma C.1, $P(T-t,X_t)/P(T,x)$ is $\sigma(X_t)$-measurable (X_t is viewed as a random element taking values in E equipped with the σ-algebra $\mathcal{E}^*$, thus $\sigma(X_t)$ is generated by inverses of universally measurable sets). Its limit in probability $B_t^\infty(x)$ can also be taken $\sigma(X_t)$-measurable and, by the Doob–Dynkin lemma, we can write it as $b^\infty(t,x,X_t)$, where, for each fixed $t>0$ and $x\in E$, $b^\infty(t,x,y)$ is a universally measurable function of y. *Q.E.D.*

By Lemma C.2, for each $t>0$ and $x\in E$, the random variables $P(T-t,X_t)/P(T,x)$ converge to the random variable $b^\infty(t,x,X_t)$ in probability under $\mathbb{P}_x$. In Theorem 4.1, we strengthen it to pointwise convergence of the function $P(T-t,y)/P(T,x)$ as T goes to infinity, that is, for each $t>0$ and $x,y\in E$,

$$\lim_{T\to\infty}\frac{P(T-t,y)}{P(T,x)}=b^\infty(t,x,y)>0. \tag{C.3}$$

Now we are ready to prove Theorem 4.1.

PROOF OF THEOREM 4.1: By Lemma C.1, $P(t,x)$ is jointly measurable with respect to $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{E}^*$. Thus, by Eq. (C.3), $P(T-t,y)/P(T,x)$ is jointly measurable with respect to $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{E}^*\otimes\mathcal{E}^*$. Thus, the function $b^\infty(t,x,y)$ is also jointly measurable with respect to $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{E}^*\otimes\mathcal{E}^*$.

For any $t,s>0$ and $x,y,z\in E$, we can write

$$\begin{aligned} b^\infty(t+s,y,z)&=\lim_{T\to\infty}\frac{P(T-t,z)}{P(T+s,y)}=\lim_{T\to\infty}\frac{P(T,x)}{P(T+s,y)}\frac{P(T-t,z)}{P(T,x)}\\ &=b^\infty(s,y,x)b^\infty(t,x,z). \end{aligned} \tag{C.4}$$

Taking $x=y=z$ in Eq. (C.4), we have

$$b^\infty(t,x,x)b^\infty(s,x,x)=b^\infty(t+s,x,x),$$

which implies that, for each fixed $x\in E$, $b^\infty(t,x,x)$ satisfies Cauchy's multiplicative functional equation as a function of time. Since $b^\infty(t,x,y)$ is jointly measurable with respect to $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{E}^*\otimes\mathcal{E}^*$, for fixed x $\ln b^\infty(t,x,x)$ is measurable with respect to $\mathcal{B}(\mathbb{R}_+)$. It is known that a Borel measurable function that satisfies Cauchy's functional equation is linear. Thus, we have that $b^\infty(t,x,x)=e^{\lambda_L(x)t}$.

Again by Eq. (C.4), for any $x,y\in E$, we have

$$b^\infty(2t,y,x)=b^\infty(t,y,x)b^\infty(t,x,x)=b^\infty(t,y,y)b^\infty(t,y,x),$$

and we have $b^\infty(t,y,y)=b^\infty(t,x,x)$. Thus, $\lambda_L(x)$ is independent of x. Taking $y=x$ in Eq. (C.4), we have $b^\infty(t+s,x,z)=e^{\lambda_L s}b^\infty(t,x,z)$. Thus, $e^{-\lambda_L t}b^\infty(t,x,z)$ is independent of t. Fix $x_0\in E$ and define $\pi_L(x):=e^{-\lambda_L t}b^\infty(t,x_0,x)$. It is independent of t and x_0 is fixed. By Eq. (C.4), $b^\infty(t,x_0,x)b^\infty(t,x,x_0)=b^\infty(2t,x_0,x_0)=e^{-2\lambda_L t}$. Thus, $b^\infty(t,x,x_0)=e^{\lambda_L t}1/\pi_L(x)$. Finally, we have

$$b^\infty(t,x,y)=b^\infty(t/2,x,x_0)b^\infty(t/2,x_0,y)=e^{\lambda_L t}\frac{\pi_L(y)}{\pi_L(x)}.$$

By Eq. (C.2) we then have

$$B_t^\infty(x)=e^{\lambda_L t}\frac{\pi(X_t)}{\pi(x)}.$$

π_L is an eigenfunction of the pricing operators $\mathscr{P}_t$ with the eigenvalues $e^{-\lambda_L t}$ from the fact that $M_t^\infty=S_tB_t^\infty$ is a martingale. Thus, we arrive at the identification of the long forward measure with an eigen-measure associated with the eigenfunction π_L, and the identification $\mathbb{L}=\mathbb{Q}^{\pi_L}$ thus follows. *Q.E.D.*

REMARK C.1: We note the difference between the setting here and the one in Qin and Linetsky (2016). Here we do not assume that the pricing operator maps Borel functions to Borel functions upfront. Since the long bond $e^{\lambda_L t}\frac{\pi_L(X_t)}{\pi_L(x)}$ is a right-continuous semimartingale, by Çinlar, Jacod, Protter, and Sharpe (1980) the function π_L is locally the difference of two 1-excessive functions. For a Borel right process, its excessive functions are generally only universally measurable, but not necessarily Borel measurable. Thus the eigenfunction π_L we find above is also not necessarily Borel measurable, but is universally measurable. Hence, after the measure change from the data-generating measure to the long forward measure, under $\mathbb{L}=\mathbb{Q}^\pi$ the Markov process X may not be a Borel right process, but it is a right process. If we explicitly assume that the pricing operator maps Borel functions to Borel functions, as is done in Qin and Linetsky (2016), then the eigenfunction π_L is automatically Borel and X is a Borel right process under $\mathbb{Q}^{\pi_L}$. Here we opted for this slightly more general setup, so not to impose further restrictions on the pricing kernel.

PROOF OF THEOREM 4.2: Let $Q^{\pi_R}(t,x,\cdot)$ denote the transition measure of X under $\mathbb{Q}^{\pi_R}$. We verify the L^1 convergence condition Eq. (3.1) with $M_t^\infty=M_t^{\pi_R}$ with the martingale associated with the recurrent eigenfunction. This then identifies $e^{\lambda_R t}\frac{\pi_R(X_t)}{\pi_R(X_0)}$ with the long bond B_t^∞ and the recurrent eigen-measure with the long forward measure, $\mathbb{Q}^{\pi_R}=\mathbb{L}$.

We note that for any valuation process V, the condition (3.1) can be written under any locally equivalent probability measure $\mathbb{Q}^V$ defined by $\mathbb{Q}^V|_{\mathcal{F}_t}=S_tR_{0,t}^V\mathbb{P}|_{\mathcal{F}_t}$:

$$\lim_{T\to\infty}\mathbb{E}^{\mathbb{Q}^V}\left[\left|B_t^T/V_t-B_t^\infty/V_t\right|\right]=0. \tag{C.5}$$

We can use this freedom to choose the measure convenient for the setting at hand. Here we choose to verify it under $\mathbb{Q}^{\pi_R}$, that is, we will verify Eq. (C.5) under $\mathbb{Q}^V = \mathbb{Q}^{\pi_R}$ due to its convenient form. Since

$$P_t^T = e^{-\lambda_R(T-t)}\pi_R(X_t)\mathbb{E}_{X_t}^{\mathbb{Q}^{\pi_R}}\left[\frac{1}{\pi_R(X_{T-t})}\right],$$

we have

$$e^{-\lambda_R t}\frac{P_t^T\pi_R(X_0)}{P_0^T\pi_R(X_t)} = \frac{\mathbb{E}_{X_t}^{\mathbb{Q}^{\pi_R}}\left[\frac{1}{\pi_R(X_{T-t})}\right]}{\mathbb{E}_{X_0}^{\mathbb{Q}^{\pi_R}}\left[\frac{1}{\pi_R(X_T)}\right]}.$$

Let $J := \int_E \varsigma(dy)\frac{1}{\pi_R(y)}$ (it is finite by Assumption 4.2). Since

$$\mathbb{E}_x^{\mathbb{Q}^{\pi_R}}\left[\frac{1}{\pi_R(X_t)}\right] = \int_E Q^{\pi_R}(t,x,dy)\frac{1}{\pi_R(y)},$$

by Eq. (4.5) we have, for $T - t \geq t_0$,

$$J - \frac{c}{\pi_R(X_t)}e^{-\alpha(T-t)} \leq \mathbb{E}_{X_t}^{\mathbb{Q}^{\pi_R}}\left[\frac{1}{\pi_R(X_{T-t})}\right] \leq J + \frac{c}{\pi_R(X_t)}e^{-\alpha(T-t)}, \tag{C.6}$$

and for each initial state $X_0 = x \in E$ and $T \geq \max(T_0, t + t_0)$,

$$J - \frac{c}{\pi_R(x)}e^{-\alpha T} \leq \mathbb{E}_x^{\mathbb{Q}^{\pi_R}}\left[\frac{1}{\pi_R(X_T)}\right] \leq J + \frac{c}{\pi_R(x)}e^{-\alpha T}. \tag{C.7}$$

For each $x \in E$, there exists T_0 such that for $T \geq T_0$, $\frac{c}{\pi_R(x)}e^{-\alpha T} \leq J/2$. We can thus write, for each $x \in E$,

$$-1 \leq e^{-\lambda_R t}\frac{P_t^T\pi_R(x)}{P_0^T\pi_R(X_t)} - 1 \leq \frac{2}{J}\left(\frac{c}{\pi_R(X_t)}e^{-\alpha(T-t)} + \frac{c}{\pi_R(x)}e^{-\alpha T}\right).$$

Thus,

$$\left|e^{-\lambda_R t}\frac{P_t^T\pi_R(x)}{P_0^T\pi_R(X_t)} - 1\right| \leq \frac{2}{J}\left(\frac{c}{\pi_R(X_t)}e^{-\alpha(T-t)} + \frac{c}{\pi_R(x)}e^{-\alpha T}\right) + 1.$$

Since, for each t, the $\mathscr{F}_t$-measurable random variable $\frac{1}{\pi_R(X_t)}$ is integrable under $\mathbb{Q}_x^{\pi_R}$ for each $x \in E$, for each t the $\mathscr{F}_t$-measurable random variable $|e^{-\lambda_R t}\frac{P_t^T\pi_R(x)}{P_0^T\pi_R(X_t)} - 1|$ is bounded by an integrable random variable. Furthermore, by Eqs. (C.6) and (C.7),

$$\lim_{T\to\infty}\left|e^{-\lambda_R t}\frac{P_t^T(\omega)\pi_R(x)}{P_0^T\pi_R\big(X_t(\omega)\big)} - 1\right| = 0$$

for each ω. Thus, by the dominated convergence theorem, Eq. (C.5) is verified with $B_t^\infty = e^{\lambda_R t}\frac{\pi_R(X_t)}{\pi_R(X_0)}$. *Q.E.D.*

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