

SUPPLEMENT TO “FOLKLORE THEOREMS, IMPLICIT MAPS,
AND INDIRECT INFERENCE”

(*Econometrica*, Vol. 80, No. 1, January 2012, 425–454)

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THIS SUPPLEMENT PROVIDES technical results and proofs.

A.1. *Some Useful Integral Asymptotic Expansions*

The following lemmas provide results on asymptotic expansions of integrals that are useful in the main arguments. In particular, these results are used to develop bias expansions for the binding function $b_n(\rho)$ (see equation (14) of the paper) for three separate fixed ρ cases ($|\rho| < 1$, $\rho = \pm 1$, and $|\rho| > 1$) and to show asymptotic behavior in the local to unity case where $\rho = 1 + \frac{c}{n}$ for fixed c and for $|c| \rightarrow \infty$ with $c = o(n)$ as $n \rightarrow \infty$.

These integral asymptotic expansion formulae are likely to have applications in other contexts.

LEMMA 1: *Let $F_n = F_n(x; \rho) = 1 - \rho^2 x + (1 - x)x^{2n-1}\rho^{2n}$ and suppose $a_1, a_2, \gamma > 0$. Then as $n \rightarrow \infty$,*

$$(A.1) \quad \int_0^1 x^{a_1 n + a_4} (1 - \rho^2 x^2)^\alpha F_n^{-\beta} dx \\ = \frac{(1 - \rho^2)^{\alpha - \beta}}{a_1 n} + O(n^{-2}), \quad |\rho| < 1,$$

$$(A.2) \quad \int_0^1 x^{a_1 n + a_4} (1 + x)^\alpha (1 + \gamma x^{a_2 n + a_3})^\beta dx \\ = \frac{2^\alpha}{a_2 n} \int_0^1 y^{(a_1 - a_2)/a_2} (1 + \gamma y)^\beta dy + O(n^{-2}),$$

$$(A.3) \quad n \int_0^1 x^{a_1 n + a_4} (1 + x)^\alpha (1 + \gamma x^{a_2 n + a_3})^\beta (1 - x) dx \\ = -\frac{2^\alpha}{a_2^2 n} \int_0^1 y^{(a_1 - a_2)/a_2} (1 + \gamma y)^\beta \log y dy + O(n^{-2}).$$

PROOF: To prove (A.1), note first that ρ^{2n} is exponentially small for $|\rho| < 1$. Then $F_n = 1 - \rho^2 x + O(\rho^{2n})$. Setting $y = x^{a_1 n + a_4}$, we have $dy = (a_1 n + a_4)x^{a_1 n + a_4 - 1} dx = (a_1 n + a_4)y^{(a_1 n + a_4 - 1)/(a_1 n + a_4)} dx$ and upon transforma-

tion,

$$\begin{aligned}
& \int_0^1 x^{a_1 n + a_4} (1 - \rho^2 x^2)^\alpha F_n^{-\beta} dx \\
&= \frac{1}{a_1 n + a_4} \int_0^1 y^{1 - (a_1 n + a_4 - 1)/(a_1 n + a_4)} (1 - \rho^2 y^{2/(a_1 n + a_4)})^\alpha \\
&\quad \times (1 - \rho^2 y^{1/(a_1 n + a_4)})^{-\beta} dy \\
&= \frac{(1 - \rho^2)^{\alpha - \beta}}{a_1 n + a_4} \int_0^1 y^{1/(a_1 n + a_4)} dy \{1 + O(n^{-1})\} \\
&= \frac{(1 - \rho^2)^{\alpha - \beta}}{a_1 n} + O(n^{-2}),
\end{aligned}$$

since $y^{b/(a_1 n + a_4)} = 1 + \frac{b}{a_1 n + a_4} \log y + O(n^{-2})$ for all $b \neq 0$ and $|\int_0^1 \log y dy| = 1$.

To prove (A.2), set $y = x^{a_2 n + a_3}$, so that $dy = (a_2 n + a_3) x^{a_2 n + a_3 - 1} dx = a_2 n y^{(a_2 n + a_3 - 1)/(a_2 n + a_3)} dx \{1 + O(n^{-1})\}$ and upon transformation,

$$\begin{aligned}
& \int_0^1 x^{a_1 n + a_4} (1 + x)^\alpha (1 + \gamma x^{a_2 n + a_3})^\beta dx \\
&= \frac{1}{a_2 n + a_3} \int_0^1 y^{((a_1 - a_2)n + a_4 - a_3 + 1)/(a_2 n + a_3)} \\
&\quad \times (1 + y^{1/(a_2 n + a_3)})^\alpha (1 + \gamma y)^\beta dy \\
&= \frac{2^\alpha}{a_2 n} \int_0^1 y^{(a_1 - a_2)/a_2} (1 + \gamma y)^\beta dy + O(n^{-2}),
\end{aligned}$$

since $y^{b/(a_2 n + a_3)} = 1 + \frac{b}{a_2 n + a_3} \log y + O(n^{-2})$ for all $y > 0$.

To prove (A.3), the same approach leads to

$$\begin{aligned}
\text{(A.4)} \quad & n \int_0^1 x^{a_1 n + a_4} (1 + x)^\alpha (1 + \gamma x^{a_2 n + a_3})^\beta (1 - x) dx \\
&= \frac{n}{a_2 n + a_3} \int_0^1 y^{((a_1 - a_2)n + a_4 - a_3 + 1)/(a_2 n + a_3)} (1 + y^{1/(a_2 n + a_3)})^\alpha \\
&\quad \times (1 + \gamma y)^\beta (1 - y^{1/(a_2 n + a_3)}) dy \\
&= -\frac{2^\alpha}{a_2^2 n} \int_0^1 y^{(a_1 - a_2)/a_2} (1 + \gamma y)^\beta \log y dy + O(n^{-2}).
\end{aligned}$$

Observe that, using the transformation $w = -\log y$, we have

$$(A.5) \quad \int_0^1 y^{a-1} |\log y|^b dy = \int_0^\infty e^{-aw} w^b dw < \infty$$

for all $a > 0$ and $b > -1$, which ensures that (A.4) is finite.

Q.E.D.

LEMMA 2: For $\rho = 1 + \frac{c}{n}$, with c fixed, $F_n = 1 - \rho^2 x + (1-x)x^{2n-1}\rho^{2n}$, $a_1 > 0$, and $\alpha - \beta > -1$, we have

$$\begin{aligned} & \int_0^1 x^{a_1 n + a_4} (1 - \rho^2 x^2)^\alpha F_n^{-\beta} dx \\ &= \begin{cases} \frac{2^\alpha}{2n} \int_0^1 y^{(a_1-2)/2} (1 + e^{2c} y)^{-\beta} dy + O(n^{-2}), & \alpha = \beta, \\ \frac{2^\alpha}{4n} \int_0^1 y^{(a_1-2)/2} (1 + e^{2c} y)^{-\beta} (-\log y)^{\alpha-\beta} dy + O(n^{-2}), & \alpha \neq \beta. \end{cases} \end{aligned}$$

PROOF: Since $\rho^{2n} = (1 + \frac{c}{n})^{2n} = e^{2c} \{1 + O(n^{-1})\}$, we have

$$(A.6) \quad \begin{aligned} F_n(x, \rho) &= 1 - x + (1-x)x^{2n-1}e^{2c} + O(n^{-1}) \\ &= (1-x)(1 + e^{2c}x^{2n-1}) + O(n^{-1}). \end{aligned}$$

Using Lemma 1, we obtain

$$\begin{aligned} & \int_0^1 x^{a_1 n + a_4} (1 - \rho^2 x^2)^\alpha F_n^{-\beta} dx \\ &= \int_0^1 x^{a_1 n + a_4} (1 - x^2)^\alpha (1-x)^{-\beta} (1 + e^{2c}x^{2n-1})^{-\beta} dx \{1 + O(n^{-1})\} \\ &= \int_0^1 x^{a_1 n + a_4} (1+x)^\alpha (1-x)^{\alpha-\beta} (1 + e^{2c}x^{2n-1})^{-\beta} dx \{1 + O(n^{-1})\} \\ &= \begin{cases} \int_0^1 x^{a_1 n + a_4} (1+x)^\alpha (1 + e^{2c}x^{2n-1})^{-\beta} dx \{1 + O(n^{-1})\}, & \alpha = \beta \\ \int_0^1 x^{a_1 n + a_4} (1+x)^\alpha (1-x)^{\alpha-\beta} (1 + e^{2c}x^{2n-1})^{-\beta} dx \{1 + O(n^{-1})\}, & \alpha \neq \beta \end{cases} \\ &= \begin{cases} \frac{2^\alpha}{2n} \int_0^1 y^{(a_1-2)/2} (1 + e^{2c}y)^{-\beta} dy + O(n^{-2}), & \alpha = \beta, \\ \frac{2^\alpha}{4n} \int_0^1 y^{(a_1-2)/2} (1 + e^{2c}y)^{-\beta} (-\log y)^{\alpha-\beta} dy + O(n^{-2}), & \alpha \neq \beta, \end{cases} \end{aligned}$$

the final integral being finite in view of (A.5) when $\alpha - \beta > -1$. *Q.E.D.*

LEMMA 3: *If $|\rho| < 1$ and $a_1 > 0$, then as $n \rightarrow \infty$,*

$$\begin{aligned} & \int_0^1 x^{a_1 n + a_2} (1 - \rho^2 x^2)^\alpha (1 - \rho^2 x)^{-\beta} dx \\ &= \frac{(1 - \rho^2)^{\alpha - \beta}}{a_1 n} + O(n^{-2}). \end{aligned}$$

PROOF: Integrating by parts, we have

$$\begin{aligned} & \int_0^1 x^{a_1 n + a_2} (1 - \rho^2 x^2)^\alpha (1 - \rho^2 x)^{-\beta} dx \\ &= \left[\frac{x^{a_1 n + a_2 + 1}}{a_1 n + a_2 + 1} (1 - \rho^2 x^2)^\alpha (1 - \rho^2 x)^{-\beta} \right]_0^1 \\ & \quad + \frac{2\alpha\rho^2}{a_1 n + a_2 + 1} \int_0^1 x^{a_1 n + a_2 + 2} (1 - \rho^2 x^2)^{\alpha - 1} (1 - \rho^2 x)^{-\beta} dx \\ & \quad - \frac{\beta\rho^2}{a_1 n + a_2 + 1} \int_0^1 x^{a_1 n + a_2 + 2} (1 - \rho^2 x^2)^{\alpha - 1} (1 - \rho^2 x)^{-\beta - 1} dx \\ &= \frac{(1 - \rho^2)^{\alpha - \beta}}{a_1 n} + O(n^{-2}). \end{aligned} \quad \text{Q.E.D.}$$

A.2. Proof of Expression (25)

We consider the first derivative of the binding function $b_n(\rho)$ when $|\rho| < 1$, namely

$$\begin{aligned} \text{(A.7)} \quad & b_n(\rho; |\rho| \leq 1) \\ &= \rho - \frac{3\rho}{2} \int_0^1 x^{(n-1)/2} (1 - \rho^2 x^2)^{1/2} F_n^{-1/2} dx \\ & \quad + \frac{\rho}{2} \int_0^1 x^{(n-3)/2} (1 - \rho^2 x^2)^{3/2} F_n^{-3/2} dx \\ & \quad - \frac{n\rho^{2n-1}}{2} \int_0^1 x^{(5n-7)/2} (1 - \rho^2 x^2)^{3/2} F_n^{-3/2} (1 - x) dx, \end{aligned}$$

given in equation (16) in the paper. As is clear from (A.7), the final term in the binding function expression is $O(\rho^n)$. The first derivative of this function is of the same order and since it is dominated by the other terms, it can be neglected

in the following calculations. For $|\rho| < 1$, we therefore have

$$\begin{aligned}
(A.8) \quad b'_n(\rho) &= 1 - \frac{\partial}{\partial \rho} \left\{ \frac{3\rho}{2} \int_0^1 x^{(n-1)/2} (1 - \rho^2 x^2)^{1/2} F_n^{-1/2} dx \right\} \\
&\quad + \frac{\partial}{\partial \rho} \left\{ \frac{\rho}{2} \int_0^1 x^{(n-3)/2} (1 - \rho^2 x^2)^{3/2} F_n^{-3/2} dx \right\} + O(\rho^n) \\
&= 1 - \frac{3}{2} \int_0^1 x^{(n-1)/2} (1 - \rho^2 x^2)^{1/2} F_n^{-1/2} dx \\
&\quad + \frac{3\rho^2}{2} \int_0^1 x^{(n-5)/2} (1 - \rho^2 x^2)^{-1/2} F_n^{-1/2} dx \\
&\quad + \frac{3\rho}{4} \int_0^1 x^{(n-1)/2} (1 - \rho^2 x^2)^{1/2} F_n^{-3/2} \frac{\partial}{\partial \rho} F_n dx \\
&\quad + \frac{1}{2} \int_0^1 x^{(n-3)/2} (1 - \rho^2 x^2)^{3/2} F_n^{-3/2} dx \\
&\quad - \frac{3\rho^2}{2} \int_0^1 x^{(n-7)/2} (1 - \rho^2 x^2)^{1/2} F_n^{-3/2} dx \\
&\quad - \frac{3\rho}{4} \int_0^1 x^{(n-3)/2} (1 - \rho^2 x^2)^{3/2} F_n^{-5/2} \frac{\partial}{\partial \rho} F_n dx + O(\rho^n).
\end{aligned}$$

Now $F_n = 1 - \rho^2 x + (1 - x)x^{2n-1}\rho^{2n}$ and

$$(A.9) \quad \frac{\partial}{\partial \rho} F_n = -2\rho x + 2n(1 - x)x^{2n-1}\rho^{2n-1} = -2\rho x + O(n\rho^{2n-1}),$$

so that substituting (A.9) into (A.8) and using (A.1) of Lemma 1 above, we deduce that for $|\rho| < 1$,

$$(A.10) \quad b'_n(\rho) = 1 + O(n^{-1}),$$

as required.

A.3. Proofs of the Main Results

PROOF OF THEOREM 1: By the mean value theorem,

$$\varphi_n(T_n) - \varphi_n(\theta) = \varphi'_n(T_n^*)(T_n - \theta)$$

for some T_n^* on the line segment connecting T_n and θ . Hence

$$\frac{d_n}{\varphi'_n(\theta)} (\varphi_n(T_n) - \varphi_n(\theta)) = \left\{ 1 + \frac{\varphi'_n(T_n^*) - \varphi'_n(\theta)}{\varphi'_n(\theta)} \right\} d_n(T_n - \theta).$$

Since $|T_n^* - \theta| \leq |T_n - \theta| = O_p(d_n^{-1})$ and $\frac{s_n}{d_n} \rightarrow 0$, it follows that $s_n|T_n^* - \theta| = o_p(1)$. Then

$$\frac{\varphi_n'(T_n^*) - \varphi_n'(\theta)}{\varphi_n'(\theta)} \rightarrow_p 0$$

by local relative equicontinuity (see equation (2) of the paper) in a shrinking neighborhood of radius $O(s_n^{-1})$, giving the required result. *Q.E.D.*

For the Proof of Lemma 2 of the paper, see Ge and Wang (2002, Lemma 1).

PROOF OF THEOREM 3: The structure of the proof follows Shenton and Johnson (1965; SJ) and Shenton and Vinod (1995; SV) by considering ratios of quadratic forms in normal variates. The starting point is to write the density and moments of $\hat{\rho}_n = \sum_{i=1}^n y_i y_{i-1} / \sum_{i=1}^n y_{i-1}^2 = U/V$ in terms of the joint moment generating function $m(u, q)$ of the quadratic forms (U, V) . White (1961)—see also Vinod and Shenton (1996)—showed that $m(u, q) = D_n^{-1/2}$, where $D_n = D_n(u, q)$ is a determinant that satisfies the second order difference equation

$$D_n = (1 + \rho^2 + 2q)D_{n-1} - (\rho + u)^2 D_{n-2}, \quad D_0 = D_1 = 1.$$

Then by direct calculation (see SJ, p. 3), we have the following expression for the bias function,

$$(A.11) \quad E(\hat{\rho}_n - \rho) = \int_0^\infty \frac{\partial}{\partial \rho} D_n(q)^{-1/2} dq,$$

where the determinant $D_n(q) = D_n(0, q)$ is evaluated explicitly as

$$(A.12) \quad D_n(q) = A\theta^n + (1 - A)\rho^{2n}\theta^{-n}, \quad A = \frac{\theta - \rho^2}{\theta^2 - \rho^2},$$

$$\theta = \theta(q) = (1 + \rho^2 + 2q + \sqrt{\Delta})/2,$$

$$(A.13) \quad \Delta = (1 + \rho^2 + 2q)^2 - 4\rho^2.$$

Observe that the inequalities

$$\theta = \theta(q) = (1 + \rho^2 + 2q + \sqrt{\Delta})/2 \geq 0,$$

$$\Delta = (1 - \rho^2)^2 + 4q^2 + 4q(1 + \rho^2) \geq (1 - \rho^2)^2 \geq 0,$$

$$\theta - \rho^2 = (1 - \rho^2 + 2q + \sqrt{\Delta})/2 \geq q \geq 0,$$

$$\theta - \rho = (1 - \rho)^2 + 2q + \sqrt{\Delta} \geq 0,$$

$$\theta + \rho = (1 + \rho)^2 + 2q + \sqrt{\Delta} \geq 0,$$

$$\theta^2 - \rho^2 = (\theta - \rho)(\theta + \rho) \geq 0$$

hold for all $q \geq 0$. It follows that the determinant (A.12) is positive for all $q > 0$ and the integral (A.11) is defined for all ρ .

Write the binding function as

$$\begin{aligned} b_n(\rho) &= E(\hat{\rho}_n) = \rho + \int_0^\infty \frac{\partial}{\partial \rho} D_n(q)^{-1/2} dq \\ &= \rho - \frac{1}{2} \int_0^\infty D_n(q)^{-3/2} \frac{\partial D_n(q)}{\partial \rho} dq. \end{aligned}$$

Define $x = 1/\theta$ and $C = 1 + \rho^2 + 2q$, so that

$$(A.14) \quad x = \frac{2}{1 + \rho^2 + 2q + \sqrt{\Delta}} = \frac{2}{C + \sqrt{\Delta}} = 2 \frac{C - \sqrt{\Delta}}{C^2 - \Delta} = \frac{C - \sqrt{\Delta}}{2\rho^2},$$

since $\Delta = (1 + \rho^2 + 2q)^2 - 4\rho^2 = C^2 - 4\rho^2$. It follows from (A.14) that

$$C + \Delta^{1/2} = \frac{2}{x} \quad \text{and} \quad C - \Delta^{1/2} = 2\rho^2 x,$$

so that $C = 1/x + \rho^2 x$, leading to

$$q = \frac{1}{2}(C - 1 - \rho^2) = \frac{1}{2}(1/x + \rho^2 x - 1 - \rho^2) = \frac{(1-x)(1-\rho^2 x)}{2x}.$$

We write

$$\begin{aligned} q &= \frac{(1-x)(1-\rho^2 x)}{2x} \\ &= \begin{cases} \frac{(1-x)(1-\rho^2 x)}{2x}, & x \in (0, 1], |\rho| \leq 1, \\ \frac{(x-1)(x\rho^2 - 1)}{2x}, & x \in [1, \infty), |\rho| > 1, \end{cases} \end{aligned}$$

with derivative

$$(A.15) \quad \frac{dq}{dx} = -\frac{(1-\rho^2 x^2)}{2x^2} \begin{cases} < 0, & x \in (0, 1], |\rho| \leq 1, \\ > 0, & x \in [1, \infty), |\rho| > 1, \end{cases}$$

so $q = q(x)$ is monotonic over the two domains of x in each case with $q \in [0, \infty)$. We may therefore change the variable of integration in (A.11) from q to x with corresponding changes in the domain of integration depending on the value of ρ as specified in (A.15). For $\rho = 1$, either domain may be used.

Using this change of variable, we have

$$A = \frac{\theta - \rho^2}{\theta^2 - \rho^2} = \frac{\frac{1}{x} - \rho^2}{\frac{1}{x^2} - \rho^2} = \frac{x - \rho^2 x^2}{1 - \rho^2 x^2} = \frac{x(1 - \rho^2 x)}{1 - \rho^2 x^2},$$

$$1 - A = 1 - \frac{x - \rho^2 x^2}{1 - \rho^2 x^2} = \frac{1 - x}{1 - \rho^2 x^2},$$

and then

$$D_n(q) = \frac{(1 - \rho^2 x)}{1 - \rho^2 x^2} \frac{1}{x^{n-1}} + \frac{1 - x}{1 - \rho^2 x^2} \rho^{2n} x^n$$

$$= \begin{cases} \frac{1 - \rho^2 x + (1 - x)x^{2n-1}\rho^{2n}}{(1 - \rho^2 x^2)x^{n-1}} = \frac{F_n(x; \rho)}{(1 - \rho^2 x^2)x^{n-1}}, & |\rho| \leq 1, \\ \frac{\rho^2 x - 1 + (x - 1)x^{2n-1}\rho^{2n}}{(\rho^2 x^2 - 1)x^{n-1}} = \frac{G_n(x; \rho)}{(\rho^2 x^2 - 1)x^{n-1}}, & |\rho| > 1, \end{cases}$$

where

$$F_n(x; \rho) := 1 - \rho^2 x + (1 - x)x^{2n-1}\rho^{2n},$$

$$G_n(x; \rho) := \rho^2 x - 1 + (x - 1)x^{2n-1}\rho^{2n}.$$

For $|\rho| \leq 1$, we have

$$(A.16) \quad E(\hat{\rho}_n - \rho) = \frac{\partial}{\partial \rho} \int_0^\infty D_n(q)^{-1/2} dq$$

$$= \frac{\partial}{\partial \rho} \int_0^1 \left(\frac{F_n(x; \rho)}{(1 - \rho^2 x^2)x^{n-1}} \right)^{-1/2} \frac{(1 - \rho^2 x^2)}{2x^2} dx$$

$$= \frac{1}{2} \frac{\partial}{\partial \rho} \left\{ \int_0^1 x^{(n-5)/2} (1 - \rho^2 x^2)^{3/2} F_n(x; \rho)^{-1/2} dx \right\}.$$

To evaluate (A.16), note that

$$\frac{\partial}{\partial \rho} \left\{ \int_0^1 x^{(n-5)/2} (1 - \rho^2 x^2)^{3/2} F_n^{-1/2} dx \right\}$$

$$= \frac{3}{2} \int_0^1 x^{(n-5)/2} (-2\rho x^2) (1 - \rho^2 x^2)^{1/2} F_n^{-1/2} dx$$

$$- \frac{1}{2} \int_0^1 x^{(n-5)/2} (1 - \rho^2 x^2)^{3/2}$$

$$\times F_n^{-3/2} \{-2\rho x + 2n(1 - x)x^{2n-1}\rho^{2n-1}\} dx,$$

so that for $|\rho| \leq 1$, we have

$$\begin{aligned}
 (A.17) \quad b_n(\rho) &= \rho + \frac{1}{2} \frac{\partial}{\partial \rho} \left\{ \int_0^1 x^{(n-5)/2} (1 - \rho^2 x^2)^{3/2} F_n^{-1/2}(x; \rho) dx \right\} \\
 &= \rho + \frac{3}{4} \int_0^1 x^{(n-5)/2} (-2\rho x^2) (1 - \rho^2 x^2)^{1/2} F_n^{-1/2} dx \\
 &\quad - \frac{1}{4} \int_0^1 x^{(n-5)/2} (1 - \rho^2 x^2)^{3/2} \\
 &\quad \times F_n^{-3/2} \{-2\rho x + 2n(1-x)x^{2n-1}\rho^{2n-1}\} dx \\
 &= \rho - \frac{3\rho}{2} \int_0^1 x^{(n-1)/2} (1 - \rho^2 x^2)^{1/2} F_n^{-1/2} dx \\
 &\quad + \frac{\rho}{2} \int_0^1 x^{(n-3)/2} (1 - \rho^2 x^2)^{3/2} F_n^{-3/2} dx \\
 &\quad - \frac{n\rho^{2n-1}}{2} \int_0^1 x^{(5n-7)/2} (1 - \rho^2 x^2)^{3/2} F_n^{-3/2} (1-x) dx.
 \end{aligned}$$

For $|\rho| > 1$, we have

$$\begin{aligned}
 (A.18) \quad E(\hat{\rho}_n - \rho) &= \frac{\partial}{\partial \rho} \int_0^\infty D_n(q)^{-1/2} dq, \\
 &= \frac{\partial}{\partial \rho} \int_1^\infty \left(\frac{G_n(x; \rho)}{(\rho^2 x^2 - 1)x^{n-1}} \right)^{-1/2} \frac{(\rho^2 x^2 - 1)}{2x^2} dx \\
 &= \frac{1}{2} \frac{\partial}{\partial \rho} \int_1^\infty x^{(n-5)/2} (\rho^2 x^2 - 1)^{3/2} G_n(x; \rho)^{-1/2} dx
 \end{aligned}$$

and, by direct evaluation,

$$\begin{aligned}
 &\frac{\partial}{\partial \rho} \left\{ \int_1^\infty x^{(n-5)/2} (\rho^2 x^2 - 1)^{3/2} G_n^{-1/2} dx \right\} \\
 &= \frac{3}{2} \int_1^\infty x^{(n-5)/2} (2\rho x^2) (\rho^2 x^2 - 1)^{1/2} G_n^{-1/2} dx \\
 &\quad - \frac{1}{2} \int_1^\infty x^{(n-5)/2} (\rho^2 x^2 - 1)^{3/2} \\
 &\quad \times G_n^{-3/2} \{2\rho x + 2n(x-1)x^{2n-1}\rho^{2n-1}\} dx
 \end{aligned}$$

$$\begin{aligned}
&= \frac{3}{2} \int_1^\infty x^{(n-5)/2} (2\rho x^2) (\rho^2 x^2 - 1)^{1/2} G_n^{-1/2} dx \\
&\quad - \rho \int_1^\infty x^{(n-3)/2} (\rho^2 x^2 - 1)^{3/2} G_n^{-3/2} dx \\
&\quad - n\rho^{2n-1} \int_1^\infty x^{(n-5)/2} (\rho^2 x^2 - 1)^{3/2} G_n^{-3/2} (x-1) x^{2n-1} dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
b_n(\rho) &= \rho + \frac{1}{2} \frac{\partial}{\partial \rho} \left\{ \int_1^\infty x^{(n-5)/2} (\rho^2 x^2 - 1)^{3/2} G_n^{-1/2} dx \right\} \\
&= \rho + \frac{3}{4} \int_1^\infty x^{(n-5)/2} (2\rho x^2) (\rho^2 x^2 - 1)^{1/2} G_n^{-1/2} dx \\
&\quad - \frac{\rho}{2} \int_1^\infty x^{(n-3)/2} (\rho^2 x^2 - 1)^{3/2} G_n^{-3/2} dx \\
&\quad - \frac{n\rho^{2n-1}}{2} \int_1^\infty x^{(n-5)/2} (\rho^2 x^2 - 1)^{3/2} G_n^{-3/2} (x-1) x^{2n-1} dx.
\end{aligned}$$

Hence, the binding formula for $|\rho| > 1$ is

$$\begin{aligned}
\text{(A.19)} \quad b_n(\rho) &= \rho + \frac{3\rho}{2} \int_1^\infty x^{(n-1)/2} (\rho^2 x^2 - 1)^{1/2} G_n^{-1/2} dx \\
&\quad - \frac{\rho}{2} \int_1^\infty x^{(n-3)/2} (\rho^2 x^2 - 1)^{3/2} G_n^{-3/2} dx \\
&\quad - \frac{n\rho^{2n-1}}{2} \int_1^\infty x^{(5n-7)/2} (\rho^2 x^2 - 1)^{3/2} G_n^{-3/2} (x-1) dx.
\end{aligned}$$

Transforming using $y = 1/x$, and noting that

$$\begin{aligned}
G_n\left(\frac{1}{y}; \rho\right) &= \rho^2 \frac{1}{y} - 1 + \left(\frac{1}{y} - 1\right) y^{-2n+1} \rho^{2n} \\
&= \frac{(\rho^2 - y)y^{2n-1} + (1-y)\rho^{2n}}{y^{2n}} =: \frac{H_n(y; \rho)}{y^{2n}},
\end{aligned}$$

we have the alternate form

$$\begin{aligned}
\text{(A.20)} \quad b_n(\rho) &= \rho + \frac{3\rho}{2} \int_0^1 y^{-(n-1)/2} \frac{(\rho^2 - y^2)^{1/2}}{y} H_n^{-1/2}(y; \rho) y^{n-2} dy \\
&\quad - \frac{\rho}{2} \int_0^1 y^{-(n-3)/2} \frac{(\rho^2 - y^2)^{3/2}}{y^3} H_n^{-3/2}(y; \rho) y^{3n-2} dy
\end{aligned}$$

$$\begin{aligned}
& - \frac{n\rho^{2n-1}}{2} \int_0^1 y^{-(5n-7)/2} \frac{(\rho^2 - y^2)^{3/2}}{y^3} H_n^{-3/2}(y; \rho) y^{3n-3} (1-y) dy \\
& = \rho + \frac{3\rho}{2} \int_0^1 y^{(n-5)/2} (\rho^2 - y^2)^{1/2} H_n^{-1/2}(y; \rho) dy \\
& \quad - \frac{\rho}{2} \int_0^1 y^{(5n-7)/2} (\rho^2 - y^2)^{3/2} H_n^{-3/2}(y; \rho) dy \\
& \quad - \frac{n\rho^{2n-1}}{2} \int_0^1 y^{(n-5)/2} (\rho^2 - y^2)^{3/2} H_n^{-3/2}(y; \rho) (1-y) dx. \quad \text{Q.E.D.}
\end{aligned}$$

PROOF OF THEOREM 4:

(i) *Case* $|\rho| < 1$. Using $F_n = 1 - \rho^2 x + (1-x)x^{2n-1}\rho^{2n} = 1 - \rho^2 x + O(\rho^{2n})$ and $n\rho^n = o(n^{-2})$, we have, for $|\rho| < 1$ from (A.17) and Lemma 3,

$$\begin{aligned}
b_n(\rho) & = \rho - \frac{3\rho}{2} \int_0^1 x^{(n-1)/2} (1 - \rho^2 x^2)^{1/2} F_n^{-1/2} dx \\
& \quad + \frac{\rho}{2} \int_0^1 x^{(n-3)/2} (1 - \rho^2 x^2)^{3/2} F_n^{-3/2} + o(n^{-1}) \\
& = \rho - \frac{3\rho}{2} \int_0^1 x^{(n-1)/2} (1 - \rho^2 x^2)^{1/2} (1 - \rho^2 x)^{-1/2} dx \\
& \quad + \frac{\rho}{2} \int_0^1 x^{(n-3)/2} (1 - \rho^2 x^2)^{3/2} (1 - \rho^2 x)^{-3/2} dx + o(n^{-2}) \\
& = \rho - \frac{2\rho}{n} + O(n^{-2}),
\end{aligned}$$

giving the well known asymptotic bias formula for $\hat{\rho}_n$ in the stationary case.

(ii) *Case* $\rho = \pm 1$. When $\rho = 1$, we have, $F_n(x; 1) = 1 - x + (1-x)x^{2n-1} = (1-x)(1+x^{2n-1})$ and so, from (A.17),

$$\begin{aligned}
b_n(1) & = 1 - \frac{3}{2} \int_0^1 x^{(n-1)/2} \frac{(1+x)^{1/2}}{(1+x^{2n-1})^{1/2}} dx \\
& \quad + \frac{1}{2} \int_0^1 x^{(n-3)/2} \frac{(1+x)^{3/2}}{(1+x^{2n-1})^{3/2}} dx \\
& \quad - \frac{n}{2} \int_0^1 x^{(5n-7)/2} \frac{(1+x)^{3/2}}{(1+x^{2n-1})^{3/2}} (1-x) dx.
\end{aligned}$$

Using (A.2) and (A.3), we have

$$\begin{aligned} \int_0^1 x^{(n-1)/2} \frac{(1+x)^{1/2}}{(1+x^{2n-1})^{1/2}} dx &= \frac{2^{1/2}}{2n} \int_0^1 y^{-3/4} (1+y)^{-1/2} dy + O(n^{-2}), \\ \int_0^1 x^{(n-3)/2} \frac{(1+x)^{3/2}}{(1+x^{2n-1})^{3/2}} dx &= \frac{2^{3/2}}{2n} \int_0^1 y^{-3/4} (1+y)^{-3/2} dy + O(n^{-2}), \\ n \int_0^1 x^{(n-1)/2} \frac{(1+x)^{3/2}}{(1+x^{2n-1})^{3/2}} (1-x)x^{2n-3} dx \\ &= -\frac{2^{3/2}}{4n} \int_0^1 y^{1/4} (1+y)^{-3/2} \log y dy + O(n^{-2}). \end{aligned}$$

It follows that

$$\begin{aligned} \text{(A.21)} \quad b_n(1) &= 1 - 3 \frac{2^{1/2}}{4n} \int_0^1 y^{-3/4} (1+y)^{-1/2} dy + \frac{2^{3/2}}{4n} \int_0^1 y^{-3/4} (1+y)^{-3/2} dy \\ &\quad - \frac{1}{2} \left\{ -\frac{2^{3/2}}{4n} \int_0^1 y^{1/4} (1+y)^{-3/2} \log y dy \right\} + O(n^{-2}), \end{aligned}$$

and numerical evaluation of the integrals gives

$$\begin{aligned} \text{(A.22)} \quad b_n(1) &= 1 - \frac{3}{4n} 2^{0.5} (3.7081) + \frac{2^{1.5}}{4n} (3.2683) - \frac{2^{1/2}}{4n} (0.45077) + O(n^{-2}) \\ &= 1 - \frac{1.7814}{n} + O(n^{-2}), \end{aligned}$$

corresponding to the result found by SJ for the unit root case using different methods. The numerical value -1.7814 is the mean of the limit distribution of $n(\hat{\rho}_n - 1)$ when $\rho = 1$.

Similar calculations apply when $\rho = -1$, in which case we have

$$\begin{aligned} \text{(A.23)} \quad b_n(-1) &= -1 + \frac{3}{4n} 2^{0.5} (3.7081) - \frac{2^{1.5}}{4n} (3.2683) \\ &\quad + \frac{2^{1/2}}{4n} (0.45077) + O(n^{-2}) \\ &= -1 + \frac{1.7814}{n} + O(n^{-2}), \end{aligned}$$

giving the mirror image of (A.22).

(iii) *Case* $\rho = 1 + \frac{c}{n}$, $c < 0$. We consider the local to unity case with $\rho = 1 + \frac{c}{n}$, $c < 0$, and $|\rho| < 1$. The following arguments allow c to be fixed and $c \rightarrow -\infty$

as $n \rightarrow \infty$ with $c = o(n)$. The relevant expression for the bias when $|\rho| < 1$ is

$$(A.24) \quad b_n(\rho) = \rho - \frac{3\rho}{2} \int_0^1 x^{(n-1)/2} (1 - \rho^2 x^2)^{1/2} F_n^{-1/2} dx \\ + \frac{\rho}{2} \int_0^1 x^{(n-3)/2} (1 - \rho^2 x^2)^{3/2} F_n^{-3/2} dx \\ - \frac{n\rho^{2n-1}}{2} \int_0^1 x^{(5n-7)/2} (1 - \rho^2 x^2)^{3/2} F_n^{-3/2} (1-x) dx.$$

As before, set $y = x^{2n-1}$ so that $dy = (2n-1)x^{2n-2} dx = (2n-1) \times y^{(2n-2)/(2n-1)} dx = (2n-1)y^{1-1/(2n-1)} dx$. Then, using $F_n = 1 - \rho^2 x + (1-x)x^{2n-1}\rho^{2n}$, we have for the first integral in (A.24),

$$\int_0^1 x^{(n-1)/2} (1 - \rho^2 x^2)^{1/2} F_n^{-1/2} dx \\ = \frac{1}{2n-1} \int_0^1 y^{(n+1)/(4n-4)-1} (1 - \rho^2 y^{2/(2n-1)})^{1/2} \\ \times \left\{ (1 - \rho^2 y^{1/(2n-1)}) + (1 - y^{1/(2n-1)})y\rho^{2n} \right\}^{-1/2} dy \\ = \frac{1}{2n-1} \int_0^1 y^{-3/4+1/(2n-2)} (1 - \rho^2 y^{2/(2n-1)})^{1/2} \\ \times \left\{ (1 - \rho^2 y^{1/(2n-1)}) + (1 - y^{1/(2n-1)})y\rho^{2n} \right\}^{-1/2} dy.$$

Since $y^{b/(2n+a)} = 1 + \frac{b}{2n+a} \log y + O(n^{-2})$, $\rho^2 = 1 + \frac{2c}{n} + \frac{c^2}{n^2}$, $\rho^{2n} = (1 + \frac{c}{n})^{2n} = e^{2c} \{1 + O(\frac{c^2}{n})\}$, and $c < 0$ with $\frac{c}{n} = o(1)$, it follows that

$$(A.25) \quad \int_0^1 x^{(n-1)/2} (1 - \rho^2 x^2)^{1/2} F_n^{-1/2} dx \\ = \frac{1}{2n-1} \int_0^1 y^{-3/4} (1 - \rho^2 y^{2/(2n-1)})^{1/2} \\ \times \left\{ (1 - \rho^2 y^{1/(2n-1)}) + (1 - y^{1/(2n-1)})y\rho^{2n} \right\}^{-1/2} dy \{1 + O(n^{-1})\} \\ = \frac{1}{2n} \int_0^1 y^{-3/4} \left\{ \frac{1 - \rho^2 y^{1/(2n-1)}}{1 - \rho^2 y^{2/(2n-1)}} + \frac{1 - y^{1/(2n-1)}}{1 - \rho^2 y^{2/(2n-1)}} y\rho^{2n} \right\}^{-1/2} dy \\ \times \{1 + O(n^{-1})\}$$

$$\begin{aligned}
&= \frac{1}{2n} \int_0^1 y^{-3/4} \left\{ \frac{1 - \rho^2 \left(1 + \frac{1}{2n} \log y\right)}{1 - \rho^2 \left(1 + \frac{1}{n} \log y\right)} \right. \\
&\quad \left. - \frac{\frac{1}{2n} \log y}{1 - \rho^2 \left(1 + \frac{1}{n} \log y\right)} y \rho^{2n} \right\}^{-1/2} dy \\
&\quad \times \{1 + O(n^{-1})\} \\
&= \frac{1}{2n} \int_0^1 y^{-3/4} \left\{ \frac{-\frac{2c}{n} - \frac{1}{2n} \log y - \left(\frac{c}{n}\right)^2}{-\frac{2c}{n} - \frac{1}{n} \log y - \left(\frac{c}{n}\right)^2} \right. \\
&\quad \left. - \frac{\frac{1}{2n} \log y}{-\frac{2c}{n} - \frac{1}{n} \log y - \left(\frac{c}{n}\right)^2} y e^{2c} \left[1 + O\left(\frac{c^2}{n}\right)\right] \right\}^{-1/2} dy \\
&\quad \times \{1 + O(n^{-1})\} \\
&= \frac{1}{2n} \int_0^1 y^{-3/4} \left\{ \frac{(4c + \log y) \left(1 + \frac{2c^2}{n(4c + \log y)}\right)}{(4c + 2 \log y) \left(1 + \frac{2c^2}{n(4c + 2 \log y)}\right)} \right. \\
&\quad \left. + \frac{\log y}{(4c + 2 \log y) \left(1 + \frac{2c^2}{n(4c + 2 \log y)}\right)} y e^{2c} \right\}^{-1/2} dy \\
&\quad \times \{1 + O(n^{-1})\} \\
&= \frac{1}{2n} \int_0^1 y^{-3/4} \left\{ \frac{(4c + \log y)}{(4c + 2 \log y)} + \frac{\log y}{(4c + 2 \log y)} y e^{2c} \right\}^{-1/2} dy \\
&\quad \times \left\{1 + O\left(\frac{c}{n}\right)\right\}.
\end{aligned}$$

The second integral in (A.24) can be reduced in the same way. Again, setting $y = x^{2n-1}$ with $dy = (2n-1)y^{1-1/(2n-1)} dx$ and $y^{b/(2n+a)} = 1 + \frac{b}{2n+a} \log y + O(n^{-2})$

and using $c = o(n)$, we have

$$\begin{aligned}
(A.26) \quad & \int_0^1 x^{(n-3)/2} (1 - \rho^2 x^2)^{3/2} F_n^{-3/2} dx \\
&= \frac{1}{2n-1} \int_0^1 y^{-3/4 - (1/4)/(2n-2)} (1 - \rho^2 y^{2/(2n-1)})^{3/2} \\
&\quad \times \left\{ (1 - \rho^2 y^{1/(2n-1)}) + (1 - y^{1/(2n-1)}) y \rho^{2n} \right\}^{-3/2} dy \\
&= \frac{1}{2n} \int_0^1 y^{-3/4} \left\{ \frac{1 - \rho^2 y^{1/(2n-1)}}{1 - \rho^2 y^{2/(2n-1)}} + \frac{1 - y^{1/(2n-1)}}{1 - \rho^2 y^{2/(2n-1)}} y \rho^{2n} \right\}^{-3/2} dy \\
&\quad \times \{1 + O(n^{-1})\} \\
&= \frac{1}{2n} \int_0^1 y^{-3/4} \left\{ \frac{1 - \rho^2 \left(1 + \frac{1}{2n} \log y\right)}{1 - \rho^2 \left(1 + \frac{1}{n} \log y\right)} \right. \\
&\quad \left. - \frac{\frac{1}{2n} \log y}{1 - \rho^2 \left(1 + \frac{1}{n} \log y\right)} y \rho^{2n} \right\}^{-3/2} dy \{1 + O(n^{-1})\} \\
&= \frac{1}{2n} \int_0^1 y^{-3/4} \left\{ \frac{-\frac{2c}{n} - \frac{1}{2n} \log y}{-\frac{2c}{n} - \frac{1}{n} \log y} - \frac{\frac{1}{2n} \log y}{-\frac{2c}{n} - \frac{1}{n} \log y} y e^{2c} \right\}^{-3/2} dy \\
&\quad \times \left\{ 1 + O\left(\frac{c}{n}\right) \right\} \\
&= \frac{1}{2n} \int_0^1 y^{-3/4} \left\{ \frac{4c + \log y}{4c + 2 \log y} + \frac{\log y}{4c + 2 \log y} y e^{2c} \right\}^{-3/2} dy \\
&\quad \times \left\{ 1 + O\left(\frac{c}{n}\right) \right\}.
\end{aligned}$$

Finally, for the third integral in (A.24), we have in the same fashion,

$$\begin{aligned}
(A.27) \quad & \int_0^1 x^{(5n-7)/2} (1 - \rho^2 x^2)^{3/2} F_n^{-3/2} (1-x) dx \\
&= \frac{1}{(2n-1)^2} \int_0^1 y^{(5n-7)/(4n-2) - 1 + 1/(2n-1)} (1 - \rho^2 y^{2/(2n-1)})^{3/2}
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ (1 - \rho^2 y^{1/(2n-1)}) + (1 - y^{1/(2n-1)}) y \rho^{2n} \right\}^{-3/2} \log y dy \{1 + O(n^{-1})\} \\
&= \frac{1}{4n^2} \int_0^1 y^{(n-3)/(4n-2)} (1 - \rho^2 y^{2/(2n-1)})^{3/2} \\
& \quad \times \left\{ (1 - \rho^2 y^{1/(2n-1)}) + (1 - y^{1/(2n-1)}) y \rho^{2n} \right\}^{-3/2} \log y dy \{1 + O(n^{-1})\} \\
&= \frac{1}{4n^2} \int_0^1 y^{1/4} \left\{ \frac{1 - \rho^2 y^{1/(2n-1)}}{1 - \rho^2 y^{2/(2n-1)}} + \frac{1 - y^{1/(2n-1)}}{1 - \rho^2 y^{2/(2n-1)}} y \rho^{2n} \right\}^{-3/2} \log y dy \\
& \quad \times \{1 + O(n^{-1})\} \\
&= \frac{1}{4n^2} \int_0^1 y^{1/4} \left\{ \frac{1 - \left(1 + \frac{2c}{n}\right) \left(1 + \frac{1}{2n} \log y\right)}{1 - \left(1 + \frac{2c}{n}\right) \left(1 + \frac{1}{n} \log y\right)} \right. \\
& \quad \left. - \frac{\frac{1}{2n} \log y}{1 - \left(1 + \frac{2c}{n}\right) \left(1 + \frac{1}{n} \log y\right)} y \rho^{2n} \right\}^{-3/2} \log y dy \left\{ 1 + O\left(\frac{c}{n}\right) \right\} \\
&= \frac{1}{4n^2} \int_0^1 y^{1/4} \left\{ \frac{-\frac{2c}{n} - \frac{1}{2n} \log y}{-\frac{2c}{n} - \frac{1}{n} \log y} - \frac{\frac{1}{2n} \log y}{-\frac{2c}{n} - \frac{1}{n} \log y} y e^{2c} \right\}^{-3/2} \\
& \quad \times \log y dy \left\{ 1 + O\left(\frac{c}{n}\right) \right\} \\
&= \frac{1}{4n^2} \int_0^1 y^{1/4} \left\{ \frac{4c + \log y}{4c + 2 \log y} + \frac{\log y}{4c + 2 \log y} y e^{2c} \right\}^{-3/2} \\
& \quad \times \log y dy \left\{ 1 + O\left(\frac{c}{n}\right) \right\}.
\end{aligned}$$

Combining results (A.25)–(A.27) gives the following approximation to the binding function for $\rho = 1 + \frac{c}{n}$ with $c < 0$, $c = o(n)$, and $|\rho| < 1$:

$$\begin{aligned}
\text{(A.28)} \quad b_n(\rho) &= \rho - \frac{3\rho}{2} \int_0^1 x^{(n-1)/2} (1 - \rho^2 x^2)^{1/2} F_n^{-1/2} dx \\
& \quad + \frac{\rho}{2} \int_0^1 x^{(n-3)/2} (1 - \rho^2 x^2)^{3/2} F_n^{-3/2} dx \\
& \quad - \frac{n\rho^{2n-1}}{2} \int_0^1 x^{(5n-7)/2} (1 - \rho^2 x^2)^{3/2} F_n^{-3/2} (1-x) dx
\end{aligned}$$

$$\begin{aligned}
&= \rho - \frac{3\rho}{4n} \int_0^1 y^{-3/4} \left\{ \frac{4c + \log y}{4c + 2\log y} + \frac{\log y}{4c + 2\log y} y e^{2c} \right\}^{-1/2} dy \\
&\quad \times \left\{ 1 + O\left(\frac{c}{n}\right) \right\} \\
&+ \frac{\rho}{4n} \int_0^1 y^{-3/4} \left\{ \frac{4c + \log y}{4c + 2\log y} + \frac{\log y}{4c + 2\log y} y e^{2c} \right\}^{-3/2} dy \\
&\quad \times \left\{ 1 + O\left(\frac{c}{n}\right) \right\} \\
&+ \frac{\rho^{2n-1}}{8n} \int_0^1 y^{1/4} \left\{ \frac{4c + \log y}{4c + 2\log y} + \frac{\log y}{4c + 2\log y} y e^{2c} \right\}^{-3/2} \\
&\quad \times \log y dy \left\{ 1 + O\left(\frac{c}{n}\right) \right\}.
\end{aligned}$$

Observe the sign change in the last term because of the transformation in the integrand that involves $\log y$, which is negative for $y \in (0, 1)$, so the whole expression is negative.

The approximation (A.28) holds with an error of $O(n^{-2})$ uniformly for all c in compact sets of $\mathbb{R}_- = (-\infty, 0)$ when $n \rightarrow \infty$. It also holds with a relative error of $O(\frac{c}{n})$ when $c \rightarrow -\infty$ provided $c = o(n)$. In this case, terms involving e^{2c} become exponentially small. The approximation to the binding function when $n \rightarrow \infty$ and $c \rightarrow -\infty$ while $|\rho| = |1 + \frac{c}{n}| < 1$ and $c = o(n)$ as $n \rightarrow \infty$ has the form

$$\begin{aligned}
\text{(A.29)} \quad b_n(\rho) &= \rho - \frac{3\rho}{4n} \int_0^1 y^{-3/4} dy + \frac{\rho}{4n} \int_0^1 y^{-3/4} dy + O\left(\frac{c}{n^2}\right) \\
&= \rho - \frac{\rho}{2n} \left[\frac{y^{1/4}}{1/4} \right]_0^1 + O\left(\frac{c}{n^2}\right) = \rho - \frac{2\rho}{n} + O\left(\frac{c}{n^2}\right),
\end{aligned}$$

so that the leading term in the approximation corresponds to the case of fixed ρ with $|\rho| < 1$. From (A.29), the binding function is linear in ρ as $n \rightarrow \infty$ and $c \rightarrow -\infty$ with $|\rho| < 1$.

Next, when $c = 0$, we have

$$\begin{aligned}
b_n(1) &= 1 - \frac{3}{4n} \int_0^1 y^{-3/4} \left\{ \frac{1}{2} + \frac{1}{2}y \right\}^{-1/2} dy \\
&\quad + \frac{1}{4n} \int_0^1 y^{-3/4} \left\{ \frac{1}{2} + \frac{1}{2}y \right\}^{-3/2} dy
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{8n} \int_0^1 y^{1/4} \left\{ \frac{1}{2} + \frac{1}{2}y \right\}^{-3/2} \log y \, dy + O(n^{-2}) \\
& = 1 - \frac{3}{4n} 2^{1/2} \int_0^1 y^{-3/4} \{1+y\}^{-1/2} \, dy \\
& \quad + \frac{2^{3/2}}{4n} \int_0^1 y^{-3/4} \{1+y\}^{-3/2} \, dy \\
& \quad + \frac{2^{1/2}}{4n} \int_0^1 y^{1/4} \{1+y\}^{-3/2} \log y \, dy + O(n^{-2}) \\
& = 1 - \frac{1.7814}{n} + O(n^{-2}),
\end{aligned}$$

corresponding to (A.21). Thus, (A.28) encompasses both the stationary and unit root cases at the limits of the domain of definition for $c < 0$.

(iv) *Case* $\rho = 1 + \frac{c}{n} > 1$, $c > 0$. We start with the local to unity case $\rho = 1 + c/n$ with $c > 0$ and later consider fixed $\rho > 1$. We allow for the case where $c \rightarrow \infty$ under the condition $c = o(n)$. The binding function formula for $\rho > 1$ is

$$\begin{aligned}
\text{(A.30)} \quad b_n(\rho) &= \rho + \frac{3\rho}{2} \int_1^\infty x^{(n-1)/2} (\rho^2 x^2 - 1)^{1/2} G_n^{-1/2} \, dx \\
&\quad - \frac{\rho}{2} \int_1^\infty x^{(n-3)/2} (\rho^2 x^2 - 1)^{3/2} G_n^{-3/2} \, dx \\
&\quad - \frac{n\rho^{2n-1}}{2} \int_1^\infty x^{(n-5)/2} (\rho^2 x^2 - 1)^{3/2} G_n^{-3/2} (x-1)x^{2n-1} \, dx.
\end{aligned}$$

We proceed by taking each term in turn. As before, set $y = x^{2n-1}$ so that

$$\begin{aligned}
dy &= (2n-1)x^{2n-2} \, dx = (2n-1)y^{(2n-2)/(2n-1)} \, dx \\
&= (2n-1)y^{1-1/(2n-1)} \, dx,
\end{aligned}$$

and use the expansion $y^{1/(2n-1)} = 1 + \frac{1}{2n-1} \log y + O(n^{-2})$. In $G_n = \rho^2 x - 1 + (x-1)x^{2n-1}\rho^{2n}$, we have $\rho = 1 + \frac{c}{n}$ with $c > 0$ and in what follows, we allow for $c \rightarrow \infty$ such that $c = o(n)$. As before, $\rho^2 = 1 + \frac{2c}{n} + \frac{c^2}{n^2}$ and $\rho^{2n} = (1 + \frac{c}{n})^{2n} = e^{2c} \{1 + O(\frac{c^2}{n})\}$. The integral in the second term of (A.30) is then

$$\begin{aligned}
\text{(A.31)} \quad & \int_1^\infty x^{(n-1)/2} (\rho^2 x^2 - 1)^{1/2} G_n^{-1/2} \, dx \\
&= \int_1^\infty x^{(n-1)/2} \left\{ \frac{\rho^2 x^2 - 1}{\rho^2 x - 1 + (x-1)x^{2n-1}\rho^{2n}} \right\}^{1/2} \, dx
\end{aligned}$$

$$\begin{aligned}
&= \int_1^\infty y^{(n-1)/2/(2n-1)} \left\{ \frac{\rho^2 y^{2/(2n-1)} - 1}{\rho^2 y^{1/(2n-1)} - 1 + (y^{1/(2n-1)} - 1)y\rho^{2n}} \right\}^{1/2} \\
&\quad \times \frac{dy}{(2n-1)y^{(2n-2)/(2n-1)}} \\
&= \frac{1}{2n} \int_1^\infty y^{-3/4} \\
&\quad \times \left\{ \frac{\left(1 + \frac{2c}{n} + \frac{c^2}{n^2}\right) \left(1 + \frac{2}{2n} \log y\right) - 1}{\left(1 + \frac{2c}{n} + \frac{c^2}{n^2}\right) \left(1 + \frac{1}{2n} \log y\right) - 1 + \frac{\rho^{2n}}{2n} y \log y} \right\}^{1/2} dy \\
&\quad \times \{1 + O(n^{-1})\} \\
&= \frac{1}{2n} \int_1^\infty y^{-3/4} \\
&\quad \times \left\{ \frac{(4c + 2 \log y) \left(1 + \frac{2c^2}{n(4c + 2 \log y)} \{1 + o(1)\}\right)}{(4c + \log y + e^{2c} y \log y) \left(1 + O\left(\frac{e^{-2c} c^2}{n}\right)\right)} \right\}^{1/2} dy \\
&\quad \times \{1 + O(n^{-1})\} \\
&= \frac{1}{2n} \int_1^\infty y^{-3/4} \left\{ \frac{(4c + 2 \log y)}{4c + \log y + e^{2c} y \log y} \right\}^{1/2} dy \left\{ 1 + O\left(\frac{c}{n}\right) \right\}.
\end{aligned}$$

Use the transformation $w = \log y$ so that $w \in [0, \infty)$ and $dy = e^w dw$, giving for the leading term of (A.31),

$$\begin{aligned}
\text{(A.32)} \quad &\frac{1}{2n} \int_1^\infty y^{-3/4} \left\{ \frac{4c + 2 \log y}{4c + \log y + e^{2c} y \log y} \right\}^{1/2} dy \\
&= \frac{1}{2n} \int_0^\infty e^{(1/4)w} \left\{ \frac{4c + 2w}{4c + w + e^{2c} w e^w} \right\}^{1/2} dw.
\end{aligned}$$

Proceeding in the same way with the integral in the third term of (A.30), we have

$$\begin{aligned}
\text{(A.33)} \quad &\int_1^\infty x^{(n-3)/2} (\rho^2 x^2 - 1)^{3/2} G_n^{-3/2} dx \\
&= \int_1^\infty x^{(n-3)/2} \left\{ \frac{\rho^2 x^2 - 1}{\rho^2 x - 1 + (x-1)x^{2n-1}\rho^{2n}} \right\}^{3/2} dx
\end{aligned}$$

$$\begin{aligned}
&= \int_1^\infty y^{((n-3)/2)/(2n-1)} \left\{ \frac{\rho^2 y^{2/(2n-1)} - 1}{\rho^2 y^{1/(2n-1)} - 1 + (y^{1/(2n-1)} - 1)y\rho^{2n}} \right\}^{3/2} \\
&\quad \times \frac{dy}{(2n-1)y^{(2n-2)/(2n-1)}} \\
&= \frac{1}{2n} \int_1^\infty y^{-3/4} \\
&\quad \times \left\{ \frac{\left(1 + \frac{2c}{n} + \frac{c^2}{n^2}\right) \left(1 + \frac{2}{2n} \log y\right) - 1}{\left(1 + \frac{2c}{n} + \frac{c^2}{n^2}\right) \left(1 + \frac{1}{2n} \log y\right) - 1 + \frac{\rho^{2n}}{2n} y \log y} \right\}^{3/2} dy \\
&\quad \times \{1 + O(n^{-1})\} \\
&= \frac{1}{2n} \int_1^\infty y^{-3/4} \left\{ \frac{4c + 2 \log y}{4c + \log y + e^{2c} y \log y} \right\}^{3/2} dy \left\{ 1 + O\left(\frac{c}{n}\right) \right\} \\
&= \frac{1}{2n} \int_0^\infty e^{1/4w} \left\{ \frac{4c + 2w}{4c + w + e^{2c} w e^w} \right\}^{3/2} dw \left\{ 1 + O\left(\frac{c}{n}\right) \right\}.
\end{aligned}$$

Finally, the integral in the fourth term of (A.30) is

$$\begin{aligned}
\text{(A.34)} \quad &\int_1^\infty x^{(n-5)/2} (\rho^2 x^2 - 1)^{3/2} G_n^{-3/2} (x-1) x^{2n-1} dx \\
&= \int_1^\infty x^{(n-5)/2} \left\{ \frac{\rho^2 x^2 - 1}{\rho^2 x - 1 + (x-1)x^{2n-1}\rho^{2n}} \right\}^{3/2} (x-1)x^{2n-1} dx \\
&= \int_1^\infty y^{((n-5)/2)/(2n-1)} \left\{ \frac{\rho^2 y^{2/(2n-1)} - 1}{\rho^2 y^{1/(2n-1)} - 1 + (y^{1/(2n-1)} - 1)y\rho^{2n}} \right\}^{3/2} \\
&\quad \times \frac{(y^{1/(2n-1)} - 1)y dy}{(2n-1)y^{(2n-2)/(2n-1)}} \\
&= \frac{1}{2n} \int_1^\infty y^{1/4} \\
&\quad \times \left\{ \frac{\left(1 + \frac{2c}{n} + \frac{c^2}{n^2}\right) \left(1 + \frac{2}{2n} \log y\right) - 1}{\left(1 + \frac{2c}{n} + \frac{c^2}{n^2}\right) \left(1 + \frac{1}{2n} \log y\right) - 1 + \frac{\rho^{2n}}{2n} y \log y} \right\}^{3/2} \\
&\quad \times \left(\frac{1}{2n} \log y\right) dy \{1 + O(n^{-1})\}
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{1}{2n}\right)^2 \int_1^\infty y^{1/4} \left\{ \frac{4c + 2 \log y}{4c + \log y + e^{2c} y \log y} \right\}^{3/2} \log y \, dy \left\{ 1 + O\left(\frac{c}{n}\right) \right\} \\
&= \left(\frac{1}{2n}\right)^2 \int_0^\infty e^{(5/4)w} \left\{ \frac{4c + 2w}{4c + w + e^{2c} w e^w} \right\}^{3/2} w \, dw \left\{ 1 + O\left(\frac{c}{n}\right) \right\}.
\end{aligned}$$

Combining (A.32)–(A.34) in (A.30), we get for $\rho = 1 + \frac{c}{n}$ with $c > 0$,

$$\begin{aligned}
b_n(\rho) &= \rho + \frac{3\rho}{4n} \int_0^\infty e^{(1/4)w} \left\{ \frac{4c + 2w}{4c + w + e^{2c} w e^w} \right\}^{1/2} dw \left\{ 1 + O\left(\frac{c}{n}\right) \right\} \\
&\quad - \frac{\rho}{4n} \int_0^\infty e^{(1/4)w} \left\{ \frac{4c + 2w}{4c + w + e^{2c} w e^w} \right\}^{3/2} dw \left\{ 1 + O\left(\frac{c}{n}\right) \right\} \\
&\quad - \frac{\rho^{2n-1}}{8n} \int_0^\infty e^{(5/4)w} \left\{ \frac{4c + 2w}{4c + w + e^{2c} w e^w} \right\}^{3/2} w \, dw \left\{ 1 + O\left(\frac{c}{n}\right) \right\}.
\end{aligned}$$

Hence the bias function to $O(n^{-1})$ in this case of local to unity on the explosive side of unity is

$$\begin{aligned}
\text{(A.35)} \quad b_n\left(1 + \frac{c}{n}\right) &= 1 + \frac{c}{n} + \frac{3}{4n} \int_0^\infty e^{(1/4)w} \left\{ \frac{4c + 2w}{4c + w + e^{2c} w e^w} \right\}^{1/2} dw \left\{ 1 + O\left(\frac{c}{n}\right) \right\} \\
&\quad - \frac{1}{4n} \int_0^\infty e^{(1/4)w} \left\{ \frac{4c + 2w}{4c + w + e^{2c} w e^w} \right\}^{3/2} dw \left\{ 1 + O\left(\frac{c}{n}\right) \right\} \\
&\quad - \frac{\rho^{2n}}{8n} \int_0^\infty e^{(5/4)w} \left\{ \frac{4c + 2w}{4c + w + e^{2c} w e^w} \right\}^{3/2} w \, dw \left\{ 1 + O\left(\frac{c}{n}\right) \right\}.
\end{aligned}$$

The approximation (A.35) holds with an error of $O(n^{-2})$ uniformly for all c in compact sets of $\mathbb{R}_+ = (0, \infty)$ when $n \rightarrow \infty$. The formula also produces a valid approximation to the binding function for $\rho = 1 + \frac{c}{n} > 1$ when $c \rightarrow \infty$ and $c = o(n)$ as $n \rightarrow \infty$. In this case, noting the relative error order in (A.35) and the behavior of $k^+(w; c)$ as $c \rightarrow \infty$, the approximation to the binding function has the form

$$\text{(A.36)} \quad b_n\left(1 + \frac{c}{n}\right) = 1 + \frac{c}{n} + O\left(\frac{1}{ne^c}\right),$$

so that for large c the binding function is approximately linear.

Combining (A.29) and (A.36), we deduce that the binding function $b_n(1 + \frac{c}{n})$ is approximately linear as $n \rightarrow \infty$ when $|c| \rightarrow \infty$ with $c = o(n)$.

(v) *Case* $|\rho| > 1$. We now turn to the case of fixed $\rho > 1$. The relevant bias expression is from (A.19):

$$(A.37) \quad b_n(\rho) = \rho + \frac{3\rho}{2} \int_1^\infty x^{(n-1)/2} (\rho^2 x^2 - 1)^{1/2} G_n^{-1/2} dx \\ - \frac{\rho}{2} \int_1^\infty x^{(n-3)/2} (\rho^2 x^2 - 1)^{3/2} G_n^{-3/2} dx \\ - \frac{n\rho^{2n-1}}{2} \int_1^\infty x^{(n-5)/2} (\rho^2 x^2 - 1)^{3/2} G_n^{-3/2} (x-1)x^{2n-1} dx.$$

We examine the order of magnitude of each term in turn as $n \rightarrow \infty$. For the first term,

$$\int_1^\infty x^{(n-1)/2} (\rho^2 x^2 - 1)^{1/2} G_n^{-1/2} dx \\ = \int_1^\infty x^{(n-1)/2} \left\{ \frac{\rho^2 x^2 - 1}{\rho^2 x - 1 + (x-1)x^{2n-1}\rho^{2n}} \right\}^{1/2} dx \\ = \frac{1}{\rho^n} \int_1^\infty \frac{x^{(n-1)/2}}{x^{n-1/2}} \left\{ \frac{\rho^2 x^2 - 1}{(x-1) + \frac{\rho^2 x - 1}{x^{2n-1}\rho^{2n}}} \right\}^{1/2} dx \\ \leq \frac{B}{\rho^n} \int_1^\infty \frac{1}{x^{n/2}} dx = O(n^{-1}\rho^{-n}).$$

In a similar way, the second term is

$$\int_1^\infty x^{(n-3)/2} (\rho^2 x^2 - 1)^{3/2} G_n^{-3/2} dx \\ = \int_1^\infty x^{(n-3)/2} \left\{ \frac{\rho^2 x^2 - 1}{\rho^2 x - 1 + (x-1)x^{2n-1}\rho^{2n}} \right\}^{3/2} dx \\ = O(n^{-1}\rho^{-3n}).$$

The third term is

$$\frac{n\rho^{2n-1}}{2} \int_1^\infty x^{(n-5)/2} (\rho^2 x^2 - 1)^{3/2} G_n^{-3/2} (x-1)x^{2n-1} dx \\ = \frac{n\rho^{2n-1}}{2} \int_1^\infty x^{(n-5)/2} \left\{ \frac{\rho^2 x^2 - 1}{\rho^2 x - 1 + (x-1)x^{2n-1}\rho^{2n}} \right\}^{3/2} (x-1)x^{2n-1} dx$$

$$\begin{aligned}
&= \frac{n\rho^{-n-1}}{2} \int_1^\infty \frac{x^{(n-5)/2+2n-1}}{x^{3n-3/2}} \left\{ \frac{\rho^2 x^2 - 1}{(x-1) + \frac{\rho^2 x - 1}{x^{2n-1} \rho^{2n}}} \right\}^{3/2} (x-1) dx \\
&= \frac{n\rho^{-n-1}}{2} \int_1^\infty \frac{1}{x^{n/2+2}} \left\{ \frac{\rho^2 x^2 - 1}{(x-1) + \frac{\rho^2 x - 1}{x^{2n-1} \rho^{2n}}} \right\}^{3/2} (x-1) dx \\
&= O(\rho^{-n}),
\end{aligned}$$

and therefore dominates (A.37). It follows that $b_n(\rho) = \rho + O(\rho^{-n})$, showing that the bias is exponentially small (and negative, in view of the sign of the final term of (A.37)) for $\rho > 1$. A similar result holds when $\rho < -1$, in which case the bias is exponentially small and positive. *Q.E.D.*

PROOF OF THE ALTERNATE FORM OF $g^-(c)$: We need to show that the leading term $g^-(c)$ in the binding function when $\rho = 1 + c/n$ and $c < 0$ has the alternate form

$$\begin{aligned}
\text{(A.38)} \quad g^-(c) &= -\frac{3}{4} \int_0^\infty e^{-(1/4)v} k^-(v; c)^{1/2} dv + \frac{1}{4n} \int_0^\infty e^{-(1/4)v} k^-(v; c)^{3/2} dv \\
&\quad - \frac{e^{2c}}{8} \int_0^\infty e^{-(5/4)v} k^-(v; c)^{3/2} v dv,
\end{aligned}$$

where

$$\text{(A.39)} \quad k^-(v; c) := \frac{4c - 2v}{4c - v - e^{2c} v e^{-v}}.$$

We start with the following expression for $g^-(c)$ which is established in the paper (see (A.28) above):

$$\begin{aligned}
g^-(c) &= -\frac{3}{4} \int_0^1 y^{-3/4} \left\{ \frac{4c + \log y}{4c + 2 \log y} + \frac{\log y}{4c + 2 \log y} y e^{2c} \right\}^{-1/2} dy \\
&\quad + \frac{1}{4} \int_0^1 y^{-3/4} \left\{ \frac{4c + \log y}{4c + 2 \log y} + \frac{\log y}{4c + 2 \log y} y e^{2c} \right\}^{-3/2} dy \\
&\quad + \frac{e^{2c}}{8} \int_0^1 y^{1/4} \left\{ \frac{4c + \log y}{4c + 2 \log y} + \frac{\log y}{4c + 2 \log y} y e^{2c} \right\}^{-3/2} \log y dy.
\end{aligned}$$

Transform $w = \log y$ so that $w \in [0, \infty)$ and $dy = e^w dw$, with range $w \in (-\infty, 0]$ for $x \in (0, 1]$, giving for $c \leq 0$,

$$\begin{aligned}
\text{(A.40)} \quad g^-(c) &= -\frac{3}{4} \int_0^1 y^{-3/4} \left\{ \frac{4c + \log y + ye^{2c} \log y}{4c + 2 \log y} \right\}^{-1/2} dy \\
&\quad + \frac{1}{4} \int_0^1 y^{-3/4} \left\{ \frac{4c + \log y + ye^{2c} \log y}{4c + 2 \log y} \right\}^{-3/2} dy \\
&\quad + \frac{e^{2c}}{8} \int_0^1 y^{1/4} \left\{ \frac{4c + \log y + ye^{2c} \log y}{4c + 2 \log y} \right\}^{-3/2} \log y dy \\
&= -\frac{3}{4} \int_{-\infty}^0 e^{(1/4)w} \left\{ \frac{4c + 2w}{4c + w + e^{2c} w e^w} \right\}^{1/2} dw \\
&\quad + \frac{1}{4} \int_{-\infty}^0 e^{(1/4)w} \left\{ \frac{4c + 2w}{4c + w + e^{2c} w e^w} \right\}^{3/2} dw \\
&\quad + \frac{e^{2c}}{8} \int_{-\infty}^0 e^{(5/4)w} \left\{ \frac{4c + 2w}{4c + w + e^{2c} w e^w} \right\}^{3/2} w dw \\
&= -\frac{3}{4} \int_0^\infty e^{-(1/4)v} \left\{ \frac{4c - 2v}{4c - v - e^{2c} v e^{-v}} \right\}^{1/2} dv \\
&\quad + \frac{1}{4} \int_0^\infty e^{-(1/4)v} \left\{ \frac{4c - 2v}{4c - v - e^{2c} v e^{-v}} \right\}^{3/2} dv \\
&\quad - \frac{e^{2c}}{8} \int_0^\infty e^{-(5/4)v} \left\{ \frac{4c - 2v}{4c - v - e^{2c} v e^{-v}} \right\}^{3/2} v dv,
\end{aligned}$$

which gives the stated result (A.38). Observe that for $v \geq 0$, we have

$$k^-(v; c) = \frac{4c - 2v}{4c - v - e^{2c} v e^{-v}} 1\{c < 0\} > 0,$$

so the expressions with fractional exponents in (A.40) are all nonnegative quantities. At $c = 0$, we have

$$\begin{aligned}
g^-(0) &= -\frac{3}{4} \int_0^\infty e^{(1/4)v} \left\{ \frac{2v}{v + ve^v} \right\}^{1/2} dv + \frac{1}{4} \int_0^\infty e^{(5/4)v} \left\{ \frac{2v}{v + ve^v} \right\}^{3/2} dv \\
&\quad - \frac{1}{8} \int_0^\infty e^{(1/4)v} \left\{ \frac{2v}{v + ve^v} \right\}^{3/2} v dv \\
&= -\frac{3}{4} 5.2441 + \frac{9.2441}{4} - \frac{1.275}{8} = -1.7814
\end{aligned}$$

upon direct evaluation, reproducing the earlier result at $c = 0$. Further note that, as $c \rightarrow -\infty$, we have

$$(A.41) \quad \lim_{c \rightarrow -\infty} g^-(c) = -\frac{2}{4} \int_0^\infty e^{-(1/4)v} dv = -2. \quad Q.E.D.$$

PROPERTIES OF $g^-(c)$: We start with some properties of the functions $k^-(v; c)$ and $g^-(c)$ in (A.39) and (A.38), namely

$$\begin{aligned} k^-(v; c) &:= \frac{4c - 2v}{4c - v - e^{2c}ve^{-v}} = \frac{2v - 4c}{v + e^{2c}ve^{-v} - 4c}, \\ g^-(c) &= -\frac{3}{4} \int_0^\infty e^{-(1/4)v} k^-(v; c)^{1/2} dv \\ &\quad + \frac{1}{4} \int_0^\infty e^{-(1/4)v} k^-(v; c)^{3/2} dv \\ &\quad - \frac{e^{2c}}{8} \int_0^\infty e^{-(5/4)v} k^-(v; c)^{3/2} v dv. \end{aligned}$$

First note the limits at the domain of definition of c ,

$$\begin{aligned} \lim_{c \rightarrow 0} k^-(v; c) &= \frac{2}{1 + e^{-v}} \leq 2, \\ \lim_{c \rightarrow -\infty} k^-(v; c) &= 1, \end{aligned}$$

and so

$$\lim_{c \rightarrow -\infty} g^-(c) = -\frac{2}{4} \int_0^\infty e^{-(1/4)v} dv = -2$$

as in (A.41) above. Next note that $v + e^{2c}ve^{-v} - 4c \leq 2v - 4c$ as $e^{2c}e^{-v} < 1$, so that

$$k^-(v; c) \geq \frac{2v - 4c}{v + ve^{-v} - 4c} \geq \frac{2v - 4c}{2v - 4c} = 1,$$

and since $e^{2c}ve^{-v} > 0$ and $-2c \geq 0$,

$$k^-(v; c) \leq \frac{2v - 4c}{v - 4c} \leq \frac{2v - 4c}{v - 2c} = 2,$$

we have

$$(A.42) \quad k^-(v; c) \in [1, 2].$$

Next consider the derivative $k_c^-(v; c) = \frac{\partial}{\partial c} k^-(v; c)$, which has the form

$$\begin{aligned}
 \text{(A.43)} \quad k_c^-(v; c) &= \frac{(v + e^{2c}ve^{-v} - 4c)(-4) - (2v - 4c)(2e^{2c}ve^{-v} - 4)}{(v + e^{2c}ve^{-v} - 4c)^2} \\
 &= \frac{(v + e^{2c}ve^{-v} - 4c)(-4) - 2(2v - 4c)e^{2c}ve^{-v} + 4(2v - 4c)}{(v + e^{2c}ve^{-v} - 4c)^2} \\
 &= \frac{4v - 4e^{2c}ve^{-v} - 4e^{2c}v^2e^{-v} + 8ce^{2c}ve^{-v}}{(v + e^{2c}ve^{-v} - 4c)^2} \\
 &= 4 \frac{v - e^{2c}v(1+v)e^{-v} + 2ce^{2c}ve^{-v}}{(v + e^{2c}ve^{-v} - 4c)^2} \\
 &= 4v \frac{1 - e^{2c}(1+v)e^{-v} + 2ce^{2c}e^{-v}}{(v + e^{2c}ve^{-v} - 4c)^2} \\
 &= 4v \frac{1 - e^{2c}e^{-v}(1+v-2c)}{(v + e^{2c}ve^{-v} - 4c)^2} \\
 &= 4v \frac{1 - e^{2c}e^{-v}(1+v-2c)}{((v-2c) + ve^{-(v-2c)} - 2c)^2} \\
 &= 4v \frac{1 - e^{2c}e^{-v}(1+v-2c)}{((v-2c) + (v-2c)e^{-(v-2c)} - 2c(1 - e^{-(v-2c)}))^2} \\
 &= 4v \frac{1 - e^{2c}e^{-v}(1+v-2c)}{((v-2c) + (v-2c)e^{-(v-2c)} - 2c(1 - e^{-(v-2c)}))^2}.
 \end{aligned}$$

This expression is bounded above as

$$\begin{aligned}
 \text{(A.44)} \quad k_c^-(v; c) &\leq 4v \frac{1 - e^{-(v-2c)}(1+v-2c)}{(v-2c)^2(1 + e^{-(v-2c)})^2} \\
 &= \frac{4v}{(v-2c)(1 + e^{-(v-2c)})^2} \frac{1 - e^{-(v-2c)}(1+v-2c)}{(v-2c)} \\
 &\leq \frac{v}{v-2c} \frac{1 - e^{-(v-2c)}(1+v-2c)}{(v-2c)} \\
 \text{(A.45)} \quad &\leq \frac{v}{v-2c} \max_{x \geq 0} \frac{1 - e^{-x}(1+x)}{x} < 0.3 \frac{v}{v-2c} \leq 0.3,
 \end{aligned}$$

since

$$\text{(A.46)} \quad \max_{x \geq 0} \frac{1 - e^{-x}(1+x)}{x} = \max_{v \geq 0, c \leq 0} \frac{1 - e^{-(v-2c)}(1+v-2c)}{(v-2c)} = 0.29843.$$

A lower bound for $k_c^-(v; c)$ is obtained from (A.43) as

$$\begin{aligned}
 k_c^-(v; c) &= 4v \frac{1 - e^{2c} e^{-v}(1 + v - 2c)}{((v - 2c) + (v - 2c)e^{-(v-2c)} - 2c(1 - e^{-(v-2c)}))^2} \\
 &\geq 4v \frac{1 - e^{2c} e^{-v}(1 + v - 2c)}{((v - 2c) + (v - 2c)e^{-(v-2c)} + v - 2c)^2} \\
 &= \frac{4v}{(v - 2c)^2} \frac{1 - e^{2c} e^{-v}(1 + v - 2c)}{(2 + e^{-(v-2c)})^2} \\
 &\geq \frac{4v}{9} \frac{1 - e^{-(v-2c)}(1 + v - 2c)}{(v - 2c)^2} \\
 &= \frac{4}{9} \frac{v}{v - 2c} \frac{1 - e^{-(v-2c)}(1 + v - 2c)}{(v - 2c)}.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \text{(A.47)} \quad \frac{4}{9} \frac{v}{v - 2c} \frac{1 - e^{-(v-2c)}(1 + v - 2c)}{(v - 2c)} &\leq k_c^-(v; c) \\
 &\leq \frac{v}{v - 2c} \times \frac{1 - e^{-(v-2c)}(1 + v - 2c)}{(v - 2c)}
 \end{aligned}$$

or

$$\text{(A.48)} \quad \frac{4}{9} K(v; c) \leq k_c^-(v; c) \leq K(v; c),$$

where

$$K(v; c) = \frac{v}{v - 2c} \frac{1 - e^{-(v-2c)}(1 + v - 2c)}{(v - 2c)}.$$

Next note the following inequalities, which follow from (A.42), (A.47), and (A.46):

$$\text{(A.49)} \quad \frac{1}{2} \leq \frac{1}{k^-(v; c)} \leq 1, \quad -1 \leq -\frac{1}{k^-(v; c)^{1/2}} \leq -\frac{1}{2^{1/2}},$$

$$\text{(A.50)} \quad 1 \leq k^-(v; c)^{1/2} \leq 2^{1/2}, \quad 1 \leq k^-(v; c)^{3/2} \leq 2^{3/2},$$

and

$$\text{(A.51)} \quad 0 \leq \frac{4}{9} K(v; c) \leq k_c^-(v; c) \leq K(v; c) \leq 0.3.$$

Now the derivative of $g^-(c)$ is

$$\begin{aligned}
 \text{(A.52)} \quad g^-(c) &= -\frac{3}{8} \int_0^\infty e^{-(1/4)v} k_c^-(v; c) k^-(v; c)^{-1/2} dv \\
 &\quad + \frac{3}{8} \int_0^\infty e^{-(1/4)v} k_c^-(v; c) k^-(v; c)^{1/2} dv \\
 &\quad - \frac{e^{2c}}{4} \int_0^\infty e^{-(5/4)v} k^-(v; c)^{3/2} v dv \\
 &\quad - \frac{3e^{2c}}{16} \int_0^\infty e^{-(5/4)v} k_c^-(v; c) k^-(v; c)^{1/2} v dv.
 \end{aligned}$$

We consider each term of (A.52) in turn. Using (A.49)–(A.51), we have

$$\begin{aligned}
 &-\frac{3}{8} \int_0^\infty e^{-(1/4)v} k_c^-(v; c) k^-(v; c)^{-1/2} dv \\
 &\geq -\frac{3}{8} \int_0^\infty e^{-(1/4)v} K(v; c) k^-(v; c)^{-1/2} dv \\
 &\geq -\frac{3}{8} \int_0^\infty e^{-(1/4)v} K(v; c) dv, \\
 &+\frac{3}{8} \int_0^\infty e^{-(1/4)v} k_c^-(v; c) k^-(v; c)^{1/2} dv \\
 &\geq \frac{43}{98} \int_0^\infty e^{-(1/4)v} K(v; c) k^-(v; c)^{1/2} dv \\
 &\geq \frac{43}{98} \int_0^\infty e^{-(1/4)v} K(v; c) dv, \\
 &-\frac{e^{2c}}{4} \int_0^\infty e^{-(5/4)v} k^-(v; c)^{3/2} v dv \\
 &\geq -\frac{2^{3/2}}{4} \int_0^\infty e^{-(5/4)v} v dv,
 \end{aligned}$$

and

$$\begin{aligned}
 &-\frac{3e^{2c}}{16} \int_0^\infty e^{-(5/4)v} k_c^-(v; c) k^-(v; c)^{1/2} v dv \\
 &\geq -\frac{3e^{2c}}{16} 2^{1/2} \int_0^\infty e^{-(5/4)v} K(v; c) v dv.
 \end{aligned}$$

Combining these inequalities, we find that

$$\begin{aligned}
g^{-'}(c) &= -\frac{3}{8} \int_0^\infty e^{-(1/4)v} k_c^-(v; c) k^-(v; c)^{-1/2} dv \\
&\quad + \frac{3}{8} \int_0^\infty e^{-(1/4)v} k_c^-(v; c) k^-(v; c)^{1/2} dv \\
&\quad - \frac{e^{2c}}{4} \int_0^\infty e^{-(5/4)v} k^-(v; c)^{3/2} dv \\
&\quad - \frac{3e^{2c}}{16} \int_0^\infty e^{-(5/4)v} k_c^-(v; c) k^-(v; c)^{1/2} v dv \\
&\geq -\frac{3}{8} \int_0^\infty e^{-(1/4)v} K(v; c) dv + \frac{4}{9} \frac{3}{8} \int_0^\infty e^{-(1/4)v} K(v; c) dv \\
&\quad - \frac{2^{3/2}}{4} \int_0^\infty e^{-(5/4)v} v dv \\
&\quad - \frac{3e^{2c}}{16} 2^{1/2} \int_0^\infty e^{-(5/4)v} K(v; c) v dv \\
&\geq -\frac{5}{24} \int_0^\infty e^{-(1/4)v} K(v; c) dv + 0 - \frac{2^{3/2}}{4} \frac{16}{25} \\
&\quad - \frac{3}{16} 2^{1/2} \int_0^\infty e^{-(5/4)v} K(v; c) v dv \\
&\geq -\frac{5}{6} 0.3 - \frac{2^{3/2} 4}{25} - 0.3 \frac{3}{16} 2^{1/2} \int_0^\infty e^{-(5/4)v} v dv \\
&= -\frac{5}{6} 0.3 - \frac{2^{3/2} 4}{25} - 0.3 \frac{3}{16} \frac{16}{25} 2^{1/2} \\
&= -0.75346.
\end{aligned}$$

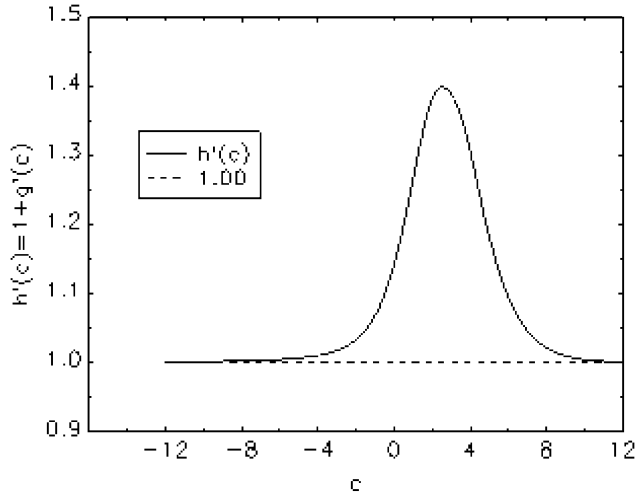
It follows that the limit function $h^-(c) = c + g^-(c)$ has derivative

$$h^{-'}(c) = 1 + g^{-'}(c) \geq 1 - 0.75346 = 0.24654 > 0,$$

so that $h^-(c)$ is an increasing function of c for all $c \in (-\infty, 0)$. This is a lower bound. Direct computation of $h^{-'}(c)$ shows that $h^{-'}(c) \geq 1$, as is evident in Figure A.1. Note also that

$$\lim_{c \rightarrow -\infty} k_c^-(v; c) = \lim_{c \rightarrow -\infty} 4v \frac{1 - e^{2c} e^{-v} (1 + v - 2c)}{(v + e^{2c} v e^{-v} - 4c)^2} = 0,$$

so that $\lim_{c \rightarrow -\infty} g^{-'}(c) = 0$ and $\lim_{c \rightarrow -\infty} h^{-'}(c) = 1$.

FIGURE A.1.—Derivative $h'(c)$ of the limit function $h(c)$.

PROPERTIES OF $g^+(c)$: We start with some properties of the functions $k^+(v; c)$ and $g^+(c)$, which we restate here for convenience:

$$k^+(w; c) := \frac{4c + 2w}{4c + w + e^{2c}we^w},$$

$$g^+(c) = \frac{3}{4} \int_0^\infty e^{(1/4)w} k^+(w; c)^{1/2} dw - \frac{1}{4} \int_0^\infty e^{(1/4)w} k^+(w; c)^{3/2} dw$$

$$- \frac{e^{2c}}{8} \int_0^\infty e^{(5/4)w} k^+(w; c)^{3/2} w dw.$$

First note that at the limits at the domain of definition of c , we have

$$\lim_{c \rightarrow 0} k^+(w; c) = \frac{2}{1 + e^w} \leq 1, \quad \lim_{c \rightarrow \infty} k^+(w; c) = 0,$$

and

$$(A.53) \quad \lim_{c \rightarrow \infty} g^+(c) = 0.$$

Next, as $e^{2c}e^w \geq 1$,

$$k^+(w; c) = \frac{4c + 2w}{4c + 2w + w(e^{w+2c} - 1)} \leq \frac{2w + 4c}{2w + 4c} = 1,$$

so we have

$$k^+(w; c) \in [0, 1].$$

Now consider the derivative $k_c^+(w; c) = \frac{\partial}{\partial c} k^+(w; c)$, which has the form

$$\begin{aligned}
 k_c^+(w; c) &= \frac{(4c + w + e^{2c}we^w)(4) - (2w + 4c)(2e^{2c}we^w + 4)}{(4c + w + e^{2c}we^w)^2} \\
 &= \frac{(-w + e^{2c}we^w)(4) - (2w + 4c)2e^{2c}we^w}{(4c + w + e^{2c}we^w)^2} \\
 &= \frac{-4w + 4e^{2c}we^w - 4e^{2c}w^2e^w - 8ce^{2c}we^w}{(4c + w + e^{2c}we^w)^2} \\
 &= 4w \frac{-1 + e^{2c}e^w - e^{2c}we^w - 2ce^{2c}e^w}{(4c + w + e^{2c}we^w)^2} \\
 &= 4w \frac{-1 + e^{2c}e^w(1 - w - 2c)}{(4c + w + e^{2c}we^w)^2} = -4w \frac{1 + e^{w+2c}(w + 2c - 1)}{(4c + w + we^{w+2c})^2},
 \end{aligned}$$

which is negative since $1 + e^x(x - 1) \geq 0$ for $x = w + 2c \geq 0$. The derivative of $g^+(c)$ is

$$\begin{aligned}
 \text{(A.54)} \quad g^{+'}(c) &= \frac{3}{8} \int_0^\infty e^{(1/4)w} k_c^+(w; c) k^+(w; c)^{-1/2} dw \\
 &\quad - \frac{3}{8} \int_0^\infty e^{(1/4)w} k_c^+(w; c) k^+(w; c)^{1/2} dw \\
 &\quad - \frac{e^{2c}}{4} \int_0^\infty e^{(5/4)w} k_c^+(w; c)^{3/2} w dw \\
 &\quad - \frac{3e^{2c}}{16} \int_0^\infty e^{(5/4)w} k_c^+(w; c) k^+(w; c)^{1/2} w dw.
 \end{aligned}$$

It follows that $h^+(c) = c + g^+(c)$ has derivative $h^{+'}(c) = 1 + g^{+'}(c)$, and, by direct computation, $h^{+'}(c) \geq 1$ and the limit function $h^+(c)$ is an increasing function of c , as is evident in Figure A.1. Since $\lim_{c \rightarrow \infty} k^+(w; c) = 0$ and

$$\lim_{c \rightarrow \infty} k_c^+(w; c) = \lim_{c \rightarrow \infty} \left\{ -4w \frac{1 + e^{w+2c}(w + 2c - 1)}{(4c + w + we^{w+2c})^2} \right\} = 0,$$

we deduce that $\lim_{c \rightarrow \infty} g^{+'}(c) = 0$ and $\lim_{c \rightarrow \infty} h^{+'}(c) = 1$.

PROOF OF CONTINUITY OF $g(c)$ AT $c = 0$: Observe that

$$\begin{aligned}
 g^-(0) &= -\frac{3}{4} \int_0^\infty e^{(1/4)v} \left\{ \frac{2}{1 + e^v} \right\}^{1/2} dv + \frac{1}{4} \int_0^\infty e^{(5/4)v} \left\{ \frac{2}{1 + e^v} \right\}^{3/2} dv \\
 &\quad - \frac{1}{8} \int_0^\infty e^{(1/4)v} \left\{ \frac{2}{1 + e^v} \right\}^{3/2} v dv
 \end{aligned}$$

and

$$\begin{aligned} g^+(0) &= \frac{3}{4} \int_0^\infty e^{(1/4)w} \left(\frac{2}{1+e^w} \right)^{1/2} dw \\ &\quad - \frac{1}{4} \int_0^\infty e^{(1/4)w} \left(\frac{2}{1+e^w} \right)^{3/2} dw \\ &\quad - \frac{1}{8} \int_0^\infty e^{(5/4)w} \left(\frac{2}{1+e^w} \right)^{3/2} w dw, \end{aligned}$$

so that

$$\begin{aligned} \text{(A.55)} \quad g^+(0) - g^-(0) &= \frac{3}{2} \int_0^\infty e^{(1/4)w} \left(\frac{2}{1+e^w} \right)^{1/2} dw \\ &\quad - \frac{1}{4} \int_0^\infty e^{(1/4)w} (1+e^w) \left(\frac{2}{1+e^w} \right)^{3/2} dw \\ &\quad - \frac{1}{8} \int_0^\infty e^{(5/4)w} \left(\frac{2}{1+e^w} \right)^{3/2} w dw \\ &\quad + \frac{1}{8} \int_0^\infty e^{(1/4)v} \left\{ \frac{2}{1+e^v} \right\}^{3/2} v dv \\ &= \int_0^\infty e^{(1/4)w} \left(\frac{2}{1+e^w} \right)^{1/2} dw \\ &\quad - \frac{1}{8} \int_0^\infty e^{(1/4)w} (e^w - 1) \left(\frac{2}{1+e^w} \right)^{3/2} w dw \\ &= \int_0^\infty e^{(1/4)w} \left(\frac{2}{1+e^w} \right)^{1/2} \left(1 - \frac{1}{4} \frac{e^w - 1}{1+e^w} w \right) dw \\ &= \frac{1}{4} \int_0^\infty e^{(1/4)w} \left(\frac{2}{1+e^w} \right)^{1/2} \frac{4 + 4e^w - we^w + w}{1+e^w} dw \\ &= \frac{1}{8} \int_0^\infty e^{(1/4)w} \left(\frac{2}{1+e^w} \right)^{3/2} (4 + w + e^w(4 - w)) dw \\ &= \frac{2^{3/2}}{8} \int_0^\infty e^{-(5/4)w} (1 + e^{-w})^{-3/2} (4 + w + e^w(4 - w)) dw. \end{aligned}$$

Now upon expansion and integration term by term, which is valid by majorization of the series, (A.55) can be written as

$$\begin{aligned}
& \int_0^\infty e^{-(5/4)w} (1 + e^{-w})^{-3/2} (4 + w + e^w(4 - w)) dw \\
&= \int_0^\infty e^{-(5/4)w} \\
&\quad \times \sum_{j=0}^\infty \frac{\binom{3}{2}_j}{j!} (-1)^j e^{-jw} (4 + w + e^w(4 - w)) dw \\
&= \sum_{j=0}^\infty \frac{\binom{3}{2}_j}{j!} (-1)^j \\
&\quad \times \int_0^\infty e^{-((5/4)+j)w} (4 + w + e^w(4 - w)) dw \\
&= \sum_{j=0}^\infty \frac{\binom{3}{2}_j}{j!} (-1)^j \\
&\quad \times \left\{ \int_0^\infty e^{-((5/4)+j)w} (4 + w) dw + \int_0^\infty e^{-((1/4)+j)w} (4 - w) dw \right\} \\
&= \sum_{j=0}^\infty \frac{\binom{3}{2}_j}{j!} (-1)^j \left\{ \frac{4(16j + 24)}{(4j + 5)^2} + \frac{4}{(4j + 1)^2} 16j \right\} \\
&= \sum_{j=0}^\infty \frac{\binom{3}{2}_j}{j!} (-1)^j \frac{4(16j + 24)}{(4j + 5)^2} + \sum_{j=0}^\infty \frac{\binom{3}{2}_j}{j!} (-1)^j \frac{4}{(4j + 1)^2} 16j \\
&= 32 \sum_{j=0}^\infty \frac{\binom{3}{2}_j}{j!} (-1)^j \frac{(2j + 3)}{(4j + 5)^2} + 32 \sum_{j=0}^\infty \frac{\binom{3}{2}_j}{j!} (-1)^j \frac{2j}{(4j + 1)^2} \\
&= 32 \sum_{j=0}^\infty \frac{\binom{3}{2}_j}{j!} (-1)^j \frac{(2j + 3)}{(4j + 5)^2} + 32 \sum_{j=1}^\infty \frac{\binom{3}{2}_j}{j!} (-1)^j \frac{2j}{(4j + 1)^2}
\end{aligned}$$

$$\begin{aligned}
&= 32 \sum_{j=0}^{\infty} \frac{\left(\frac{3}{2}\right)_j}{j!} (-1)^j \frac{(2j+3)}{(4j+5)^2} + 32 \sum_{j=1}^{\infty} \frac{\left(\frac{3}{2}\right)_j}{j!} (-1)^{(j-1)+1} \\
&\quad \times \frac{2(j-1)+2}{(4(j-1)+5)^2} \\
&= 32 \sum_{j=0}^{\infty} \frac{\left(\frac{3}{2}\right)_j}{j!} (-1)^j \frac{(2j+3)}{(4j+5)^2} - 32 \sum_{k=0}^{\infty} \frac{\left(\frac{3}{2}\right)_{k+1}}{(k+1)!} (-1)^k \frac{2k+2}{(4k+5)^2} \\
&= 32 \sum_{j=0}^{\infty} \frac{(-1)^j}{(4j+5)^2} \left\{ \frac{\left(\frac{3}{2}\right)_j (2j+3)}{j!} - \frac{\left(\frac{3}{2}\right)_{j+1} (2j+2)}{(j+1)!} \right\} \\
&= 32 \sum_{j=0}^{\infty} \frac{(-1)^j}{(4j+5)^2} \left\{ \frac{\left(\frac{3}{2}\right)_j (2j+3)}{j!} - \frac{2\left(\frac{3}{2}\right)_{j+1}}{j!} \right\} \\
&= 32 \sum_{j=0}^{\infty} \frac{(-1)^j}{(4j+5)^2 j!} \left\{ \left(\frac{3}{2}\right)_j (2j+3) - 2\left(\frac{3}{2}\right)_{j+1} \right\} \\
&= \frac{32}{\Gamma\left(\frac{3}{2}\right)} \sum_{j=0}^{\infty} \frac{(-1)^j}{(4j+5)^2 j!} \left\{ \Gamma\left(\frac{3}{2}+j\right) (2j+3) - 2\Gamma\left(\frac{3}{2}+j+1\right) \right\} \\
&= \frac{32}{\Gamma\left(\frac{3}{2}\right)} \sum_{j=0}^{\infty} \frac{(-1)^j}{(4j+5)^2 j!} \left\{ 2\Gamma\left(\frac{3}{2}+j+1\right) - 2\Gamma\left(\frac{3}{2}+j+1\right) \right\} = 0,
\end{aligned}$$

which proves that $g^+(0) - g^-(0) = 0$, as required.

Q.E.D.

PROPERTIES OF THE LIMIT FUNCTION $h(c)$: Combining (A.54) with (A.52), we have

$$g'(c) = g^-(c)1_{\{c \leq 0\}} + g^+(c)1_{\{c > 0\}}.$$

Write $h(c) = c + g(c) = h^-(c)1_{\{c \leq 0\}} + h^+(c)1_{\{c > 0\}}$. The derivative is then

$$\begin{aligned}
h'(c) &= 1 + g'(c) = h^-(c)1_{\{c \leq 0\}} + h^+(c)1_{\{c > 0\}} \\
&= 1 + g^-(c)1_{\{c \leq 0\}} + g^+(c)1_{\{c > 0\}}
\end{aligned}$$

and tends to unity as $|c| \rightarrow \infty$, as seen in Figure A.1 The limit function $h(c)$ is monotonically increasing over $c \in (-\infty, \infty)$. Moreover, in view of (A.41) and (A.53), we have

$$(A.56) \quad h(c) \sim \begin{cases} c - 2, & \text{as } c \rightarrow -\infty, \\ c, & \text{as } c \rightarrow \infty, \end{cases}$$

so that $h(c)$ is linear in c for large $|c|$.

PROPERTIES OF THE DERIVATIVES $b_n^{(j)}(\rho)$: For $\rho = 1 + \frac{c}{n}$ and for large n , we have

$$\frac{\partial}{\partial c} b_n \left(1 + \frac{c}{n} \right) = \frac{1}{n} b_n^{(1)} \left(1 + \frac{c}{n} \right) = \frac{1}{n} (1 + g'(c)) + O(n^{-2}) > 0,$$

and so

$$(A.57) \quad \frac{\partial}{\partial \rho} b_n(\rho) = 1 + g'(n(\rho - 1)) + O(n^{-1}) > 0.$$

Hence, the binding function $b_n(\rho)$ is monotonic and increasing for large enough n . Furthermore,

$$\frac{\partial^j}{\partial c^j} b_n \left(1 + \frac{c}{n} \right) = \frac{1}{n^j} b_n^{(j)} \left(1 + \frac{c}{n} \right) = \frac{1}{n} (1 + g^{(j)}(c)) \{1 + o(1)\}$$

and then

$$\begin{aligned} \frac{\partial^j}{\partial \rho^j} b_n(\rho) &= n^{j-1} (1 + g^{(j)}(c)) \{1 + o(1)\} \\ &= n^{j-1} (1 + g^{(j)}(n(\rho - 1))) \{1 + o(1)\} \\ &= O(n^{j-1}). \end{aligned}$$

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Manuscript received June, 2010; final revision received June, 2011.