Supplementary Material for "Semiparametric Estimation of the Canonical Permanent-Transitory Model of Earnings Dynamics"

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Abstract

This supplementary material presents (i) proofs for the identification, (ii) further details on estimation, (iii) large sample theories for the estimators proposed in the main text of the paper, and (iv) additional estimation result in the application to the earnings dynamics of U.S. men.

A Proofs for the Identification

A.1 Proof of Lemma 1

Proof. By (2.1), (2.2) and (2.3), we can write $V_{t+\tau}$ as

$$V_{t+\tau} = V_t + Y_{t+\tau} - Y_t - \sum_{\tau'=1}^{\tau} \eta_{t+\tau'}. \tag{A.1}$$

Also, by (2.1), (2.2) and (2.3), we can write the first difference $Y_{t+q+1} - Y_{t+q}$ by

$$Y_{t+q+1} - Y_{t+q} = (\rho_{t+q+1,1} - 1)V_{t+q} + \sum_{p'=2}^{p} \rho_{t+q+1,p'} V_{t+q+1-p'} + G_{t+q+1}\left(\varepsilon_{t+q+1}, \varepsilon_{t+q}, \cdots, \varepsilon_{t+1}\right) + \eta_{t+q+1}. \tag{A.2}$$

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Substituting (A.1) for each $\tau \in \{1, \cdots, q\}$ in (A.2) and rearranging terms, we obtain

$$\mu^Y_{t+q+1}(\rho_{t+q+1}) = \kappa(\rho_{t+q+1}) V_t + \nu^{\eta,\varepsilon}_{t+q+1}(\rho_{t+q+1}). \tag{A.3}$$

Therefore, (4.1) follows under Assumption 8. □

A.2 Proof of Lemma 2

Proof. By Lemma 1, (4.1) holds and thus we have

$$\mathrm{E}\left[\frac{\mu^Y_{t+q+1}(\rho_{t+q+1})}{\kappa(\rho_{t+q+1})}\ e^{isY_t}\right] = \mathrm{E}\left[V_t\ e^{isY_t}\right] + \mathrm{E}\left[\frac{\nu^{\eta,\varepsilon}_{t+q+1}(\rho_{t+q+1})}{\kappa(\rho_{t+q+1})}\ e^{isY_t}\right],$$

where the expectation exists under Assumption 9 (i). The first term on the right-hand side is rewritten as $\mathrm{E}\left[e^{isU_t}\right]\mathrm{E}\left[V_t\ e^{isV_t}\right]$ by $U_t \perp\!\!\!\perp V_t$. The second term on the right-hand side is zero by (4.2) and (4.3) that hold under Assumption 7. Therefore, by Assumption 9 (ii), we obtain

$$\frac{d}{ds}\log\phi_{V_t}(s) = \frac{i\ \mathrm{E}\left[V_t\ e^{isV_t}\right]}{\mathrm{E}\left[e^{isV_t}\right]} = \frac{i\ \mathrm{E}\left[\frac{\mu^Y_{t+q+1}(\rho_{t+q+1})}{\kappa(\rho_{t+q+1})}\ e^{isY_t}\right]}{\mathrm{E}\left[e^{isY_t}\right]} = \frac{i\ \mathrm{E}\left[\mu^Y_{t+q+1}(\rho_{t+q+1})\ e^{isY_t}\right]}{\kappa(\rho_{t+q+1})\,\mathrm{E}\left[e^{isY_t}\right]}.$$

By Assumption 9 and Picard-Lindelöf theorem, we therefore obtain (4.4). Next, using (2.1) and (4.4), we obtain (4.5) under Assumption 9 (ii). Finally, using (2.2) and (4.5), we obtain (4.6) under Assumption 9 (ii). □

A.3 Proof of Theorem 1

Proof. Lemma 2 shows that ϕ_{η_t} is identified up to the finite-dimensional parameters ρ_{t+q} and ρ_{t+q+1} by (4.6) under the current assumptions. By Assumption 10 (i) and (ii), we identify the density function f_{U_t} up to the finite-dimensional parameters ρ_{t+q+1} by

$$f_{U_t}(\eta;\rho_{t+q+1}) \ = \ \frac{1}{2\pi}\int e^{-isu}\ \phi_{U_t}(s;\rho_{t+q+1})ds. \tag{A.4}$$

where $\phi_{U_t}(s;\rho_{t+q+1})$ is given by (4.5). Similarly, by Assumption 10 (iii) and (iv), we identify the density function f_{η_t} up to the finite-dimensional parameters ρ_{t+q} and ρ_{t+q+1} by

$$f_{\eta_t}(\eta;\rho_{t+q},\rho_{t+q+1}) \ = \ \frac{1}{2\pi}\int e^{-is\eta}\ \phi_{\eta_t}(s;\rho_{t+q},\rho_{t+q+1})ds. \tag{A.5}$$

where $\phi_{\eta_t}(s;\rho_{t+q},\rho_{t+q+1})$ is given by (4.6). Since U_t is first-order Markov under (2.2) and (4.8) that holds under Assumption 7 (i), the joint density of $(U_{t+\tau},\cdots,U_t)$ is written as

$$f_{U_t,\cdots,U_{t+\tau}}(u_t,\cdots,u_{t+\tau}) = f_{U_t}(u_t)\prod_{\tau'=1}^{\tau} f_{\eta_{t+\tau'}}(u_{t+\tau'} - u_{t+\tau'-1}). \tag{A.6}$$

Combining (A.4), (A.5) and (A.6) together yields (4.9).

Furthermore, by using (A.6), we identify the joint characteristic function $\phi_{U_t,\cdots,U_{t+\tau}}$ by

$$\begin{aligned}&\phi_{U_t,\cdots,U_{t+\tau}}(s_t,\cdots,s_{t+\tau}) = \\ &\int\cdots\int \exp\left(i\sum_{\tau'=0}^{\tau} s_{t+\tau'}u_{t+\tau'}\right) f_{U_t}(u_t)\prod_{\tau'=1}^{\tau} f_{\eta_{t+\tau'}}(u_{t+\tau'}-u_{t+\tau'-1})du_t\cdots du_{t+\tau}.\end{aligned} \tag{A.7}$$

Thus, by (2.1) and (4.7) that holds under Assumption 7 (i), we in turn identify the joint characteristic function $\phi_{V_t,\cdots,V_{t+\tau}}$ by

$$\begin{aligned}&\phi_{V_t,\cdots,V_{t+\tau}}(s_t,\cdots,s_{t+\tau}) = \phi_{Y_t,\cdots,Y_{t+\tau}}(s_t,\cdots,s_{t+\tau})/\phi_{U_t,\cdots,U_{t+\tau}}(s_t,\cdots,s_{t+\tau}) = \\ &\frac{\mathrm{E}\left[\exp\left(i\sum_{\tau'=0}^{\tau} s_{t+\tau'}Y_{t+\tau'}\right)\right]}{\int\cdots\int\exp\left(i\sum_{\tau'=0}^{\tau} s_{t+\tau'}u_{t+\tau'}\right) f_{U_t}(u_t)\prod_{\tau'=1}^{\tau} f_{\eta_{t+\tau'}}(u_{t+\tau'}-u_{t+\tau'-1})du_t\cdots du_{t+\tau}}.\end{aligned}$$

Under Assumption 10 (v), we can then recover the joint characteristic function $\phi_{V_t,\cdots,V_{t+\tau}}$ by

$$\begin{aligned}&f_{V_t,\cdots,V_{t+\tau}}(v_t,\cdots,v_{t+\tau}) = \frac{1}{(2\pi)^{\tau+1}}\int\cdots\int \\ &\frac{\mathrm{E}\left[\exp\left(i\sum_{\tau'=0}^{\tau} s_{t+\tau'}(Y_{t+\tau'}-v_{t+\tau'})\right)\right]}{\int\cdots\int\exp\left(i\sum_{\tau'=0}^{\tau} s_{t+\tau'}u_{t+\tau'}\right) f_{U_t}(u_t)\prod_{\tau'=1}^{\tau} f_{\eta_{t+\tau'}}(u_{t+\tau'}-u_{t+\tau'-1})du_t\cdots du_{t+\tau}}ds_t\cdots ds_{t+\tau}.\end{aligned} \tag{A.8}$$

Combining (A.4), (A.5) and (A.8) together yields (4.10). □

A.4 Proof of Proposition 3

Proof. Substitute (2.1) and (2.2) in (A.3) with the time subscript reduced by q to get

$$\mu^Y_{t+1}(\rho_{t+1}) = \kappa(\rho_{t+1})(V_{t-q-1} + Y_{t-q} - Y_{t-q-1} - \eta_{t-q}) + \nu^{\eta,\varepsilon}_{t+1}(\rho_{t+1}).$$

Further decrementing the time subscript in (A.3) yields

$$\mu^Y_t(\rho_t) = \kappa(\rho_t)V_{t-q-1} + \nu^{\eta,\varepsilon}_t(\rho_t).$$

Using these two equations to eliminate V_{t-q-1}, we obtain the new restriction

$$\begin{aligned}&\kappa(\rho_t)\mu^Y_{t+1}(\rho_{t+1}) - \kappa(\rho_{t+1})\mu^Y_t(\rho_t) = \\ &\kappa(\rho_{t+1})\kappa(\rho_t)\left[Y_{t-q} - Y_{t-q-1} - \eta_{t-q}\right] + \kappa(\rho_t)\nu^{\eta,\varepsilon}_{t+1}(\rho_{t+1}) - \kappa(\rho_{t+1})\nu^{\eta,\varepsilon}_t(\rho_t) - \kappa(\rho_{t+1})\kappa(\rho_t)\eta_{t-q}\end{aligned}$$

By (4.2) and (4.3) that hold under Assumption 7, we have

$$\mathrm{E}\left[\kappa(\rho_t)\nu^{\eta,\varepsilon}_{t+1}(\rho_{t+1}) - \kappa(\rho_{t+1})\nu^{\eta,\varepsilon}_t(\rho_t) - \kappa(\rho_{t+1})\kappa(\rho_t)\eta_{t-q}\middle|\mathcal{I}_{t-q-1}\right] = 0.$$

Therefore, the moment equality (4.11) follows. □

B Further Details on Estimation

B.1 Estimator of ρ_t under Example 4

In this section, we describe estimation of the AR parameters under the parametric life-cycle specification of Example 4. For each $j \in \{1, \cdots, N\}$ and $t \in \{1+p+q, \cdots, T-1\}$, define

$$g_{j,t}(\theta) := (Y_{j,t-q-1}, \cdots, Y_{j,t-q-p})' \left\{ \kappa(h(t,\theta))\mu^Y_{j,t+1}(h(t+1,\theta)) - \kappa(h(t+1,\theta))\mu^Y_{j,t}(h(t,\theta)) \right.$$
$$\left. -\kappa(h(t,\theta))\kappa(h(t+1,\theta)) \left(Y_{j,t-q} - Y_{j,t-q-1}\right) \right\}$$

The GMM estimator for $\theta_0 \in \Theta$ is defined by

$$\widehat{\theta} = \arg\min_{\theta \in \Theta} \left[\frac{1}{N(T-p-q-1)} \sum_{j=1}^{N} \sum_{t=1+p+q}^{T-1} g_{j,t}(\theta) \right]' W \left[\frac{1}{N(T-p-q-1)} \sum_{j=1}^{N} \sum_{t=1+p+q}^{T-1} g_{j,t}(\theta) \right]$$

for a suitable weighting matrix W. The AR parameters may then be estimated by $\rho_t = h(t, \widehat{\theta})$ for each t. Since the asymptotic properties of GMM estimators is standard in the literature, we refer readers to Newey and McFadden (1994; Theorems 2.6 and 3.4).

B.2 Estimation of the MA Structure under Linearity

As remarked at the end of Section 3.4, the MA structure can be explicitly identified under the additional parametric linearity assumption. The MA parameter λ_t can be identified by imposing a restriction on (3.18). Like Example 4, we may impose a parametric life-cycle restriction $\lambda_t = l(t, \vartheta)$. By eliminating $var(\varepsilon_t)$ and $var(\varepsilon_{t+1})$ from (3.18), we obtain the restriction

$$\lambda_{t+2} var\left(V_{t+1} - \rho_{t+1}V_t\right) - \lambda_{t+2}\lambda_{t+1} cov\left(V_t - \rho_t V_{t-1}, V_{t+1} - \rho_{t+1}V_t\right)$$
$$= cov\left(V_{t+1} - \rho_{t+1}V_t, V_{t+2} - \rho_{t+2}V_{t+1}\right).$$

Substituting $\lambda_t = l(t, \vartheta)$, we obtain a minimum distance estimator

$$\widehat{\vartheta} = \arg\min_{\vartheta} d(\widehat{L}(\,\cdot\,, \vartheta), \widehat{L})$$

for some metric d, where $\widehat{L}(\,\cdot\,, \theta)$ and $\widehat{L}$ are given by

$$\begin{aligned} \widehat{L}(t,\vartheta) &= l(t+2,\vartheta)\widehat{var}\left(V_{t+1} - \widehat{\rho}_{t+1}V_t\right) - l(t+2,\vartheta)l(t+1,\vartheta)\widehat{cov}\left(V_t - \widehat{\rho}_t V_{t-1}, V_{t+1} - \widehat{\rho}_{t+1}V_t\right) \\ \widehat{L}(t) &= \widehat{cov}\left(V_{t+1} - \widehat{\rho}_{t+1}V_t, V_{t+2} - \widehat{\rho}_{t+2}V_{t+1}\right) \end{aligned}$$

for $t \in \{2, \cdots, T-q-3\}$. The variance and covariance estimates, $\widehat{var}$ and $\widehat{cov}$, can be computed by integration with respect to the multivariate density estimates $\widehat{f}_{V_t,V_{t+1},V_{t+2}}$, $\widehat{f}_{V_{t-1},V_t,V_{t+1}}$, and $\widehat{f}_{V_t,V_{t+1}}$ obtained in the previous estimation step. The MA parameter estimates are then given by $\widehat{\lambda}_t = l(t, \widehat{\vartheta})$ for each t.

Once λ_{t+1} is estimated, we can then use the analog estimator for (3.17), given by

$$\widehat{f}_{\varepsilon_t}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\widehat{\phi}_{\varepsilon_t}(s)\phi_K(hs)ds, \qquad \text{where}$$

$$\widehat{\phi}_{\varepsilon_t}(s) = \exp\left[\int_0^s \frac{i\widehat{E}\left[\left(\frac{V_{t+1}-\rho_{t+1}V_t}{\widehat{\lambda}_{t+1}}\right)\exp\left(is\left(V_t-\rho_t V_{t-1}\right)\right)\right]}{\widehat{E}\left[\exp\left(V_t-\widehat{\rho}_t V_{t-1}\right)\right]}ds\right]$$

The estimated expectations, $\widehat{\mathrm{E}}$, can be computed by integration with respect to the multivariate density estimates, $\widehat{f}_{V_{t-1},V_t}$ and $\widehat{f}_{V_t,V_{t+1}}$, obtained in the earlier estimation step.

B.3 Closed-Form Moment Estimators

Following the discussion of Section 5, we provide the closed-form estimators for the first four moments of V_t as follows.

$$\frac{\widehat{\phi}^{(1)}_{V_t}(0;\widehat{\rho}_{t+q+1})}{i^1} = \frac{\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})]}{\kappa(\widehat{\rho}_{t+q+1})},$$

$$\frac{\widehat{\phi}^{(2)}_{V_t}(0;\widehat{\rho}_{t+q+1})}{i^2} = \frac{\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})]^2}{\kappa(\widehat{\rho}_{t+q+1})^2} + \frac{\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})Y_{j,t}] - \widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})]\widehat{\mathrm{E}}_N[Y_{j,t}]}{\kappa(\widehat{\rho}_{t+q+1})},$$

$$\begin{aligned}\frac{\widehat{\phi}^{(3)}_{V_t}(0;\widehat{\rho}_{t+q+1})}{i^3} &= \frac{\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})]^3}{\kappa(\widehat{\rho}_{t+q+1})^3}\\ &+\frac{3\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})]\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})Y_{j,t}] - 3\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})]^2\widehat{\mathrm{E}}_N[Y_{j,t}]}{\kappa(\widehat{\rho}_{t+q+1})^2}\\ &+\frac{\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})Y^2_{j,t}] + 2\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})]\widehat{\mathrm{E}}_N[Y_{j,t}]^2}{\kappa(\widehat{\rho}_{t+q+1})}\\ &+\frac{-\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})]\widehat{\mathrm{E}}_N[Y^2_{j,t}] - 2\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})Y_{j,t}]\widehat{\mathrm{E}}_N[Y_{j,t}]}{\kappa(\widehat{\rho}_{t+q+1})},\end{aligned}$$

and

$$
\begin{aligned}
\frac{\widehat{\phi}^{(4)}_{V_t}(0;\widehat{\rho}_{t+q+1})}{i^4} &= \frac{\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})]^4}{\kappa(\widehat{\rho}_{t+q+1})^4} \\
&+ \frac{6\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})]^2\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})Y_{j,t}] - 6\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})]^3\widehat{\mathrm{E}}_N[Y_{j,t}]}{\kappa(\widehat{\rho}_{t+q+1})^3} \\
&+ \frac{4\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})]\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})Y^2_{j,t}] - 4\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})]^2\widehat{\mathrm{E}}_N[Y^2_{j,t}]}{\kappa(\widehat{\rho}_{t+q+1})^2} \\
&+ \frac{3\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})Y_{j,t}]^2 + 11\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})]^2\widehat{\mathrm{E}}_N[Y_{j,t}]^2}{\kappa(\widehat{\rho}_{t+q+1})^2} \\
&+ \frac{-14\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})]\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})Y_{j,t}]\widehat{\mathrm{E}}_N[Y_{j,t}]}{\kappa(\widehat{\rho}_{t+q+1})^2} \\
&+ \frac{\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})Y^3_{j,t}] - \widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})]\widehat{\mathrm{E}}_N[Y^3_{j,t}]}{\kappa(\widehat{\rho}_{t+q+1})} \\
&+ \frac{-3\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})Y^2_{j,t}]\widehat{\mathrm{E}}_N[Y_{j,t}] - 3\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})Y_{j,t}]\widehat{\mathrm{E}}_N[Y^2_{j,t}]}{\kappa(\widehat{\rho}_{t+q+1})} \\
&+ \frac{6\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})Y_{j,t}]\widehat{\mathrm{E}}_N[Y_{j,t}]^2 - 6\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})]\widehat{\mathrm{E}}_N[Y_{j,t}]^3}{\kappa(\widehat{\rho}_{t+q+1})} \\
&+ \frac{6\widehat{\mathrm{E}}_N[\mu^Y_{j,t+q+1}(\widehat{\rho}_{t+q+1})]\widehat{\mathrm{E}}_N[Y_{j,t}]\widehat{\mathrm{E}}_N[Y^2_{j,t}]}{\kappa(\widehat{\rho}_{t+q+1})}
\end{aligned}
$$

where $\widehat{\mathrm{E}}_N$ is a short-hand notation for the sample mean operator $\frac{1}{N}\sum_{j=1}^N$.

Furthermore, letting $\widehat{\phi}^{(k)}_{Y_t} = i^k\widehat{\mathrm{E}}_N[Y^k_{j,t}]$, we provide the closed-form estimators for the first four moments of U_t as follows.

$$
\begin{aligned}
\frac{\widehat{\phi}^{(1)}_{U_t}(0;\widehat{\rho}_{t+q+1}))}{i^1} &= \frac{\widehat{\phi}^{(1)}_{Y_t}(0) - \widehat{\phi}^{(1)}_{V_t}(0;\widehat{\rho}_{t+q+1})),}{i^1} \\
\frac{\widehat{\phi}^{(2)}_{U_t}(0;\widehat{\rho}_{t+q+1}))}{i^2} &= \frac{\widehat{\phi}^{(2)}_{Y_t}(0) - 2\widehat{\phi}^{(1)}_{Y_t}(0)\widehat{\phi}^{(1)}_{V_t}(0;\widehat{\rho}_{t+q+1})) - \widehat{\phi}^{(2)}_{V_t}(0;\widehat{\rho}_{t+q+1})) + 2\widehat{\phi}^{(1)}_{V_t}(0;\widehat{\rho}_{t+q+1}))^2,}{i^2} \\
\frac{\widehat{\phi}^{(3)}_{U_t}(0;\widehat{\rho}_{t+q+1}))}{i^3} &= \frac{\widehat{\phi}^{(3)}_{Y_t}(0) - 3\widehat{\phi}^{(2)}_{Y_t}(0)\widehat{\phi}^{(1)}_{V_t}(0;\widehat{\rho}_{t+q+1})) - 3\widehat{\phi}^{(1)}_{Y_t}(0)\widehat{\phi}^{(2)}_{V_t}(0;\widehat{\rho}_{t+q+1}))}{i^3} \\
&\quad \frac{+6\widehat{\phi}^{(1)}_{Y_t}(0)\widehat{\phi}^{(1)}_{V_t}(0;\widehat{\rho}_{t+q+1}))^2 - \widehat{\phi}^{(3)}_{V_t}(0;\widehat{\rho}_{t+q+1}))}{i^3} \\
&\quad \frac{+6\widehat{\phi}^{(1)}_{V_t}(0;\widehat{\rho}_{t+q+1}))\widehat{\phi}^{(2)}_{V_t}(0;\widehat{\rho}_{t+q+1})) - 6\widehat{\phi}^{(1)}_{V_t}(0;\widehat{\rho}_{t+q+1}))^3,}{i^3}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\widehat{\phi}_{U_t}^{(4)}(0;\widehat{\rho}_{t+q+1}))}{i^4} = & \frac{\widehat{\phi}_{Y_t}^{(4)}(0)-4\widehat{\phi}_{Y_t}^{(3)}(0)\widehat{\phi}_{V_t}^{(1)}(0;\widehat{\rho}_{t+q+1}))-6\widehat{\phi}_{Y_t}^{(2)}(0)\widehat{\phi}_{V_t}^{(2)}(0;\widehat{\rho}_{t+q+1}))}{i^4} \\
& \frac{+12\widehat{\phi}_{Y_t}^{(2)}(0)\widehat{\phi}_{V_t}^{(1)}(0;\widehat{\rho}_{t+q+1}))^2-4\widehat{\phi}_{Y_t}^{(1)}(0)\widehat{\phi}_{V_t}^{(3)}(0;\widehat{\rho}_{t+q+1}))-\widehat{\phi}_{V_t}^{(4)}(0;\widehat{\rho}_{t+q+1}))}{i^4} \\
& \frac{+8\widehat{\phi}_{V_t}^{(1)}(0;\widehat{\rho}_{t+q+1}))\widehat{\phi}_{V_t}^{(3)}(0;\widehat{\rho}_{t+q+1}))+6\widehat{\phi}_{V_t}^{(2)}(0;\widehat{\rho}_{t+q+1}))^2}{i^4} \\
& \frac{-36\widehat{\phi}_{V_t}^{(1)}(0;\widehat{\rho}_{t+q+1}))^2\widehat{\phi}_{V_t}^{(2)}(0;\widehat{\rho}_{t+q+1}))+24\widehat{\phi}_{V_t}^{(1)}(0;\widehat{\rho}_{t+q+1}))^4}{i^4}.
\end{aligned}
$$

C Large Sample Properties

Asymptotic properties for nonparametric deconvolution estimators in repeated measurement models have been studied in the literature (e.g., Li and Vuong, 1998). The uniform convergence rates for the estimators $\widehat{f}_{U_t}$ and $\widehat{f}_{V_t}$ for the marginal density function can be obtained by extending their results with an additional accounting for the pre-estimation of the AR parameters ρ. Our discussions are based on the ρ_t estimator under Example 1, but the same conclusions will hold under Examples 2 and 3 that similarly yield the $\sqrt{N}$ convergence rate for the parametric estimation of ρ_t. The following assumption ensures that we can ignore the effect of the pre-estimation on the second-step nonparametric estimation of the marginal densities.

Assumption 1. *(i) $\{Y_{j,t-q-p},\cdots,Y_{j,t+q}\}$ is independently and identically distributed across j. (ii) $(Y_{j,t-q-1},\cdots,Y_{j,t-q-p})'(\Delta_{j,t,1},\cdots,\Delta_{j,t,p})$ has a finite first moment that is non-singular. (iii) $(Y_{j,t-q-1},\cdots,Y_{j,t-q-p})'\Delta_{j,t,0}$ has a finite $(2+\delta)$-th moment for some $\delta>0$. (iv) $Y_{j,t+\tau}-Y_{j,t}$ has a finite first moment for each $\tau\in\{1,\cdots,q\}$.*

Parts (i), (ii) and (iii) of Assumption 1 are used to guarantee the root-N convergence of the estimator $\widehat{\rho}$ of the AR parameters ρ – see Lemma 1. Part (iv) in addition ensures that substitutions of this parametric estimator $\widehat{\rho}$ in the nonparametric estimators, $\widehat{\phi}_{V_t}$, $\widehat{\phi}_{U_t}$ and $\widehat{\phi}_{\eta_t}$, can be ignored in terms of the uniform convergence rates – see Lemma 2.

Following Li and Vuong (1998), we consider the following four cases of smoothness of the distributions of V_t and U_t:

(1) $d_{v_t}^0|s|^{-\beta_{v_t}}\leqslant|\phi_{V_t}(s)|\leqslant d_{v_t}^1|s|^{-\beta_{v_t}}$ and $d_{u_t}^0|s|^{-\beta_{u_t}}\leqslant|\phi_{U_t}(s)|\leqslant d_{u_t}^1|s|^{-\beta_{u_t}}$

(2) $d_{v_t}^0|s|^{-\beta_{v_t}}\leqslant|\phi_{V_t}(s)|\leqslant d_{v_t}^1|s|^{-\beta_{v_t}}$ and
$d_{u_t}^0|s|^{-\beta_{u_t}}\exp(-|s|^{\beta_u^*}/\gamma_u)\leqslant|\phi_{U_t}(s)|\leqslant d_{u_t}^1|s|^{-\beta_{u_t}}\exp(-|s|^{\beta_u^*}/\gamma_u)$

(3) $d_{v_t}^0|s|^{-\beta_{v_t}}\exp(-|s|^{\beta_v^*}/\gamma_v)\leqslant|\phi_{V_t}(s)|\leqslant d_{v_t}^1|s|^{-\beta_{v_t}}\exp(-|s|^{\beta_v^*}/\gamma_v)$ and
$d_{u_t}^0|s|^{-\beta_{u_t}}\leqslant|\phi_{U_t}(s)|\leqslant d_{u_t}^1|s|^{-\beta_{u_t}}$

(4) $d^0_{v_t}\,|s|^{-\beta_{v_t}}\exp(-\,|s|^{\beta^*_v}/\gamma_v) \leqslant |\phi_{V_t}(s)| \leqslant d^1_{v_t}\,|s|^{-\beta_{v_t}}\exp(-\,|s|^{\beta^*_v}/\gamma_v)$ and
$d^0_{u_t}\,|s|^{-\beta_{u_t}}\exp(-\,|s|^{\beta^*_u}/\gamma_u) \leqslant |\phi_{U_t}(s)| \leqslant d^1_{u_t}\,|s|^{-\beta_{u_t}}\exp(-\,|s|^{\beta^*_u}/\gamma_u)$

The marginal distribution of V_t is ordinary-smooth in Cases 1 and 2, while it is super-smooth in Cases 3 and 4. The marginal distribution of U_t is ordinary-smooth in Cases 1 and 3, while it is super-smooth in Cases 2 and 4. For each of these four cases, we derive the uniform convergence rates of our characteristic function estimators, $\widehat{\phi}_{V_t}(\,\cdot\,;\widehat{\rho}_{t+q+1})$, $\widehat{\phi}_{U_t}(\,\cdot\,;\widehat{\rho}_{t+q+1})$, and $\widehat{\phi}_{\eta_t}(\,\cdot\,;\widehat{\rho}_{t+q},\widehat{\rho}_{t+q+1})$, by combining our auxiliary result (Lemma 2) with the results of Li and Vuong (1998) – see Lemmas 3–6. To use the results by Li and Vuong, we also make the following assumption in addition.

Assumption 2. *$U_{j,t}$ and $V_{j,t}$ have bounded supports for each t.*

In Li and Vuong, they use the uniform function for the regularizer ϕ_K, but it is known in the statistical literature that we can replace it by a more general class of functions For our purpose to analyze the convergence rates of the estimated density functions, f_{V_t}, f_{U_t} and f_{η_t}, we make the following assumptions on the kernel function K.

Assumption 3. *(i) $\phi_K(s) = 1$ for all $s \in [-c, c]$ for some $c \in (0,\infty)$. (ii) $\phi_K(s) \in [0,1]$ for all $s \in \mathbb{R}$. (iii) $\phi_K(s) = 0$ for all $s \in \mathbb{R}\backslash[-1,1]$. (iv) $\int |\phi_K(s)|\,ds < \infty$.*

In our application, we specifically choose the kernel function

$$\phi_K(u) = \begin{cases} 1 & \text{if } |u| \leqslant c \\ \exp\left[\dfrac{-b\exp\left\{\frac{-b}{(|u|-c)^2}\right\}}{(|u|-c)^2}\right] & \text{if } c < |u| < 1 \\ 0 & \text{if } 1 \leqslant |u| \end{cases} \tag{C.1}$$

with $b = 0.10$ and $c = 0.10$ (Politis and Romano, 1999). For uniform convergence rates of our density estimators, parts (i) and (ii) of Assumption 3 helps to control the bias of our estimators, whereas parts (iii) and (iv) of this assumption help to control the variance of our estimators – see Lemma 7. Combining our previous auxiliary lemmas, (Lemmas 3–6 tailored to Cases (1)–(4), respectively) on the convergence rates of the characteristic function estimators, with this Lemma 7, we obtain the following uniform convergence results.

Theorem 1. *If Assumptions 1, 2 and 3 are satisfied in addition to the identifying assumptions under Case (1), then*

$$(i)\qquad \sup_{v\in\mathbb{R}}\left|\widehat{f}_{V_t}(v) - f_{V_t}(v)\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha}{2(1+\beta_{v_t}+\beta_{u_t})}}\right) + O\left(\left(\frac{N}{\log\log N}\right)^{\frac{\alpha(1-\beta_{v_t})}{2(1+\beta_{v_t}+\beta_{u_t})}}\right)$$

$$(ii)\qquad \sup_{u\in\mathbb{R}}\left|\widehat{f}_{U_t}(u) - f_{U_t}(u)\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha(1+\beta_{v_t})}{2(1+\beta_{v_t}+\beta_{u_t})}}\right) + O\left(\left(\frac{N}{\log\log N}\right)^{\frac{\alpha(1-\beta_{u_t})}{2(1+\beta_{v_t}+\beta_{u_t})}}\right)$$

holds with $h_N^{-1} = O\left(\left(\frac{N}{\log\log N}\right)^{\frac{\alpha}{2(1+\beta_{v_t}+\beta_{u_t})}}\right)$ *for* $0 < \alpha < (1+\beta_{v_t}+\beta_{u_t})/(2+3\beta_{v_t}+2\beta_{u_t})$. *Furthermore,*

$$(iii) \qquad \sup_{\eta\in\mathbb{R}}\left|\widehat{f}_{\eta_t}(\eta) - f_{\eta_t}(\eta)\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha\left(1+\max\left\{\beta_{v_t},\beta_{v_{t-1}}\right\}+\beta_{u_{t-1}}\right)}{2\left(1+\max\left\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\right\}\right)}}\right) + O\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha\left(1-\left(\beta_{u_t}-\beta_{u_{t-1}}\right)\right)}{2\left(1+\max\left\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\right\}\right)}}\right)$$

holds with $h_N^{-1} = O\left(\left(\frac{N}{\log\log N}\right)^{\frac{\alpha}{2\left(1+\max\left\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\right\}\right)}}\right)$ *for* $0 < \alpha < (1+\max\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\})/(2+2\max\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\}+\beta_{v_{t-1}}+\beta_{u_{t-1}})$.

Theorem 2. *If Assumptions 1, 2 and 3 are satisfied in addition to the identifying assumptions under Case (2), then*

$$(i) \qquad \sup_{v\in\mathbb{R}}\left|\widehat{f}_{V_t}(v) - f_{V_t}(v)\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{3+2\beta_{v_t}+2\beta_{u_t}}{\beta_u^*}} + \left(\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{1-\beta_{v_t}}{\beta_u^*}}$$

$$(ii) \qquad \sup_{u\in\mathbb{R}}\left|\widehat{f}_{U_t}(u) - f_{U_t}(u)\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{3+3\beta_{v_t}+2\beta_{u_t}}{\beta_u^*}} + O\left(\left(\frac{N}{\log\log N}\right)^{-\frac{\alpha}{2}}\right)\left(\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{1-\beta_{u_t}-\beta_u^*}{\beta_u^*}}$$

holds with $h_N^{-1} = \left[\frac{\alpha\gamma_u}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta_u^*}}$ *for* $0 < \alpha < 1/2$. *Furthermore,*

$$(iii) \qquad \sup_{\eta\in\mathbb{R}}\left|\widehat{f}_{\eta_t}(\eta) - f_{\eta_t}(\eta)\right| = O_p\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\frac{3\alpha}{2}}\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{3+\max\left\{3\beta_{v_t}+2\beta_{u_t},3\beta_{v_{t-1}}+2\beta_{u_{t-1}}\right\}+\beta_{u_{t-1}}}{\beta_u^*}} + \left(\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{1-\left(\beta_{u_t}-\beta_{u_{t-1}}\right)}{\beta_u^*}}$$

holds with $h_N^{-1} = \left[\frac{\alpha\gamma_u}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta_u^*}}$ *for* $0 < \alpha < 1/3$.

Theorem 3. *If Assumptions 1, 2 and 3 are satisfied in addition to the identifying assumptions under Case (3), then*

$$(i) \qquad \sup_{v\in\mathbb{R}}\left|\widehat{f}_{V_t}(v)-f_{V_t}(v)\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{3+2\beta_{v_t}+2\beta_{u_t}}{\beta_v^*}} + O\left(\left(\frac{N}{\log\log N}\right)^{-\frac{\alpha}{2}}\right)\left(\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{1-\beta_{v_t}-\beta_v^*}{\beta_v^*}}$$

$$(ii) \qquad \sup_{u\in\mathbb{R}}\left|\widehat{f}_{U_t}(u)-f_{U_t}(u)\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\frac{3\alpha}{2}}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{3+3\beta_{v_t}+2\beta_{u_t}}{\beta_v^*}} + \left(\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{1-\beta_{u_t}}{\beta_v^*}}$$

$$(iii) \qquad \sup_{\eta\in\mathbb{R}}\left|\widehat{f}_{\eta_t}(\eta)-f_{\eta_t}(\eta)\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\frac{3\alpha}{2}}\right) \times \left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{3+\max\left\{3\beta_{v_t}+2\beta_{u_t},3\beta_{v_{t-1}}+2\beta_{u_{t-1}}\right\}+\beta_{u_{t-1}}}{\beta_v^*}} + \left(\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{1-\left(\beta_{u_t}-\beta_{u_{t-1}}\right)}{\beta_v^*}}$$

holds with $h_N^{-1}=\left[\frac{\alpha\gamma_v}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta_v^*}}$ *for* $0<\alpha<1/3$.

Theorem 4. *If Assumptions 1, 2 and 3 are satisfied in addition to the identifying assumptions under Case (4), then*

$$(i) \qquad \sup_{v\in\mathbb{R}}\left|\widehat{f}_{V_t}(v)-f_{V_t}(v)\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{3+2\beta_{v_t}+2\beta_{u_t}}{\beta}} + \exp\left(-\frac{1}{\gamma_v}\left(\frac{\alpha\gamma}{2}\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{\beta_v^*}{\beta}}\right)\left(\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{1-\beta_{v_t}-\beta_v^*}{\beta}}$$

$$(ii) \qquad \sup_{u\in\mathbb{R}}\left|\widehat{f}_{U_t}(u)-f_{U_t}(u)\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha\gamma}{2\gamma_v}}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{3+3\beta_{v_t}+2\beta_{u_t}}{\beta}} + \exp\left(-\frac{1}{\gamma_u}\left(\frac{\alpha\gamma}{2}\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{\beta_u^*}{\beta}}\right)\left(\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{1-\beta_{u_t}-\beta_u^*}{\beta}}$$

$$(iii) \qquad \sup_{\eta\in\mathbb{R}}\left|\widehat{f}_{\eta_t}(\eta)-f_{\eta_t}(\eta)\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha\gamma}{2\gamma_v}+\frac{\alpha\gamma}{2\gamma_u}}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{3+\max\left\{3\beta_{v_t}+2\beta_{u_t},3\beta_{v_{t-1}}+2\beta_{u_{t-1}}\right\}+\beta_{u_{t-1}}}{\beta}} + \left(\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{1-\left(\beta_{u_t}-\beta_{u_{t-1}}\right)}{\beta}}$$

holds with $h_N^{-1} = \left[\frac{\alpha\gamma}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta}}$ *for* $0 < \alpha < \min\{1/2, \gamma_v/(2\gamma_v+\gamma), 2\gamma_v\gamma_u/(2\gamma_v\gamma_u + \gamma\gamma_v + \gamma\gamma_u)\}$*, where* $\beta = \max\{\beta_v^*, \beta_u^*\}$ *and*

$$\gamma = \begin{cases} \gamma_u & \text{if } \beta_v^* < \beta_u^* \\ \frac{\gamma_u\gamma_v}{\gamma_u+\gamma_v} & \text{if } \beta_v^* = \beta_u^* \\ \gamma_v & \text{if } \beta_v^* > \beta_u^* \end{cases}.$$

Theorem 1 follows from Lemmas 3 and 7. Likewise, Theorem 2 (respectively, 3 and 4) follows from Lemmas 4 (respectively, 5 and 6) and 7, together with Lemma 4.2 of Li and Vuong (1998). The convergence rates are decomposed into two parts, namely the stochastic part indicated by O_p and the bias part indicated by O. While other papers (e.g., Li and Vuong, 1998; Theorems 3.1–3.4) obtain deterministic rates for the variance part, we only obtain stochastic rates because of the pre-estimation of ρ by $\widehat{\rho}$.

From the uniform consistency results for $\widehat{f}_{U_t}$ and $\widehat{f}_{\eta t}$ in Theorems 1–4, it follows that the estimator $\widehat{f}_{U_tU_{t+1}}$ of the joint density function of (U_t, U_{t+1}) is also uniformly consistent – see Corollaries 1–4. Furthermore, these results in turn imply the uniform consistency of the estimator $\widehat{f}_{V_tV_{t+1}}$ of the joint density function of (V_t, V_{t+1}) – see Corollary 5.

C.1 Lemma 1: First Step Parametric Estimation

Lemma 1. *If Assumption 1 (i), (ii) and (iii) are satisfied in addition to the identifying assumptions, then* $\|\widehat{\rho} - \rho\| = O_p(N^{-1/2})$ *holds.*

Proof. The conclusion follows from Khintchin's weak law of large numbers under Assumption 1 (i), (ii), Lyapunov's central limit theorem under Assumption 1 (i), (iii), and the continuous mapping theorem under Assumption 1 (ii). □

C.2 Lemma 2: Effect of First Step Estimation

Lemma 2. *If Assumption 1 is satisfied in addition to the identifying assumptions, then*

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1}) - \widehat{\phi}_{V_t}(s;\rho_{t+q+1})\right| \leqslant \frac{S_N\tilde{A}_N}{\left[\inf_{|s|\leqslant S_N}|\phi_{Y_t}(s)|\right]\left[\inf_{|s|\leqslant S_N}|\phi_{Y_t}(s)| + \tilde{B}_N\right]}$$

holds for $\tilde{A}_N = O_p(N^{-1/2})$ *and* $\tilde{B}_N = O_p\left(\sqrt{\frac{\log\log N}{N}}\right)$ *as* $N \to \infty$.

Proof. We introduce the following short-hand notations.

$$
\begin{aligned}
\widehat{A}_N(s) &= N^{-1}\sum_{j=1}^{N}\mu^{Y}_{j,t+q+1}(\widehat{\rho}_{t+q+1})e^{isY_{j,t}} & \widehat{B}_N(s) &= \kappa(\widehat{\rho}_{t+q+1})N^{-1}\sum_{j=1}^{N}e^{isY_{j,t}} \\
A_N(s) &= N^{-1}\sum_{j=1}^{N}\mu^{Y}_{j,t+q+1}(\rho_{t+q+1})e^{isY_{j,t}} & B_N(s) &= \kappa(\rho_{t+q+1})N^{-1}\sum_{j=1}^{N}e^{isY_{j,t}}.
\end{aligned}
$$

First, note that

$$
N^{-1}\sum_{j=1}^{N}|Y_{j,t+q+1-p'}-Y_{j,t}| = O_p(1) \tag{C.2}
$$

by Assumption 1 (i) and (iv) and Khintchin's week law of large numbers. Because we can write

$$
\widehat{A}_N(s)-A_N(s) = \sum_{p'=1}^{p}(\widehat{\rho}_{t+q+1,p'}-\rho_{t+q+1,p'})\ N^{-1}\sum_{j=1}^{N}(Y_{j,t+q+1-p'}-Y_{j,t})\,e^{isY_{j,t}},
$$

it follows that

$$
\sup_s\left|\widehat{A}_N(s)-A_N(s)\right| \leqslant \sum_{p'=1}^{p}|\widehat{\rho}_{t+q+1,p'}-\rho_{t+q+1,p'}|\ N^{-1}\sum_{j=1}^{N}|Y_{j,t+q+1-p'}-Y_{j,t}| = O_p(N^{-1/2}) \tag{C.3}
$$

by Lemma 1 and (C.2).

Second, because we can write

$$
\widehat{B}_N(s)-B_N(s) = \sum_{p'=1}^{p}(\widehat{\rho}_{t+q+1,p'}-\rho_{t+q+1,p'})\,N^{-1}\sum_{j=1}^{N}e^{isY_{j,t}},
$$

it follows that

$$
\sup_s\left|\widehat{B}_N(s)-B_N(s)\right| \leqslant \sum_{p'=1}^{p}|\widehat{\rho}_{t+q+1,p'}-\rho_{t+q+1,p'}| = O_p(N^{-1/2}) \tag{C.4}
$$

by Lemma 1. Furthermore, we also have

$$
\sup_s|B_N(s)-\mathrm{E}\,B_N(s)| = O\left(\sqrt{\frac{\log\log N}{N}}\right) \tag{C.5}
$$

by the law of iterated logarithm under Assumption 1 (i).

Third, note that we can write

$$
\begin{aligned}
\log\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\log\phi_{V_t}(s;\rho_{t+q+1}) &= i\int_0^s\left\{\frac{\widehat{A}_N(s')}{\widehat{B}_N(s')}-\frac{A_N(s')}{B_N(s')}\right\}ds' = \\
& i\int_0^s\frac{\widehat{A}_N(s)-A_N(s)}{\mathrm{E}\,B_N(s)+(B_N(s)-\mathrm{E}\,B_N(s))-(\widehat{B}_N(s)-B_N(s))}ds' \\
-\ & i\int_0^s\frac{A_N(s)}{\mathrm{E}\,B_N(s)+(B_N(s)-\mathrm{E}\,B_N(s))}\cdot\frac{\widehat{B}_N(s)-B_N(s)}{\mathrm{E}\,B_N(s)+(B_N(s)\,\mathrm{E}\,B_N(s))-(\widehat{B}_N(s)-B_N(s))}ds'.
\end{aligned}
$$

Therefore, for $\tilde{A}_N = O_p(N^{-1/2})$ and $\tilde{B}_N = O_p\left(\sqrt{\frac{\log\log N}{N}}\right)$, we have by Taylor expansion for N large enough

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1}) - \widehat{\phi}_{V_t}(s;\rho_{t+q+1})\right| \leqslant \frac{S_N\tilde{A}_N}{\left[\inf_{|s|\leqslant S_N}|\mathrm{E}\,B_N(s)|\right]\ \left[\inf_{|s|\leqslant S_N}|\mathrm{E}\,B_N(s)| + \tilde{B}_N\right]}$$

by (C.3), (C.4) and (C.5). The conclusion follows by noting that $|\mathrm{E}\,B_N(s)| = |\kappa(\rho_{t+q+1})|\cdot|\phi_{Y_t}(s)|$. $\square$

C.3 Lemma 3: Characteristic Function under Case 1

Lemma 3. *If Assumptions 1 and 2 are satisfied in addition to the identifying assumptions under Case (1), then*

$$(i)\qquad \sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1}) - \phi_{V_t}(s;\rho_{t+q+1})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)$$

$$(ii)\qquad \sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1}) - \phi_{U_t}(s;\rho_{t+q+1})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha\beta_{v_t}}{2(1+\beta_{v_t}+\beta_{u_t})}}\right)$$

holds with $S_N = O\left(\left(\frac{N}{\log\log N}\right)^{\frac{\alpha}{2(1+\beta_{v_t}+\beta_{u_t})}}\right)$ *for* $0 < \alpha < (1+\beta_{v_t}+\beta_{u_t})/(2+3\beta_{v_t}+2\beta_{u_t})$. *Furthermore,*

$$\begin{aligned}(iii)\qquad &\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{\eta_t}(s;\widehat{\rho}_{t+q},\widehat{\rho}_{t+q+1}) - \phi_{\eta_t}(s;\rho_{t+q},\rho_{t+q+1})\right| \\ &\qquad = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha(\max\left\{\beta_{v_t},\beta_{v_{t-1}}\right\}+\beta_{u_{t-1}})}{2\left(1+\max\left\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\right\}\right)}}\right)\end{aligned}$$

holds with $S_N = O\left(\left(\frac{N}{\log\log N}\right)^{\frac{\alpha}{2\left(1+\max\left\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\right\}\right)}}\right)$ *for* $0 < \alpha < (1+\max\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\})/(2+2\max\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\}+\beta_{v_{t-1}}+\beta_{u_{t-1}})$.

Remark 1. *For parts (i) and (ii), one can use the same rate of S_N as in (iii). In this case,* *we instead obtain the following convergence rates.*

$$(i) \qquad \sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)$$

$$(i) \qquad \sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_{t-1}}(s;\widehat{\rho}_{t+q})-\phi_{V_{t-1}}(s;\rho_{t+q})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)$$

$$(ii) \qquad \sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1})-\phi_{U_t}(s;\rho_{t+q+1})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha\beta_{v_t}}{2\left(1+\max\left\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\right\}\right)}}\right)$$

$$(ii) \qquad \sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q})-\phi_{U_{t-1}}(s;\rho_{t+q})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha\beta_{v_{t-1}}}{2\left(1+\max\left\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\right\}\right)}}\right)$$

holds with $S_N = O\left(\left(\frac{N}{\log\log N}\right)^{\frac{\alpha}{2\left(1+\max\left\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\right\}\right)}}\right)$ *for* $0<\alpha<(1+\max\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\})/(2+2\max\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\}+\max\left\{\beta_{v_t},\beta_{v_{t-1}}\right\})$.

Proof. Applying Lemma 2 to Case (1) with $S_N = O\left(\left(\frac{N}{\log\log N}\right)^{\frac{\alpha}{2(1+\beta_{v_t}+\beta_{u_t})}}\right)$, we obtain

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\widehat{\phi}_{V_t}(s;\rho_{t+q+1})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha\frac{1+2\beta_{v_t}+2\beta_{u_t}}{2(1+\beta_{v_t}+\beta_{u_t})}}\right)$$

On the other hand, Li and Vuong (1998; Lemma 3.1) shows that

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\rho_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right| = O\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)$$

holds under our identifying assumptions and our Assumption 2. Therefore, it follows that

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right). \tag{C.6}$$

This proves the first part of the lemma. Now, we note that

$$\begin{aligned}\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1})-\phi_{U_t}(s;\rho_{t+q+1}) &= \frac{\widehat{\phi}_{Y_t}(s)}{\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})}-\frac{\phi_{Y_t}(s)}{\phi_{V_t}(s;\rho_{t+q+1})}\\ &= \frac{\widehat{\phi}_{Y_t}(s)-\phi_{Y_t}(s)}{\phi_{V_t}(s;\rho_{t+q+1})+\left(\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right)}\\ &\quad-\frac{\phi_{U_t}(s;\rho_{t+q+1})\cdot\left(\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right)}{\phi_{V_t}(s;\rho_{t+q+1})+\left(\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right)}.\end{aligned}$$

Since $\frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{V_t}(s;\rho_{t+q+1})\right|} \xrightarrow{p} 0$ as $N\to\infty$ under $\alpha<(1+\beta_{v_t}+\beta_{u_t})/(2+3\beta_{v_t}+2\beta_{u_t})$, we have

$$\begin{aligned}
&\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1})-\phi_{U_t}(s;\rho_{t+q+1})\right| \\
\leqslant\ & \frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{Y_t}(s)-\phi_{Y_t}(s)\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{V_t}(s;\rho_{t+q+1})\right|-\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right|} \\
&+\frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{V_t}(s;\rho_{t+q+1})\right|-\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right|}
\end{aligned}$$

with probability approaching one as $N\to\infty$. It follows from $S_N=O\left(\left(\frac{N}{\log\log N}\right)^{\frac{\alpha}{2(1+\beta_{v_t}+\beta_{u_t})}}\right)$ and (C.6) that

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1})-\phi_{U_t}(s;\rho_{t+q+1})\right|=O_p\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha\beta_{v_t}}{2(1+\beta_{v_t}+\beta_{u_t})}}.$$

This proves the second part of the lemma. Note the proof of the above two parts follows similarly with $S_N=O\left(\left(\frac{N}{\log\log N}\right)^{\frac{\alpha}{2\left(1+\max\left\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\right\}\right)}}\right)$ by applying Lemma 4.1 (i) of Li and Vuong (1998) with $\beta:=\max\left\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\right\}$ to yield the convergence rates displayed in Remark 1. We thus use $S_N=O\left(\left(\frac{N}{\log\log N}\right)^{\frac{\alpha}{2\left(1+\max\left\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\right\}\right)}}\right)$ now to prove the third part of the lemma. Note that

$$\begin{aligned}
\widehat{\phi}_{\eta_t}(s;\widehat{\rho}_{t+q},\widehat{\rho}_{t+q+1})-\phi_{\eta_t}(s;\rho_{t+q},\rho_{t+q+1}) &= \frac{\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1})}{\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q})}-\frac{\phi_{U_t}(s;\rho_{t+q+1})}{\phi_{U_{t-1}}(s;\rho_{t+q})} \\
&= \frac{\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1})-\phi_{U_t}(s;\rho_{t+q+1})}{\phi_{U_{t-1}}(s;\rho_{t+q})+\left(\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q})-\phi_{U_{t-1}}(s;\rho_{t+q})\right)} \\
&\quad -\frac{\phi_{\eta_{t-1}}\cdot\left(\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q})-\phi_{U_{t-1}}(s;\rho_{t+q})\right)}{\phi_{U_{t-1}}(s;\rho_{t+q})+\left(\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q})-\phi_{U_{t-1}}(s;\rho_{t+q})\right)}.
\end{aligned}$$

Since $\frac{\sup_{|s|\leqslant S_N}\left|\left(\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q})-\phi_{U_{t-1}}(s;\rho_{t+q})\right)\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{U_{t-1}}(s;\rho_{t+q})\right|} \xrightarrow{p} 0$ as $N\to\infty$ under $\alpha<(1+\max\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\})/(1+2\max\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\}+\beta_{v_{t-1}}+\beta_{u_{t-1}})$, we have

$$\begin{aligned}
&\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{\eta_t}(s;\widehat{\rho}_{t+q},\widehat{\rho}_{t+q+1})-\phi_{\eta_t}(s;\rho_{t+q},\rho_{t+q+1})\right| \\
\leqslant\ & \frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1})-\phi_{U_t}(s;\rho_{t+q+1})\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{U_{t-1}}(s;\rho_{t+q})\right|-\sup_{|s|\leqslant S_N}\left|\left(\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q})-\phi_{U_{t-1}}(s;\rho_{t+q})\right)\right|} \\
&-\frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q})-\phi_{U_{t-1}}(s;\rho_{t+q})\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{U_{t-1}}(s;\rho_{t+q})\right|-\sup_{|s|\leqslant S_N}\left|\left(\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q})-\phi_{U_{t-1}}(s;\rho_{t+q})\right)\right|}
\end{aligned}$$

with probability approaching one as $N \to \infty$. It follows from

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1}) - \phi_{U_t}(s;\rho_{t+q+1})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha\beta_{v_t}}{2\left(1+\max\left\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\right\}\right)}}\right)$$

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q}) - \phi_{U_{t-1}}(s;\rho_{t+q})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha\beta_{v_{t-1}}}{2\left(1+\max\left\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\right\}\right)}}\right)$$

$$\text{and} \qquad S_N = O\left(\left(\frac{N}{\log\log N}\right)^{\frac{\alpha}{2\left(1+\max\left\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\right\}\right)}}\right)$$

that

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{\eta_t}(s;\widehat{\rho}_{t+q},\widehat{\rho}_{t+q+1}) - \phi_{\eta_t}(s;\rho_{t+q},\rho_{t+q+1})\right| = O_p\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha(\beta_{v_t}+\beta_{u_{t-1}})}{2\left(1+\max\left\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\right\}\right)}}$$
$$+O_p\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha(\beta_{v_{t-1}}+\beta_{u_{t-1}})}{2\left(1+\max\left\{\beta_{v_t}+\beta_{u_t},\beta_{v_{t-1}}+\beta_{u_{t-1}}\right\}\right)}}$$

This proves the third part of the lemma. $\square$

C.4 Lemma 4: Characteristic Function under Case 2

Lemma 4. *If Assumptions 1 and 2 are satisfied in addition to the identifying assumptions under Case (2), then*

$$(i) \qquad \sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1}) - \phi_{V_t}(s;\rho_{t+q+1})\right|$$
$$= O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+2\beta_{v_t}+2\beta_{u_t}}{\beta_u^*}}$$

$$(ii) \qquad \sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1}) - \phi_{U_t}(s;\rho_{t+q+1})\right|$$
$$= O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+3\beta_{v_t}+2\beta_{u_t}}{\beta_u^*}}$$

holds with $S_N = \left[\frac{\alpha\gamma_u}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta_u^*}}$ *for* $0 < \alpha < 1/2$. *Furthermore,*

$$(iii) \qquad \sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{\eta_t}(s;\widehat{\rho}_{t+q},\widehat{\rho}_{t+q+1}) - \phi_{\eta_t}(s;\rho_{t+q},\rho_{t+q+1})\right|$$
$$= O_p\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\frac{3\alpha}{2}}\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+\max\left\{3\beta_{v_t}+2\beta_{u_t},3\beta_{v_{t-1}}+2\beta_{u_{t-1}}\right\}+\beta_{u_{t-1}}}{\beta_u^*}}$$

holds with $S_N = \left[\frac{\alpha\gamma_u}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta_u^*}}$ *for* $0 < \alpha < 1/3$.

Proof. Applying Lemma 2 to Case (2) with $S_N = \left[\frac{\alpha\gamma_u}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta_u^*}}$, we obtain

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1}) - \widehat{\phi}_{V_t}(s;\rho_{t+q+1})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{1+2\beta_{v_t}+2\beta_{u_t}}{\beta_u^*}}$$

On the other hand, Li and Vuong (1998; Lemma 3.2) shows that

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\rho_{t+q+1}) - \phi_{V_t}(s;\rho_{t+q+1})\right| = O\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)\left(\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+2\beta_{v_t}+2\beta_{u_t}}{\beta_u^*}}$$

holds under our identifying assumptions and our Assumption 2. Therefore, it follows that

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1}) - \phi_{V_t}(s;\rho_{t+q+1})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+2\beta_{v_t}+2\beta_{u_t}}{\beta_u^*}}. \tag{C.7}$$

This proves the first part of the lemma. Since $\frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{V_t}(s;\rho_{t+q+1})\right|} \xrightarrow{p} 0$ as $N\to\infty$ under $\alpha < 1/2$, we have

$$\begin{aligned}
&\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1}) - \phi_{U_t}(s;\rho_{t+q+1})\right| \\
\leqslant\ & \frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{Y_t}(s) - \phi_{Y_t}(s)\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{V_t}(s;\rho_{t+q+1})\right| - \sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1}) - \phi_{V_t}(s;\rho_{t+q+1})\right|} \\
&+\frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1}) - \phi_{V_t}(s;\rho_{t+q+1})\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{V_t}(s;\rho_{t+q+1})\right| - \sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1}) - \phi_{V_t}(s;\rho_{t+q+1})\right|}
\end{aligned}$$

with probability approaching one as $N\to\infty$. It follows from $S_N = \left[\frac{\alpha\gamma_u}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta_u^*}}$ and (C.7) that

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1}) - \phi_{U_t}(s;\rho_{t+q+1})\right| = O_p\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+2\beta_{u_t}+3\beta_{v_t}}{\beta_u^*}}. \tag{C.8}$$

This proves the second part of the lemma. Since $\frac{\sup_{|s|\leqslant S_N}\left|\left(\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q})-\phi_{U_{t-1}}(s;\rho_{t+q})\right)\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{U_{t-1}}(s;\rho_{t+q})\right|} \xrightarrow{p} 0$ as $N\to\infty$ under $\alpha < 1/3$, we have

$$\begin{aligned}
&\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{\eta_t}(s;\widehat{\rho}_{t+q},\widehat{\rho}_{t+q+1}) - \phi_{\eta_t}(s;\rho_{t+q},\rho_{t+q+1})\right| \\
\leqslant\ & \frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1}) - \phi_{U_t}(s;\rho_{t+q+1})\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{U_{t-1}}(s;\rho_{t+q})\right| - \sup_{|s|\leqslant S_N}\left|\left(\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q}) - \phi_{U_{t-1}}(s;\rho_{t+q})\right)\right|} \\
&-\frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q}) - \phi_{U_{t-1}}(s;\rho_{t+q})\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{U_{t-1}}(s;\rho_{t+q})\right| - \sup_{|s|\leqslant S_N}\left|\left(\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q}) - \phi_{U_{t-1}}(s;\rho_{t+q})\right)\right|}
\end{aligned}$$

with probability approaching one as $N \to \infty$. It follows from (C.8) and our choice of $S_N = \left[\frac{\alpha\gamma_u}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta_u^*}}$ that

$$\begin{aligned}
&\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{\eta_t}(s;\widehat{\rho}_{t+q},\widehat{\rho}_{t+q+1}) - \phi_{\eta_t}(s;\rho_{t+q},\rho_{t+q+1})\right| \\
= \; & O_p\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\frac{3\alpha}{2}}\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+3\beta_{v_t}+2\beta_{u_t}+\beta_{u_{t-1}}}{\beta_u^*}} + \\
& O_p\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\frac{3\alpha}{2}}\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+3\beta_{v_{t-1}}+3\beta_{u_{t-1}}}{\beta_u^*}}.
\end{aligned}$$

This proves the third part of the lemma. $\square$

C.5 Lemma 5: Characteristic Function under Case 3

Lemma 5. *If Assumptions 1 and 2 are satisfied in addition to the identifying assumptions under Case (3), then*

$$\begin{aligned}
(i) \quad & \sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1}) - \phi_{V_t}(s;\rho_{t+q+1})\right| \\
& = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+2\beta_{v_t}+2\beta_{u_t}}{\beta_v^*}} \\
(ii) \quad & \sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1}) - \phi_{U_t}(s;\rho_{t+q+1})\right| \\
& = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\frac{3\alpha}{2}}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+3\beta_{v_t}+2\beta_{u_t}}{\beta_v^*}} \\
(iii) \quad & \sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{\eta_t}(s;\widehat{\rho}_{t+q},\widehat{\rho}_{t+q+1}) - \phi_{\eta_t}(s;\rho_{t+q},\rho_{t+q+1})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\frac{3\alpha}{2}}\right) \\
& \times\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+\max\left\{3\beta_{v_t}+2\beta_{u_t},3\beta_{v_{t-1}}+2\beta_{u_{t-1}}\right\}+\beta_{u_{t-1}}}{\beta_v^*}}
\end{aligned}$$

holds with $S_N = \left[\frac{\alpha\gamma_v}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta_v^*}}$ *for* $0 < \alpha < 1/3$.

Proof. Applying Lemma 2 to Case (3) with $S_N = \left[\frac{\alpha\gamma_v}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta_v^*}}$, we obtain

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1}) - \widehat{\phi}_{V_t}(s;\rho_{t+q+1})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{1+2\beta_{v_t}+2\beta_{u_t}}{\beta_v^*}}$$

On the other hand, Li and Vuong (1998; Lemma 3.3) shows that

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\rho_{t+q+1}) - \phi_{V_t}(s;\rho_{t+q+1})\right| = O\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)\left(\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+2\beta_{v_t}+2\beta_{u_t}}{\beta_v^*}}$$

holds under our identifying assumptions and our Assumption 2. Therefore, it follows that

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+2\beta_{v_t}+2\beta_{u_t}}{\beta_v^*}}. \tag{C.9}$$

This proves the first part of the lemma. Since $\frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{V_t}(s;\rho_{t+q+1})\right|}\xrightarrow{p}0$ as $N\to\infty$ under $\alpha<1/3$, we have

$$\begin{aligned}
&\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1})-\phi_{U_t}(s;\rho_{t+q+1})\right| \\
\leqslant\ & \frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{Y_t}(s)-\phi_{Y_t}(s)\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{V_t}(s;\rho_{t+q+1})\right|-\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right|} \\
&+\frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{V_t}(s;\rho_{t+q+1})\right|-\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right|}
\end{aligned}$$

with probability approaching one as $N\to\infty$. It follows from $S_N=\left[\frac{\alpha\gamma_v}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta_v^*}}$ and (C.9) that

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1})-\phi_{U_t}(s;\rho_{t+q+1})\right| = O_p\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\frac{3\alpha}{2}}\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+3\beta_{v_t}+2\beta_{u_t}}{\beta_v^*}}. \tag{C.10}$$

This proves the second part of the lemma. Since $\frac{\sup_{|s|\leqslant S_N}\left|\left(\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q})-\phi_{U_{t-1}}(s;\rho_{t+q})\right)\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{U_{t-1}}(s;\rho_{t+q})\right|}\xrightarrow{p}0$ as $N\to\infty$, we have

$$\begin{aligned}
&\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{\eta_t}(s;\widehat{\rho}_{t+q},\widehat{\rho}_{t+q+1})-\phi_{\eta_t}(s;\rho_{t+q},\rho_{t+q+1})\right| \\
\leqslant\ & \frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1})-\phi_{U_t}(s;\rho_{t+q+1})\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{U_{t-1}}(s;\rho_{t+q})\right|-\sup_{|s|\leqslant S_N}\left|\left(\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q})-\phi_{U_{t-1}}(s;\rho_{t+q})\right)\right|} \\
&-\frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q})-\phi_{U_{t-1}}(s;\rho_{t+q})\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{U_{t-1}}(s;\rho_{t+q})\right|-\sup_{|s|\leqslant S_N}\left|\left(\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q})-\phi_{U_{t-1}}(s;\rho_{t+q})\right)\right|}
\end{aligned}$$

with probability approaching one as $N\to\infty$. It follows from (C.10) and our choice of $S_N=\left[\frac{\alpha\gamma_v}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta_v^*}}$ that

$$\begin{aligned}
&\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{\eta_t}(s;\widehat{\rho}_{t+q},\widehat{\rho}_{t+q+1})-\phi_{\eta_t}(s;\rho_{t+q},\rho_{t+q+1})\right| \\
=\ & O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\frac{3\alpha}{2}}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+3\beta_{v_t}+2\beta_{u_t}+\beta_{u_{t-1}}}{\beta_v^*}} \\
&+O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\frac{3\alpha}{2}}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+3\beta_{v_{t-1}}+3\beta_{u_{t-1}}}{\beta_v^*}}
\end{aligned}$$

This proves the third part of the lemma. $\square$

C.6 Lemma 6: Characteristic Function under Case 4

Lemma 6. *If Assumptions 1 and 2 are satisfied in addition to the identifying assumptions under Case (4), then*

$$(i)\qquad \sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)\times \left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+2\beta_{v_t}+2\beta_{u_t}}{\beta}}$$

$$(ii)\qquad \sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1})-\phi_{U_t}(s;\rho_{t+q+1})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha\gamma}{2\gamma_v}}\right)\times \left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+3\beta_{v_t}+2\beta_{u_t}}{\beta}}$$

$$(iii)\qquad \sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{\eta_t}(s;\widehat{\rho}_{t+q},\widehat{\rho}_{t+q+1})-\phi_{\eta_t}(s;\rho_{t+q},\rho_{t+q+1})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha\gamma}{2\gamma_v}+\frac{\alpha\gamma}{2\gamma_u}}\right)\times \left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+\max\left\{3\beta_{v_t}+2\beta_{u_t},3\beta_{v_{t-1}}+2\beta_{u_{t-1}}\right\}+\beta_{u_{t-1}}}{\beta}}$$

holds with $S_N=\left[\frac{\alpha\gamma}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta}}$ *for* $0<\alpha<\min\{1/2,\gamma_v/(2\gamma_v+\gamma),2\gamma_v\gamma_u/(2\gamma_v\gamma_u+\gamma\gamma_v+\gamma\gamma_u)\}$*, where* $\beta=\max\{\beta_v^*,\beta_u^*\}$ *and*

$$\gamma=\begin{cases}\gamma_u & \text{if } \beta_v^*<\beta_u^*\\ \frac{\gamma_u\gamma_v}{\gamma_u+\gamma_v} & \text{if } \beta_v^*=\beta_u^*\\ \gamma_v & \text{if } \beta_v^*>\beta_u^*\end{cases}.$$

Proof. Applying Lemma 2 to Case (4) with $S_N=\left[\frac{\alpha\gamma}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta}}$, we obtain

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\widehat{\phi}_{V_t}(s;\rho_{t+q+1})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{1+2\beta_{v_t}+2\beta_{u_t}}{\beta}}$$

On the other hand, Li and Vuong (1998; Lemma 3.4) shows that

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\rho_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right| = O\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)\left(\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+2\beta_{v_t}+2\beta_{u_t}}{\beta}}$$

holds under our identifying assumptions and our Assumption 2. Therefore, it follows that

$$\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right| = O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+2\beta_{v_t}+2\beta_{u_t}}{\beta}}. \tag{C.11}$$

This proves the first part of the lemma. Since $\frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{V_t}(s;\rho_{t+q+1})\right|} \xrightarrow{p} 0$ as $N \to \infty$ under $\alpha < \min\{1/2, \gamma_v/(2\gamma_v+\gamma)\}$, we have

$$
\begin{aligned}
&\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1})-\phi_{U_t}(s;\rho_{t+q+1})\right| \\
\leqslant\ & \frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{Y_t}(s)-\phi_{Y_t}(s)\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{V_t}(s;\rho_{t+q+1})\right|-\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right|} \\
&+\frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{V_t}(s;\rho_{t+q+1})\right|-\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right|}
\end{aligned}
$$

with probability approaching one as $N \to \infty$. It follows from $S_N = \left[\frac{\alpha\gamma}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta}}$ and (C.11) that

$$
\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1})-\phi_{U_t}(s;\rho_{t+q+1})\right| = O_p\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha\gamma}{2\gamma_v}}\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+3\beta_{v_t}+2\beta_{u_t}}{\beta}}. \tag{C.12}
$$

This proves the second part of the lemma. Since $\frac{\sup_{|s|\leqslant S_N}\left|\left(\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q})-\phi_{U_{t-1}}(s;\rho_{t+q})\right)\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{U_{t-1}}(s;\rho_{t+q})\right|} \xrightarrow{p} 0$ as $N \to \infty$ under $\alpha < 2\gamma_v\gamma_u/(2\gamma_v\gamma_u+\gamma\gamma_v+\gamma\gamma_u)$, we have

$$
\begin{aligned}
&\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{\eta_t}(s;\widehat{\rho}_{t+q},\widehat{\rho}_{t+q+1})-\phi_{\eta_t}(s;\rho_{t+q},\rho_{t+q+1})\right| \\
\leqslant\ & \frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1})-\phi_{U_t}(s;\rho_{t+q+1})\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{U_{t-1}}(s;\rho_{t+q})\right|-\sup_{|s|\leqslant S_N}\left|\left(\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q})-\phi_{U_{t-1}}(s;\rho_{t+q})\right)\right|} \\
&-\frac{\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q})-\phi_{U_{t-1}}(s;\rho_{t+q})\right|}{\inf_{|s|\leqslant S_N}\left|\phi_{U_{t-1}}(s;\rho_{t+q})\right|-\sup_{|s|\leqslant S_N}\left|\left(\widehat{\phi}_{U_{t-1}}(s;\widehat{\rho}_{t+q})-\phi_{U_{t-1}}(s;\rho_{t+q})\right)\right|}
\end{aligned}
$$

with probability approaching one as $N \to \infty$. It follows from (C.12) and our choice of $S_N = \left[\frac{\alpha\gamma}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta}}$ that

$$
\begin{aligned}
&\sup_{|s|\leqslant S_N}\left|\widehat{\phi}_{\eta_t}(s;\widehat{\rho}_{t+q},\widehat{\rho}_{t+q+1})-\phi_{\eta_t}(s;\rho_{t+q},\rho_{t+q+1})\right| \\
=\ & O_p\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha\gamma}{2\gamma_v}+\frac{\alpha\gamma}{2\gamma_u}}\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+3\beta_{v_t}+2\beta_{u_t}+\beta_{u_{t-1}}}{\beta}} \\
+\ & O_p\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha\gamma}{2\gamma_v}+\frac{\alpha\gamma}{2\gamma_u}}\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{2+3\beta_{v_{t-1}}+3\beta_{u_{t-1}}}{\beta}}.
\end{aligned}
$$

This proves the third part of the lemma. □

C.7 Lemma 7: Marginal Density Functions

Lemma 7. *If Assumptions 1, 2 and 3 are satisfied in addition to the identifying assumptions, then*

$$\sup_{v\in\mathbb{R}}\left|\widehat{f}_{V_t}(v)-f_{V_t}(v)\right| \leqslant C\left(h_N^{-1}\sup_{|s|\leqslant h_N^{-1}}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right| + \int_{ch_N^{-1}}^{\infty}\left|\phi_{V_t}(s;\rho_{t+q+1})\right| ds\right)$$

$$\sup_{u\in\mathbb{R}}\left|\widehat{f}_{U_t}(u)-f_{U_t}(u)\right| \leqslant C\left(h_N^{-1}\sup_{|s|\leqslant h_N^{-1}}\left|\widehat{\phi}_{U_t}(s;\widehat{\rho}_{t+q+1})-\phi_{U_t}(s;\rho_{t+q+1})\right| + \int_{ch_N^{-1}}^{\infty}\left|\phi_{U_t}(s;\rho_{t+q+1})\right| ds\right)$$

and

$$\sup_{\eta\in\mathbb{R}}\left|\widehat{f}_{\eta_t}(\eta)-f_{\eta_t}(\eta)\right| \leqslant C\left(h_N^{-1}\sup_{|s|\leqslant h_N^{-1}}\left|\widehat{\phi}_{\eta_t}(s;\widehat{\rho}_{t+q},\widehat{\rho}_{t+q+1})-\phi_{\eta_t}(s;\rho_{t+q},\rho_{t+q+1})\right| + \int_{ch_N^{-1}}^{\infty}\left|\phi_{\eta_t}(s;\rho_{t+q},\rho_{t+q+1})\right| ds\right)$$

hold for some $C\in(0,\infty)$.

Proof. We can write $\widehat{f}_{V_t}(v)-f_{V_t}(v)$ as

$$\begin{aligned}\widehat{f}_{V_t}(v)-f_{V_t}(v) &= \frac{1}{2\pi}\int_{-h_N^{-1}}^{h_N^{-1}} e^{-isv}\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})\phi_K(hs)ds - \frac{1}{2\pi}\int_{-h_N^{-1}}^{h_N^{-1}} e^{-isv}\phi_{V_t}(s;\rho_{t+q+1})\phi_K(hs)ds \\ &\quad + \frac{1}{2\pi}\int e^{-isv}\phi_{V_t}(s;\rho_{t+q+1})\phi_K(hs)ds - \frac{1}{2\pi}\int e^{-isv}\phi_{V_t}(s;\rho_{t+q+1})ds\end{aligned}$$

under Assumption 3 (iii). The first line on the right hand side is uniformly bounded in absolute value as

$$\begin{aligned}&\sup_{v\in\mathbb{R}}\left|\frac{1}{2\pi}\int_{-h_N^{-1}}^{h_N^{-1}} e^{-isv}\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})\phi_K(hs)ds - \frac{1}{2\pi}\int_{-h_N^{-1}}^{h_N^{-1}} e^{-isv}\phi_{V_t}(s;\rho_{t+q+1})\phi_K(hs)ds\right| \\ \leqslant\ & Ch_N^{-1}\sup_{|s|\leqslant h_N^{-1}}\left|\widehat{\phi}_{V_t}(s;\widehat{\rho}_{t+q+1})-\phi_{V_t}(s;\rho_{t+q+1})\right|\end{aligned}$$

for some $C\in(0,\infty)$ under Assumption 3 (iv). On the other hand, the second line is uniformly bounded in absolute value as

$$\begin{aligned}&\sup_{v\in\mathbb{R}}\left|\frac{1}{2\pi}\int e^{-isv}\phi_{V_t}(s;\rho_{t+q+1})\phi_K(hs)ds - \frac{1}{2\pi}\int e^{-isv}\phi_{V_t}(s;\rho_{t+q+1})ds\right| \\ \leqslant\ & \frac{1}{\pi}\int_{ch_N^{-1}}^{\infty}\left|\phi_{V_t}(s;\rho_{t+q+1})\right| ds\end{aligned}$$

under Assumption 3 (i) and (ii). This proves the first part of the lemma. The second and third parts follow by analogous arguments replacing V_t by U_t and η_t, respectively. $\square$

C.8 Uniform Consistency of $\widehat{f}_{U_t U_{t+1}}$

As corollaries, it follows from Theorems 1–4 and the equality $f_{U_t U_{t+1}}(u_t, u_{t+1}) = f_{U_t}(u_t) \cdot f_{\eta_{t+1}}(u_{t+1} - u_t)$ the following uniform convergence rates for our estimator $\widehat{f}_{U_t U_{t+1}}$ of the joint density function of (U_t, U_{t+1}).

Assumption 4. *The density functions f_{U_t} and $f_{\eta_{t+1}}$ are uniformly bounded.*

Corollary 1. *If Assumptions 1, 2, 3 and 4 are satisfied in addition to the identifying assumptions under Case (1), then*

$$\sup_{(u_t, u_{t+1})} \left| \widehat{f}_{U_t U_{t+1}}(u_t, u_{t+1}) - f_{U_t U_{t+1}}(u_t, u_{t+1}) \right| = O\left(\left(\frac{N}{\log\log N} \right)^{\frac{\alpha(1-\beta_{u_t})}{2(1+\beta_{v_t}+\beta_{u_t})}} \right)$$
$$+ O_p\left(\left(\frac{N}{\log\log N} \right)^{-\frac{1}{2}+\alpha+\frac{\alpha\left(1+\max\left\{\beta_{v_{t+1}},\beta_{v_t}\right\}+\beta_{u_t}\right)}{2\left(1+\max\left\{\beta_{v_{t+1}}+\beta_{u_{t+1}},\beta_{v_t}+\beta_{u_t}\right\}\right)}} \right)$$
$$+ O\left(\left(\frac{N}{\log\log N} \right)^{-\frac{1}{2}+\alpha+\frac{\alpha\left(1-\left(\beta_{u_{t+1}}-\beta_{u_t}\right)\right)}{2\left(1+\max\left\{\beta_{v_{t+1}}+\beta_{u_{t+1}},\beta_{v_t}+\beta_{u_t}\right\}\right)}} \right)$$

holds with $h_N^{-1} = O\left(\left(\frac{N}{\log\log N} \right)^{\frac{\alpha}{2\left(1+\max\left\{\beta_{v_{t+1}}+\beta_{u_{t+1}},\beta_{v_t}+\beta_{u_t}\right\}\right)}} \right)$ *for* $0 < \alpha < (1 + \max\{\beta_{v_{t+1}} + \beta_{u_{t+1}}, \beta_{v_t} + \beta_{u_t}\})/(2 + 2\max\{\beta_{v_{t+1}} + \beta_{u_{t+1}}, \beta_{v_t} + \beta_{u_t}\} + \beta_{v_t} + \beta_{u_t})$.

Corollary 2. *If Assumptions 1, 2, 3 and 4 are satisfied in addition to the identifying assumptions under Case (2), then*

$$\sup_{(u_t, u_{t+1})} \left| \widehat{f}_{U_t U_{t+1}}(u_t, u_{t+1}) - f_{U_t U_{t+1}}(u_t, u_{t+1}) \right| = O\left(\left(\frac{N}{\log\log N} \right)^{-\frac{\alpha}{2}} \right) \left(\log O\left(\frac{N}{\log\log N} \right) \right)^{\frac{1-\beta_{u_t}-\beta_u^*}{\beta_u^*}}$$
$$O_p\left(\frac{N}{\log\log N} \right)^{-\frac{1}{2}+\frac{3\alpha}{2}} \left(\log O_p\left(\frac{N}{\log\log N} \right) \right)^{\frac{3+\max\left\{3\beta_{v_{t+1}}+2\beta_{u_{t+1}},3\beta_{v_t}+2\beta_{u_t}\right\}+\beta_{u_t}}{\beta_u^*}}$$
$$+ \left(\log O\left(\frac{N}{\log\log N} \right) \right)^{\frac{1-\left(\beta_{u_{t+1}}-\beta_{u_t}\right)}{\beta_u^*}}$$

holds with $h_N^{-1} = \left[\frac{\alpha\gamma_u}{2} \log O\left(\frac{N}{\log\log N} \right) \right]^{\frac{1}{\beta_u^*}}$ *for* $0 < \alpha < 1/3$.

Corollary 3. *If Assumptions 1, 2, 3 and 4 are satisfied in addition to the identifying assump-*

tions under Case (3), then

$$\sup_{(u_t,u_{t+1})}\left|\widehat{f}_{U_tU_{t+1}}(u_t,u_{t+1})-f_{U_tU_{t+1}}(u_t,u_{t+1})\right|=\left(\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{1-\beta_{u_t}}{\beta_v^*}}+$$

$$O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\frac{3\alpha}{2}}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{3+\max\left\{3\beta_{v_{t+1}}+2\beta_{u_{t+1}},3\beta_{v_t}+2\beta_{u_t}\right\}+\beta_{u_t}}{\beta_v^*}}$$

$$+\left(\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{1-\left(\beta_{u_{t+1}}-\beta_{u_t}\right)}{\beta_v^*}}$$

holds with $h_N^{-1}=\left[\frac{\alpha\gamma_v}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta_v^*}}$ *for* $0<\alpha<1/3$.

Corollary 4. *If Assumptions 1, 2, 3 and 4 are satisfied in addition to the identifying assumptions under Case (4), then*

$$\sup_{(u_t,u_{t+1})}\left|\widehat{f}_{U_tU_{t+1}}(u_t,u_{t+1})-f_{U_tU_{t+1}}(u_t,u_{t+1})\right|=$$

$$\exp\left(-\frac{1}{\gamma_u}\left(\frac{\alpha\gamma}{2}\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{\beta_u^*}{\beta}}\right)\left(\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{1-\beta_{u_t}-\beta_u^*}{\beta}}$$

$$+O_p\left(\left(\frac{N}{\log\log N}\right)^{-\frac{1}{2}+\alpha+\frac{\alpha\gamma}{2\gamma_v}+\frac{\alpha\gamma}{2\gamma_u}}\right)\left(\log O_p\left(\frac{N}{\log\log N}\right)\right)^{\frac{3+\max\left\{3\beta_{v_{t+1}}+2\beta_{u_{t+1}},3\beta_{v_t}+2\beta_{u_t}\right\}+\beta_{u_t}}{\beta}}$$

$$+\left(\log O\left(\frac{N}{\log\log N}\right)\right)^{\frac{1-\left(\beta_{u_{t+1}}-\beta_{u_t}\right)}{\beta}}$$

holds with $h_N^{-1}=\left[\frac{\alpha\gamma}{2}\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{1}{\beta}}$ *for* $0<\alpha<\min\{1/2,\gamma_v/(2\gamma_v+\gamma),2\gamma_v\gamma_u/(2\gamma_v\gamma_u+\gamma\gamma_v+\gamma\gamma_u)\}$, *where* $\beta=\max\{\beta_v^*,\beta_u^*\}$ *and*

$$\gamma=\begin{cases}\gamma_u & \text{if } \beta_v^*<\beta_u^*\\ \frac{\gamma_u\gamma_v}{\gamma_u+\gamma_v} & \text{if } \beta_v^*=\beta_u^*\\ \gamma_v & \text{if } \beta_v^*>\beta_u^*\end{cases}.$$

C.9 Uniform Consistency of $\widehat{f}_{V_tV_{t+1}}$

With Assumption 2, we can estimate the joint characteristic function $\phi_{U_tU_{t+1}}$ by

$$\widehat{\phi}_{U_tU_{t+1}}(s_t,s_{t+1})=\int_{\mathcal{U}}\int_{\mathcal{U}}e^{is_tu_t+is_{t+1}u_{t+1}}\widehat{f}_{U_tU_{t+1}}(u_t,u_{t+1})du_tdu_{t+1}$$

integrated over a bounded set $\mathcal{U}$ containing the bounded supports of U_t and U_{t+1}. In addition, we can estimate the joint characteristic function $\phi_{Y_tY_{t+1}}$ by

$$\widehat{\phi}_{Y_tY_{t+1}}(s_t,s_{t+1})=\frac{1}{N}\sum_{j=1}^{N}e^{is_t(Y_{j,t}-v_t)+is_{t+1}(Y_{j,t+1}-v_{t+1})}$$

which is uniformly root-N consistent under Assumption 1. With these short-hand notations, $\widehat{\phi}_{U_tU_{t+1}}(s_t, s_{t+1})$ and $\widehat{\phi}_{Y_tY_{t+1}}(s_t, s_{t+1})$, we obtain the estimator

$$\widehat{f}_{V_tV_{t+1}}(v_t, v_{t+1}) = \frac{1}{(2\pi)^2} \int\int \widehat{\phi}_{V_tV_{t+1}}(s_t, s_{t+1})\phi_K(H_N s_t)\phi_K(H_N s_{t+1}) ds_t ds_{t+1}$$

for the joint density function $f_{V_tV_{t+1}}(v_t, v_{t+1})$, where

$$\widehat{\phi}_{V_tV_{t+1}}(s_t, s_{t+1}) = \frac{\widehat{\phi}_{Y_tY_{t+1}}(s_t, s_{t+1})}{\widehat{\phi}_{U_tU_{t+1}}(s_t, s_{t+1})}$$

The symbol H_N denotes the bandwidth parameter, who rate of convergence will be discussed later. We use the upper case notation H_N to distinguish it from the previous bandwidth parameter h_N. The asymptotic behavior of the estimator $\widehat{f}_{V_tV_{t+1}}(v_t, v_{t+1})$ relies on the shape of the joint characteristic function $\phi_{U_tU_{t+1}}$, which can be further decomposed as

$$\phi_{U_tU_{t+1}}(s_t, s_{t+1}) = \phi_{U_t}(s_t + s_{t+1}) \cdot \phi_{\eta_{t+1}}(s_{t+1}).$$

Specifically, the uniform convergence rates of $\widehat{f}_{U_tU_{t+1}}$ obtained in Corollaries 1–4 translate into the uniform convergence rates of $\widehat{\phi}_{V_tV_{t+1}}$ through its shape in the following manner.

Lemma 8. *If Assumptions 1 and 2 are satisfied in addition to the identifying assumptions under any of Cases (1)–(4), then*

$$\sup_{|s_t+s_{t+1}|\leqslant S_N, |s_{t+1}|\leqslant S_N} \left|\widehat{\phi}_{V_tV_{t+1}}(s_t, s_{t+1}) - \phi_{V_tV_{t+1}}(s_t, s_{t+1})\right| \leqslant$$
$$S_N^{\beta_{U_{t+1}}} \left(O_p\left(\frac{1}{\sqrt{N}}\right) + B \sup_{(u_t,u_{t+1})} \left|\widehat{f}_{U_tU_{t+1}}(u_t, u_{t+1}) - f_{U_tU_{t+1}}(u_t, u_{t+1})\right|\right)$$

holds.

Proof. First, by the definition of our estimator $\widehat{\phi}_{U_tU_{t+1}}(s_t, s_{t+1})$, we have

$$\begin{aligned}&\sup_{(s_t,s_{t+1})} \left|\widehat{\phi}_{U_tU_{t+1}}(s_t, s_{t+1}) - \phi_{U_tU_{t+1}}(s_t, s_{t+1})\right| \\ \leqslant\ & B \sup_{(u_t,u_{t+1})} \left|\widehat{f}_{U_tU_{t+1}}(u_t, u_{t+1}) - f_{U_tU_{t+1}}(u_t, u_{t+1})\right|\end{aligned}$$

where $B = m(\mathcal{U} \times \mathcal{U})$ is the area of $\mathcal{U} \times \mathcal{U}$. Under each of the four smoothness cases, the characteristic function $\phi_{U_tU_{t+1}}$ is bounded in absolute value as

$$\frac{d^0_{U_t} d^0_{U_{t+1}}}{d^1_{U_t}} |s_t + s_{t+1}|^{-\beta_{U_t}} |s_{t+1}|^{-(\beta_{U_{t+1}}-\beta_{U_t})} \leqslant \left|\phi_{U_tU_{t+1}}(s_t, s_{t+1})\right|$$
$$\leqslant \frac{d^1_{U_t} d^1_{U_{t+1}}}{d^0_{U_t}} |s_t + s_{t+1}|^{-\beta_{U_t}} |s_{t+1}|^{-(\beta_{U_{t+1}}-\beta_{U_t})}$$

Note that we can write

$$\widehat{\phi}_{V_tV_{t+1}}(s_t,s_{t+1})-\phi_{V_tV_{t+1}}(s_t,s_{t+1})=$$
$$\frac{\phi_{U_tU_{t+1}}(s_t,s_{t+1})\left[\widehat{\phi}_{Y_tY_{t+1}}(s_t,s_{t+1})-\phi_{Y_tY_{t+1}}(s_t,s_{t+1})\right]}{\phi_{U_tU_{t+1}}(s_t,s_{t+1})\left[\phi_{U_tU_{t+1}}(s_t,s_{t+1})+\left(\widehat{\phi}_{U_tU_{t+1}}(s_t,s_{t+1})-\phi_{U_tU_{t+1}}(s_t,s_{t+1})\right)\right]}-$$
$$\frac{\phi_{Y_tY_{t+1}}(s_t,s_{t+1})\left[\widehat{\phi}_{U_tU_{t+1}}(s_t,s_{t+1})-\phi_{U_tU_{t+1}}(s_t,s_{t+1})\right]}{\phi_{U_tU_{t+1}}(s_t,s_{t+1})\left[\phi_{U_tU_{t+1}}(s_t,s_{t+1})+\left(\widehat{\phi}_{U_tU_{t+1}}(s_t,s_{t+1})-\phi_{U_tU_{t+1}}(s_t,s_{t+1})\right)\right]}$$

Therefore, we have

$$\sup_{|s_t+s_{t+1}|\leqslant S_N,|s_{t+1}|\leqslant S_N}\left|\widehat{\phi}_{V_tV_{t+1}}(s_t,s_{t+1})-\phi_{V_tV_{t+1}}(s_t,s_{t+1})\right|\leqslant$$
$$S_N^{\beta_{U_t}}S_N^{\beta_{U_{t+1}}-\beta_{U_t}}\left(O_p\left(\frac{1}{\sqrt{N}}\right)+B\sup_{(u_t,u_{t+1})}\left|\widehat{f}_{U_tU_{t+1}}(u_t,u_{t+1})-f_{U_tU_{t+1}}(u_t,u_{t+1})\right|\right)$$

as claimed. □

The following lemma provides the asymptotic rate of uniform convergence for the stochastic and bias parts of $\widehat{f}_{V_tV_{t+1}}$, where the stochastic part depends on the uniform convergence rate of $\widehat{\phi}_{V_tV_{t+1}}$ provided in the previous lemma.

Lemma 9. *If Assumptions 1, 2 and 3 are satisfied in addition to the identifying assumptions, then*

$$\begin{aligned}&\sup_{(v_t,v_{t+1})}\left|\widehat{f}_{V_tV_{t+1}}(v_t,v_{t+1})-f_{V_tV_{t+1}}(v_t,v_{t+1})\right|\\ \leqslant\ & C\left(H_N^{-1}\sup_{|s_t|\leqslant H_N^{-1},|s_{t+1}|\leqslant H_N^{-1}}\left|\widehat{\phi}_{V_tV_{t+1}}(s_t,s_{t+1})-\phi_{V_tV_{t+1}}(s_t,s_{t+1})\right|\right.\\ &\left.+\int\int_{\mathbb{R}^2\setminus\left[-cH_N^{-1},cH_N^{-1}\right]^2}\left|\phi_{V_tV_{t+1}}(s_t,s_{t+1})\right|ds_tds_{t+1}\right)\end{aligned}$$

holds for some $C\in(0,\infty)$.

Proof. We can write $\widehat{f}_{V_tV_{t+1}}(v_t,v_{t+1})-f_{V_tV_{t+1}}(v_t,v_{t+1})$ as

$$\begin{aligned}&\widehat{f}_{V_tV_{t+1}}(v_t,v_{t+1})-f_{V_tV_{t+1}}(v_t,v_{t+1})\\ =\ &\frac{1}{(2\pi)^2}\int_{-H_N^{-1}}^{H_N^{-1}}\int_{-H_N^{-1}}^{H_N^{-1}}e^{-is_tv_t-is_{t+1}v_{t+1}}\widehat{\phi}_{V_tV_{t+1}}(s_t,s_{t+1})\phi_K(H_Ns_t)\phi_K(H_Ns_{t+1})ds_tds_{t+1}\\ -\ &\frac{1}{(2\pi)^2}\int_{-H_N^{-1}}^{H_N^{-1}}\int_{-H_N^{-1}}^{H_N^{-1}}e^{-is_tv_t-is_{t+1}v_{t+1}}\phi_{V_tV_{t+1}}(s_t,s_{t+1})\phi_K(H_Ns_t)\phi_K(H_Ns_{t+1})ds_tds_{t+1}\\ +\ &\frac{1}{(2\pi)^2}\int\int e^{-is_tv_t-is_{t+1}v_{t+1}}\phi_{V_tV_{t+1}}(s_t,s_{t+1})\phi_K(H_Ns_t)\phi_K(H_Ns_{t+1})ds_tds_{t+1}\\ -\ &\frac{1}{(2\pi)^2}\int\int e^{-is_tv_t-is_{t+1}v_{t+1}}\phi_{V_tV_{t+1}}(s_t,s_{t+1})ds_tds_{t+1}\end{aligned}$$

under Assumption 3 (iii). The difference of the first two terms on the right hand side is uniformly bounded in absolute value as

$$\begin{aligned}
&\sup_{(v_t,v_{t+1})}\left|\frac{1}{(2\pi)^2}\int_{-H_N^{-1}}^{H_N^{-1}}\int_{-H_N^{-1}}^{H_N^{-1}} e^{-is_tv_t-is_{t+1}v_{t+1}}\widehat{\phi}_{V_tV_{t+1}}(s_t,s_{t+1})\phi_K(H_Ns_t)\phi_K(H_Ns_{t+1})ds_tds_{t+1}\right.\\
&\qquad\left.-\frac{1}{(2\pi)^2}\int_{-H_N^{-1}}^{H_N^{-1}}\int_{-H_N^{-1}}^{H_N^{-1}} e^{-is_tv_t-is_{t+1}v_{t+1}}\phi_{V_tV_{t+1}}(s_t,s_{t+1})\phi_K(H_Ns_t)\phi_K(H_Ns_{t+1})ds_tds_{t+1}\right|\\
\leqslant\ & CH_N^{-1}\sup_{|s_t|,|s_{t+1}|\leqslant H_N^{-1}}\left|\widehat{\phi}_{V_tV_{t+1}}(s_t,s_{t+1})-\phi_{V_tV_{t+1}}(s_t,s_{t+1})\right|
\end{aligned}$$

for some $C\in(0,\infty)$ under Assumption 3 (iv). On the other hand, the difference of the last two terms is uniformly bounded in absolute value as

$$\begin{aligned}
&\sup_{(v_t,v_{t+1})}\left|\frac{1}{(2\pi)^2}\int\int e^{-is_tv_t-is_{t+1}v_{t+1}}\phi_{V_tV_{t+1}}(s_t,s_{t+1})\phi_K(H_Ns_t)\phi_K(H_Ns_{t+1})ds_tds_{t+1}\right.\\
&\qquad\left.-\frac{1}{(2\pi)^2}\int\int e^{-is_tv_t-is_{t+1}v_{t+1}}\phi_{V_tV_{t+1}}(s_t,s_{t+1})ds_tds_{t+1}\right|\\
\leqslant\ & \frac{1}{(2\pi)^2}\int\int_{\mathbb{R}^2\setminus[-cH_N^{-1},cH_N^{-1}]^2}\left|\phi_{V_tV_{t+1}}(s_t,s_{t+1})\right|ds_tds_{t+1}
\end{aligned}$$

under Assumption 3 (i) and (ii). This proves the first part of the lemma. □

From Lemmas 8 and 9, it follows that the estimator $\widehat{f}_{V_tV_{t+1}}$ of the joint density function of (V_t,V_{t+1}) is uniformly consistent by choosing H_N tending to zero slowly enough so that it satisfies

$$H_N^{-1}\left(\sup_{(u_t,u_{t+1})}\left|\widehat{f}_{U_tU_{t+1}}(u_t,u_{t+1})-f_{U_tU_{t+1}}(u_t,u_{t+1})\right|\right)^{\frac{1}{1+\beta_{U_{t+1}}}}=o_p(1)$$

as $N\to\infty$, where the convergence rate of $\sup_{(u_t,u_{t+1})}\left|\widehat{f}_{U_tU_{t+1}}(u_t,u_{t+1})-f_{U_tU_{t+1}}(u_t,u_{t+1})\right|$ is derived in Corollaries 1–4 under Cases 1–4, respectively. Specifically, it is sufficient to choose H_N by

$$H^{-1}=\left[\log O\left(\frac{N}{\log\log N}\right)\right]^{\frac{\delta}{1+\beta_{U_{t+1}}}}, \tag{C.13}$$

where an admissible choice of $\delta>0$ varies across Cases 1–4 in the following manner: (1) $\delta<\infty$; (2) $\delta<\frac{1-(\beta_{U_{t+1}}-\beta_{U_t})}{\beta_u^*}$; (3) $\delta<\min\left\{\frac{1-\beta_{U_t}}{\beta_v^*},\frac{1-(\beta_{U_{t+1}}-\beta_{U_t})}{\beta_v^*}\right\}$; (4) $\delta<\frac{1-(\beta_{U_{t+1}}-\beta_{U_t})}{\beta}$. We summarize this result as a corollary below.

Corollary 5. *If Assumptions 1, 2, 3 and 4 are satisfied in addition to the identifying assumptions, then*

$$\sup_{(v_t,v_{t+1})}\left|\widehat{f}_{V_tV_{t+1}}(v_t,v_{t+1})-f_{V_tV_{t+1}}(v_t,v_{t+1})\right|=o_p(1)$$

holds with the choice of H_N given in equation (C.13).

We remark that the specific rate of convergence under each of Cases 1–4 can be derived by further specifying the tail behavior of the joint characteristic function of (V_t, V_{t+1}). Unlike the permanent component U_t, however, the transitory component V_t follows a complicated dynamic process, and thus the tail behavior of the joint characteristic function $\phi_{V_tV_{t+1}}$ does not follow from the tail behaviors of the marginal characteristic functions, ϕ_{V_t} and $\phi_{V_{t+1}}$. We thus leave the shape of $\phi_{V_tV_{t+1}}$ and derive only the uniform consistency in the above corollary for the joint transitory components.

D Additional Results

This section presents additional results of the empirical application to earnings dynamics.

D.1 Results: Baseline Sample

Figure 1 displays estimates for the marginal densities of the permanent earnings U_t, the transitory earnings V_t, the cumulative permanent shocks $\sum_t \eta_t$ and the composite MA shocks $\varepsilon_t + \lambda_t \varepsilon_{t-1}$ under the ARMA(1,1) model. Figure 2 compares the marginal densities of the permanent earnings U_t and the transitory earnings V_t that we obtain under the ARMA(0,0) model (left) and the ARMA(1,1) model (right). Figure 3 compares the marginal densities of the permanent earnings U_t and the transitory earnings V_t that we obtain under the ARMA(2,2) model (left) and the ARMA(4,4) model (right). Figure 4 displays estimates (solid curves) for the marginal densities of the permanent earnings U_t and the transitory earnings V_t under the ARMA(1,1) together with Gaussian references (dashed curves).

Some density figures show bumps near the tails of the distributions, particularly for the transitory components. These bumps are common features of deconvolution density estimates. For example, a closely related paper by Bonhomme and Robin (2010; Figure 5) also exhibit similar bumps near the tails, especially for transitory shocks as we do similarly. They are the artefact of the choice of h – when h is chosen to be large, a wider spectrum of waves are truncated for the purpose of reducing the variance, and hence low-frequency bumps remain. Removing these bumps will require non-optimal choice of h. With this said, these bumps will not anyway affect the statistical inference based on moments, as those statistics do not rely on h. In other words, the statistics displayed in the main text as well as in the current supplementary material are invariant from tuning of h.

Figure 5 displays the long-run joint densities of the permanent earnings U_t and the transitory earnings V_t under the ARMA(0,0) model (left) and the benchmark ARMA(1,1) model (right). Similarly, Figure 6 displays the long-run joint densities of the permanent earnings U_t and the transitory earnings V_t under the ARMA(2,2) model (left) and the benchmark ARMA(4,4) model (right). These figures contain important information about model implications for life-cycle earnings dynamics, but the contour curves are not the most effective way to present the information. Therefore, we extract some important features behind these long-run joint densities, and present them in terms of the lower tail dependence measure presented in the main text.

D.2 Results: Workers with Strong Labor Force Attachment

Figure 7 displays estimates for the marginal densities of the permanent earnings U_t, the transitory earnings V_t, the cumulative permanent shocks $\sum_t \eta_t$ and the composite MA shocks $\varepsilon_t + \lambda_t \varepsilon_{t-1}$ under the ARMA(1,1) model. Figure 8 compares the marginal densities of the permanent earnings U_t and the transitory earnings V_t that we obtain under the ARMA(0,0) model (left) and the ARMA(1,1) model (right). Figure 9 compares the marginal densities of the permanent earnings U_t and the transitory earnings V_t that we obtain under the ARMA(2,2) model (left) and the ARMA(4,4) model (right). Figure 10 displays estimates (solid curves) for the marginal densities of the permanent earnings U_t and the transitory earnings V_t under the ARMA(1,1) together with Gaussian references (dashed curves).

Figure 11 displays the long-run joint densities of the permanent earnings U_t and the transitory earnings V_t under the ARMA(0,0) model (left) and the benchmark ARMA(1,1) model (right). Similarly, Figure 12 displays the long-run joint densities of the permanent earnings U_t and the transitory earnings V_t under the ARMA(2,2) model (left) and the benchmark ARMA(4,4) model (right).

Figure 13 displays trajectories of the lower tail dependence measure $\lambda^l_{30,t}(q) = P(U_t \leq F^{-1}_{U_t}(q) | U_{30} \leq F^{-1}_{U_{30}}(q))$ of permanent earnings following the event of permanent earnings less than or equal to the q-th quantile at age 30 for $q \in \{0.10, 0.05, 0.01\}$. The solid lines represent the trajectories under our semiparametric model. The dashed lines represent those under the bivariate normal distribution. The results are displayed under each of the ARMA(1,1) and ARMA(2,2) specifications with time-varying coefficients.

Figure 14 displays trajectories of the lower tail dependence measures $\lambda^l_{30,t}(q) = P(U_t \leq F^{-1}_{U_t}(q) | U_{30} \leq F^{-1}_{U_{30}}(q))$ and $\lambda^l_{40,t}(q) = P(U_t \leq F^{-1}_{U_t}(q) | U_{40} \leq F^{-1}_{U_{40}}(q))$ of permanent earnings following the event of permanent earnings less than or equal to the q-th quantile at age 30 and 40, respectively, for $q \in \{0.10, 0.05, 0.01\}$. The solid lines represent the trajectories under our semiparametric model. The dashed lines represent those under the bivariate normal distribution. The results are displayed under the ARMA(4,4) specification with time-varying coefficients.

D.3 Results: Married Workers

Tables 1, 2, 3,4 and 5 summarize estimated marginal distributional indices under the ARMA(0,0), ARMA(1,1), ARMA(2,2), ARMA(3,3) and ARMA(4,4) models with time-varying and time-invariant AR coefficients. These indices are the mean, the standard deviation, the skewness, and the kurtosis. The numbers in parentheses indicate the standard errors of the respective estimates. The last column shows the p-values for the one-sided test of the null hypothesis that kurtosis is less than equal to three, against the alternative hypothesis that it is greater than three.

Figures 15 and 16 displays trajectories of the lower tail dependence measure $\lambda^l_{30,t}(0.01) = P(U_t \leq F^{-1}_{U_t}(0.01)|U_{30} \leq F^{-1}_{U_{30}}(0.01))$ of permanent earnings following the event of permanent earnings less than or equal to the 1 percentile at age 30. The solid lines represent the trajectories under our semiparametric model. The dashed lines represent those under the bivariate normal distribution. The results are displayed under each of the ARMA(0,0), ARMA(1,1), ARMA(2,2), ARMA(3,3) and ARMA(4,4) specifications with time-varying coefficients and time-invariant coefficients.

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Li, T. and Q. Vuong (1998) "Nonparametric Estimation of the Measurement Error Model Using Multiple Indicators," *Journal of Multivariate Analysis*, 65 (2), 139–165.

Politis, D.N. and J.P. Romano (1999) Multivariate Density Estimation with General Flat-Top Kernels of Infinite Order. *Journal of Multivariate Analysis,* 68 (1), pp. 1-25.

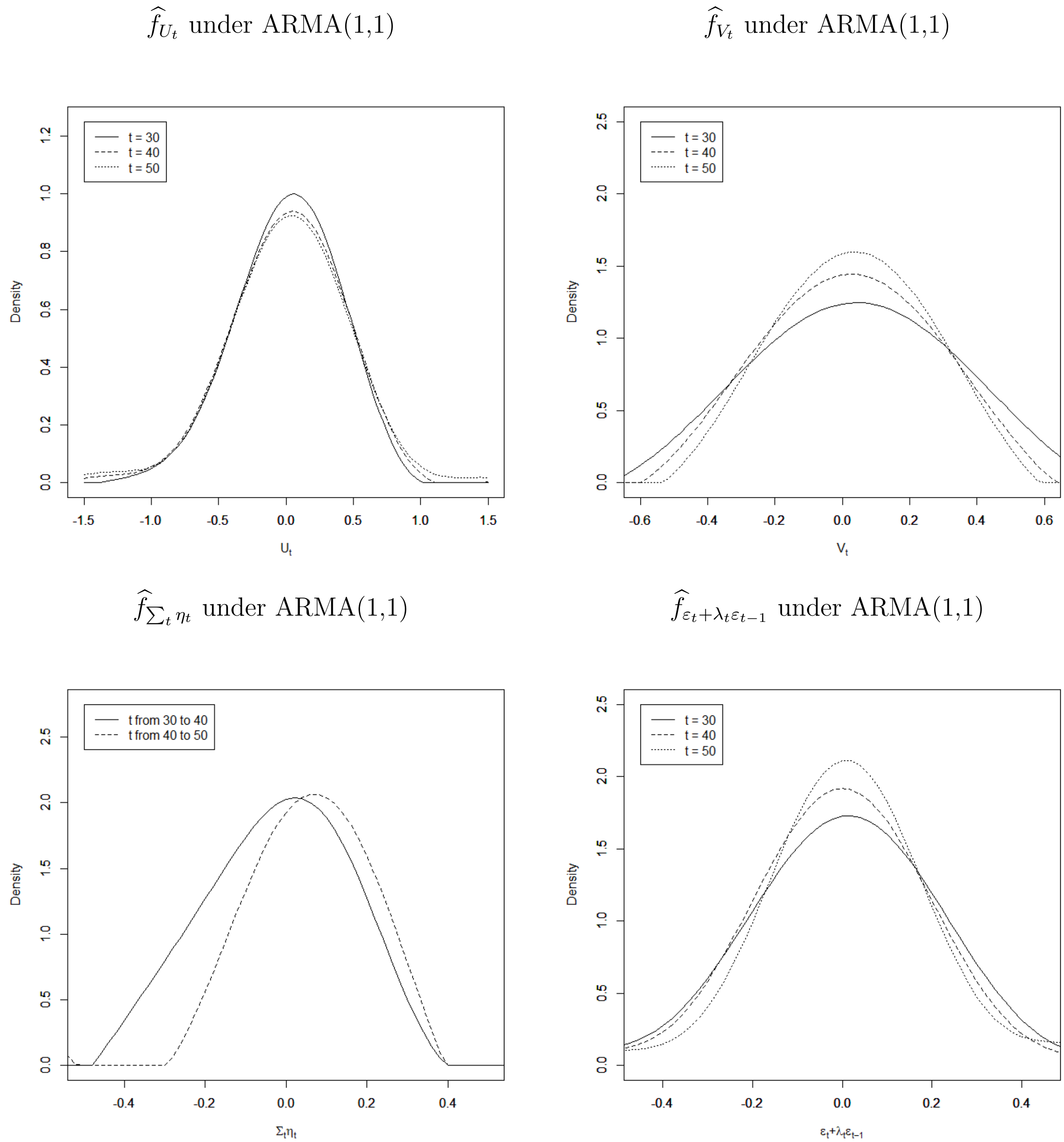


Figure 1: Nonparametric estimates of the marginal densities of the permanent earnings (top left), the transitory earnings (top right), the cumulative permanent shocks (bottom left), and the composite MR errors (bottom right) under the ARMA(1,1) specification. The results are based on the baseline sample.

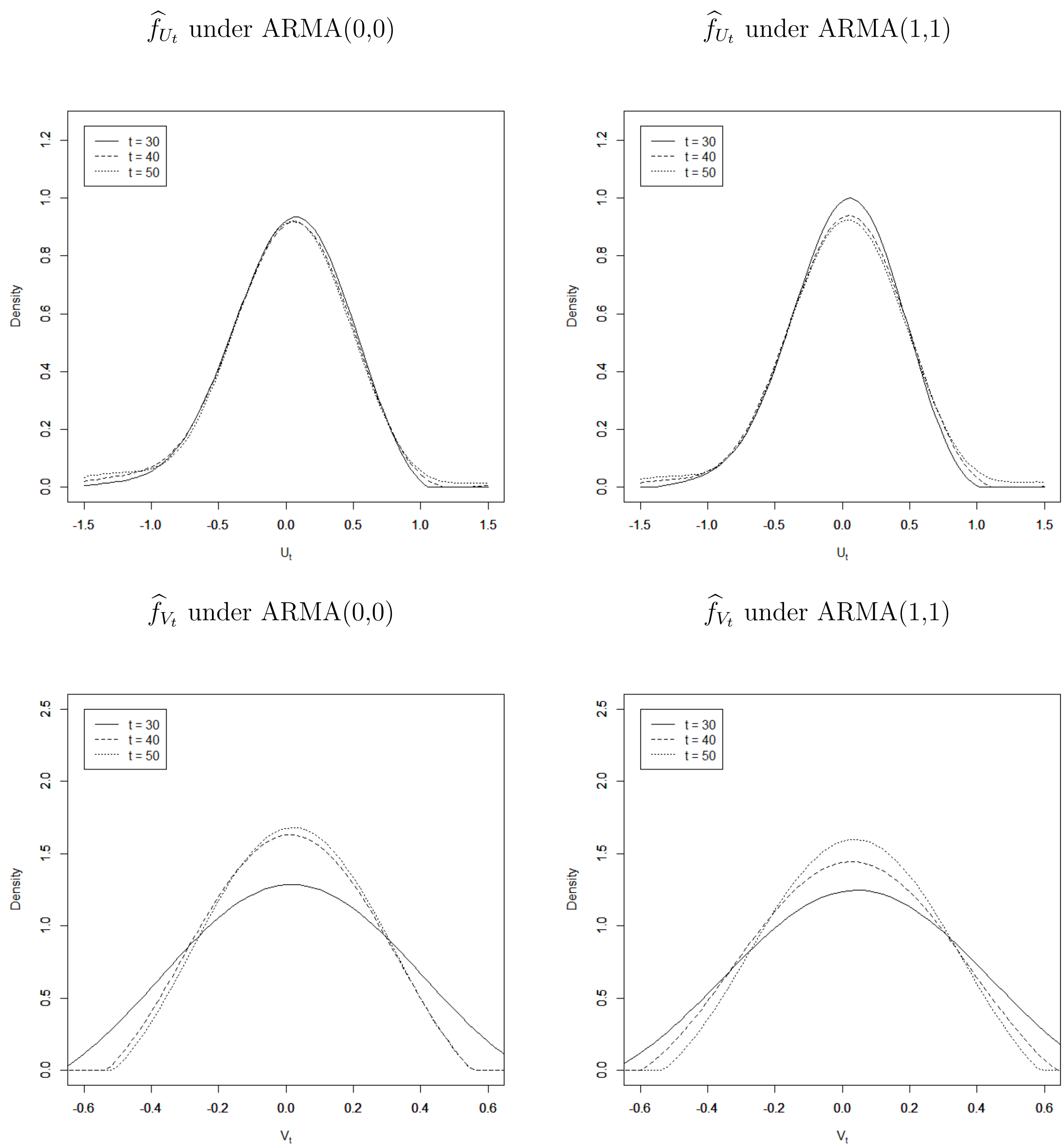


Figure 2: Nonparametric estimates of the marginal densities of the permanent earnings (top row) and the transitory earnings (bottom row) under each of the ARMA(0,0) specification (left column) and the ARMA(1,1) specification (right column). The results are based on the baseline sample.

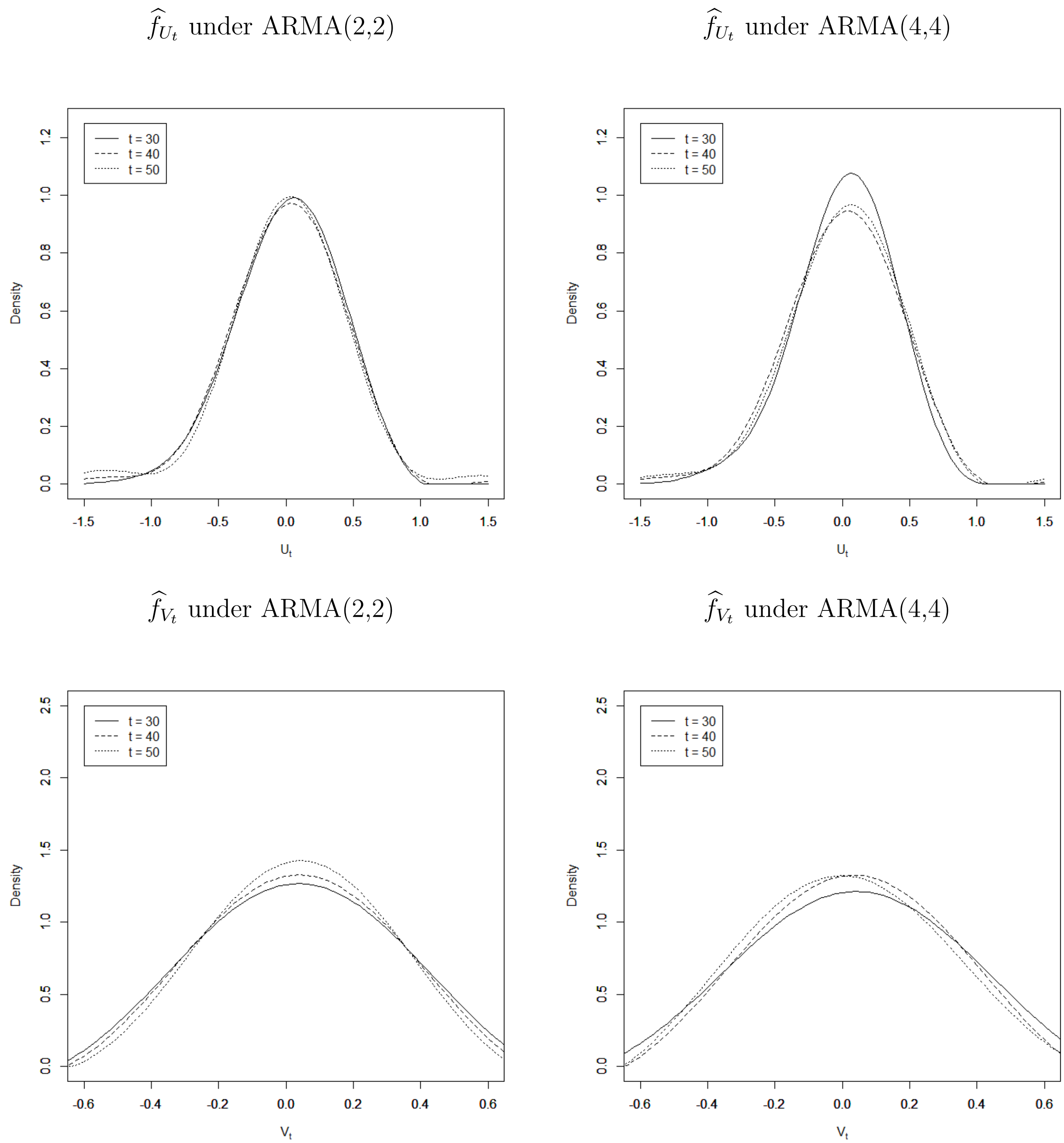


Figure 3: Nonparametric estimates of the marginal densities of the permanent earnings (top row) and the transitory earnings (bottom row) under each of the ARMA(2,2) specification (left column) and the ARMA(4,4) specification (right column). The results are based on the baseline sample.

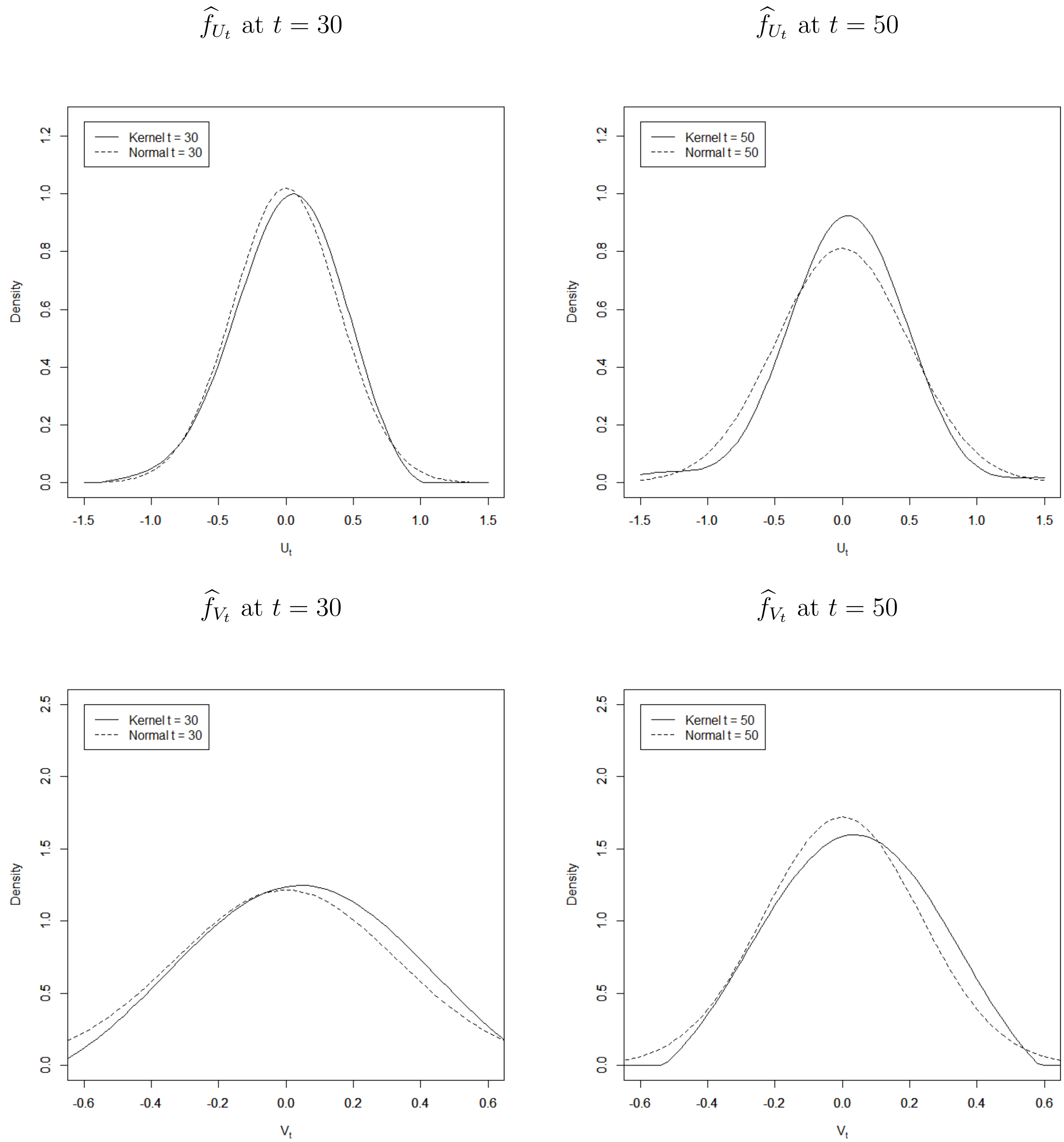


Figure 4: Nonparametric estimates of the marginal densities of the permanent earnings (top row) and the transitory earnings (bottom row) for $t = 30$ (left column) and $t = 50$ (right column) under the ARMA(1,1) specification with Gaussian references (dashed curves). The results are based on the baseline sample.

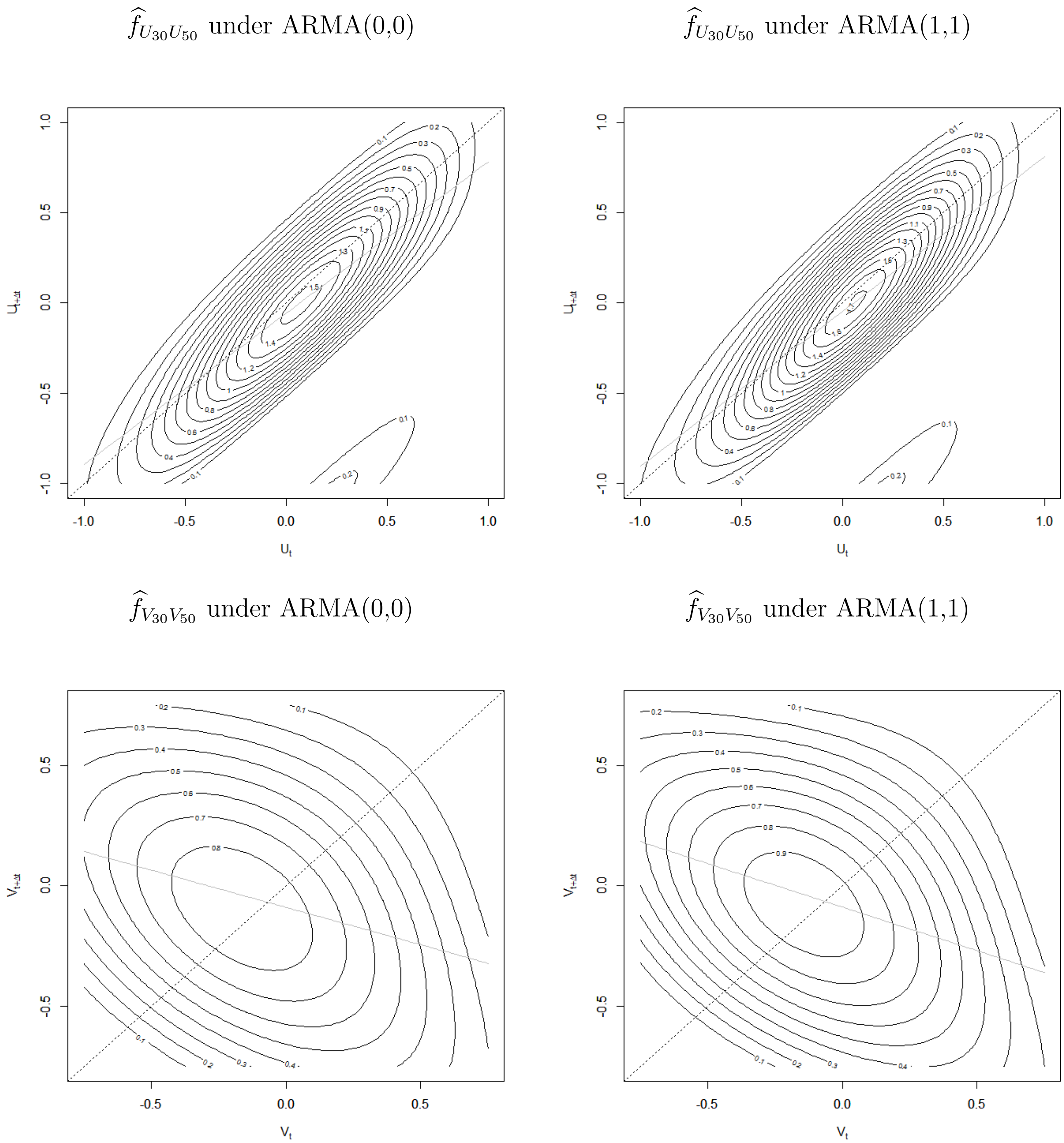


Figure 5: Nonparametric estimates of the long-run joint densities of the permanent earnings (top row) and the transitory earnings (bottom row) under each of the ARMA(0,0) specification (left column) and the ARMA(1,1) specification (right column). The results are based on the baeline sample.

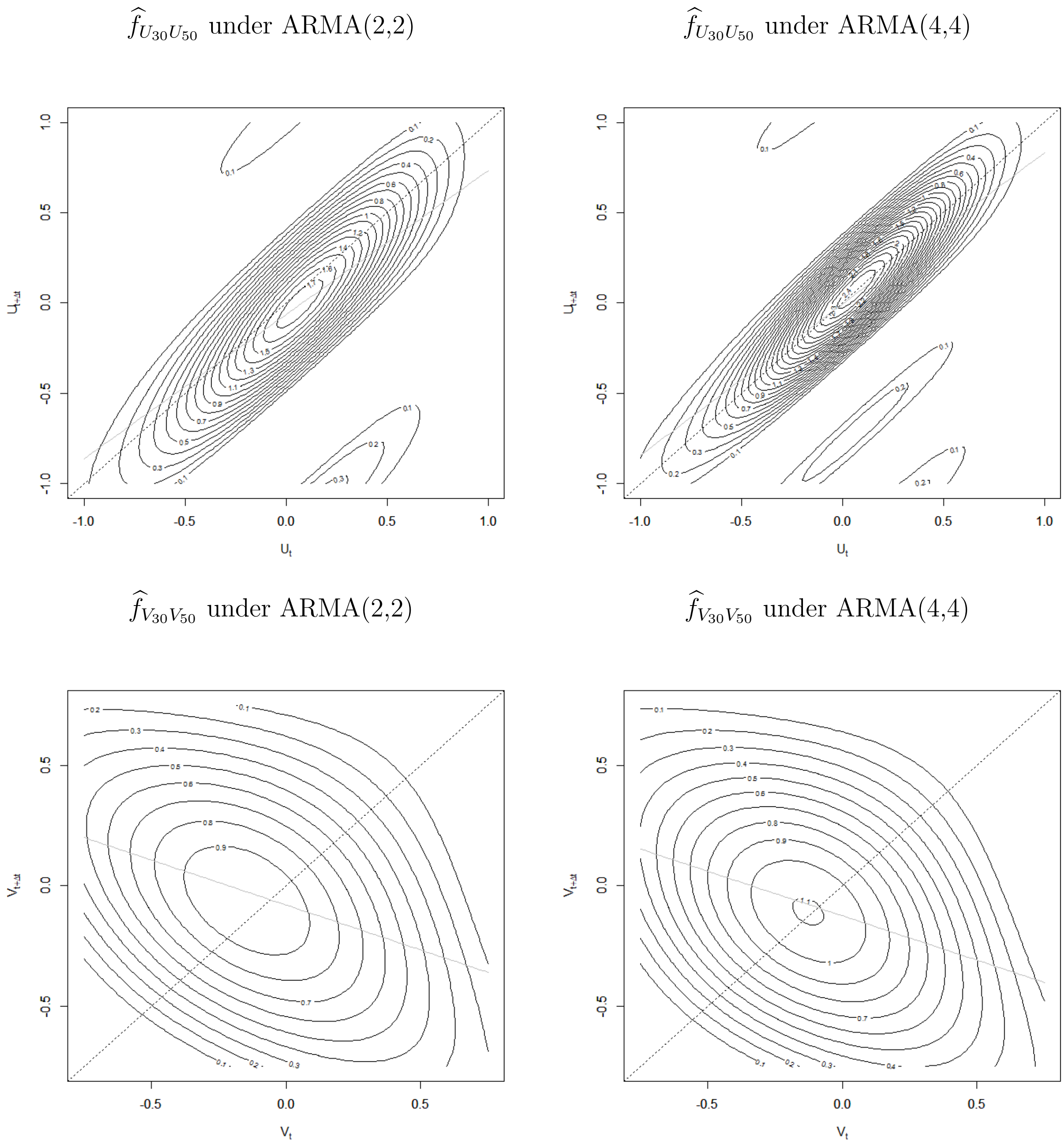


Figure 6: Nonparametric estimates of the long-run joint densities of the permanent earnings (top row) and the transitory earnings (bottom row) under each of the ARMA(2,2) specification (left column) and the ARMA(4,4) specification (right column). The results are based on the baeline sample.

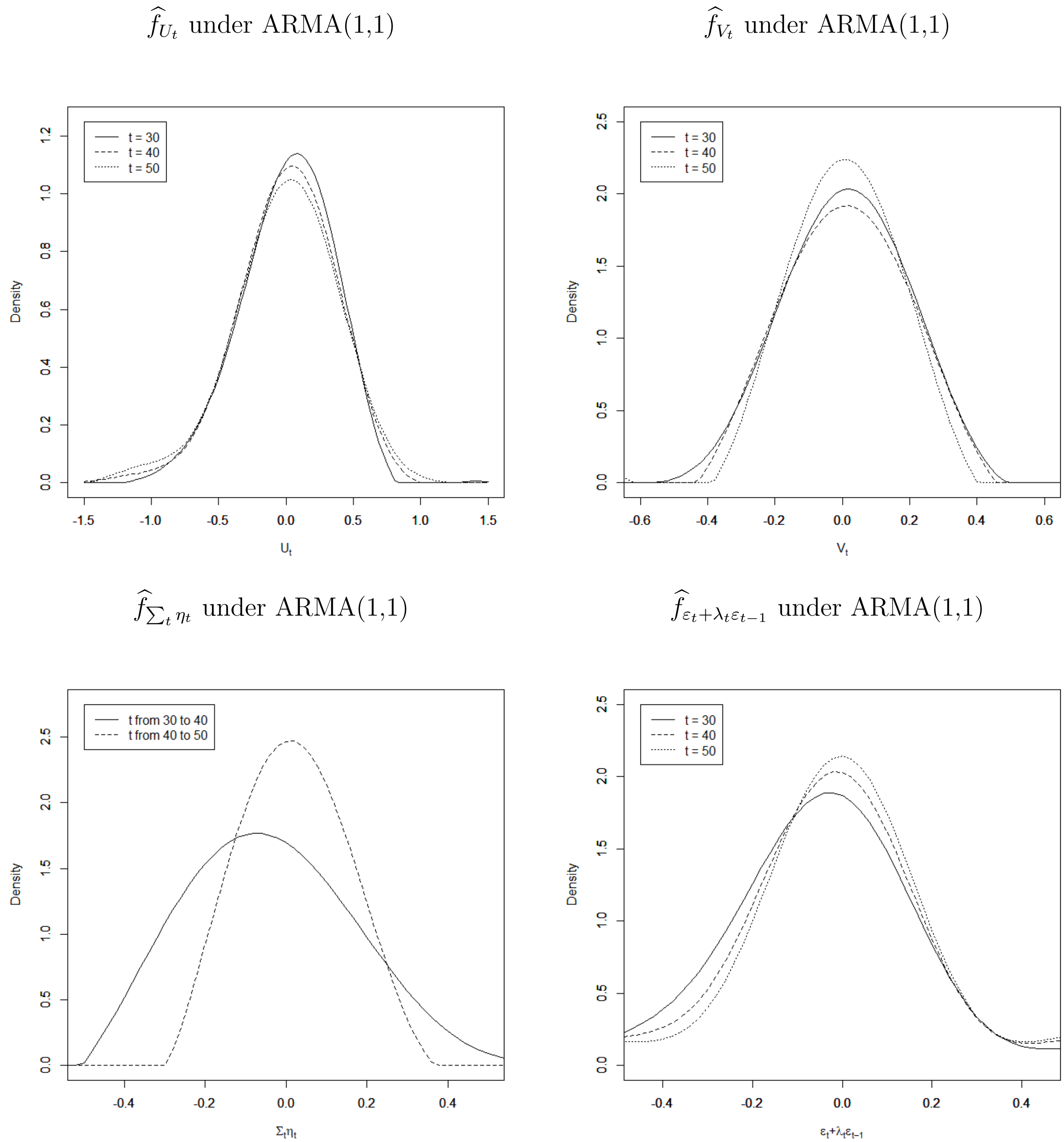


Figure 7: Nonparametric estimates of the marginal densities of the permanent earnings (top left), the transitory earnings (top right), the cumulative permanent shocks (bottom left), and the composite MR errors (bottom right) under the ARMA(1,1) specification. The sample consists of individuals with strong labor force attachment.

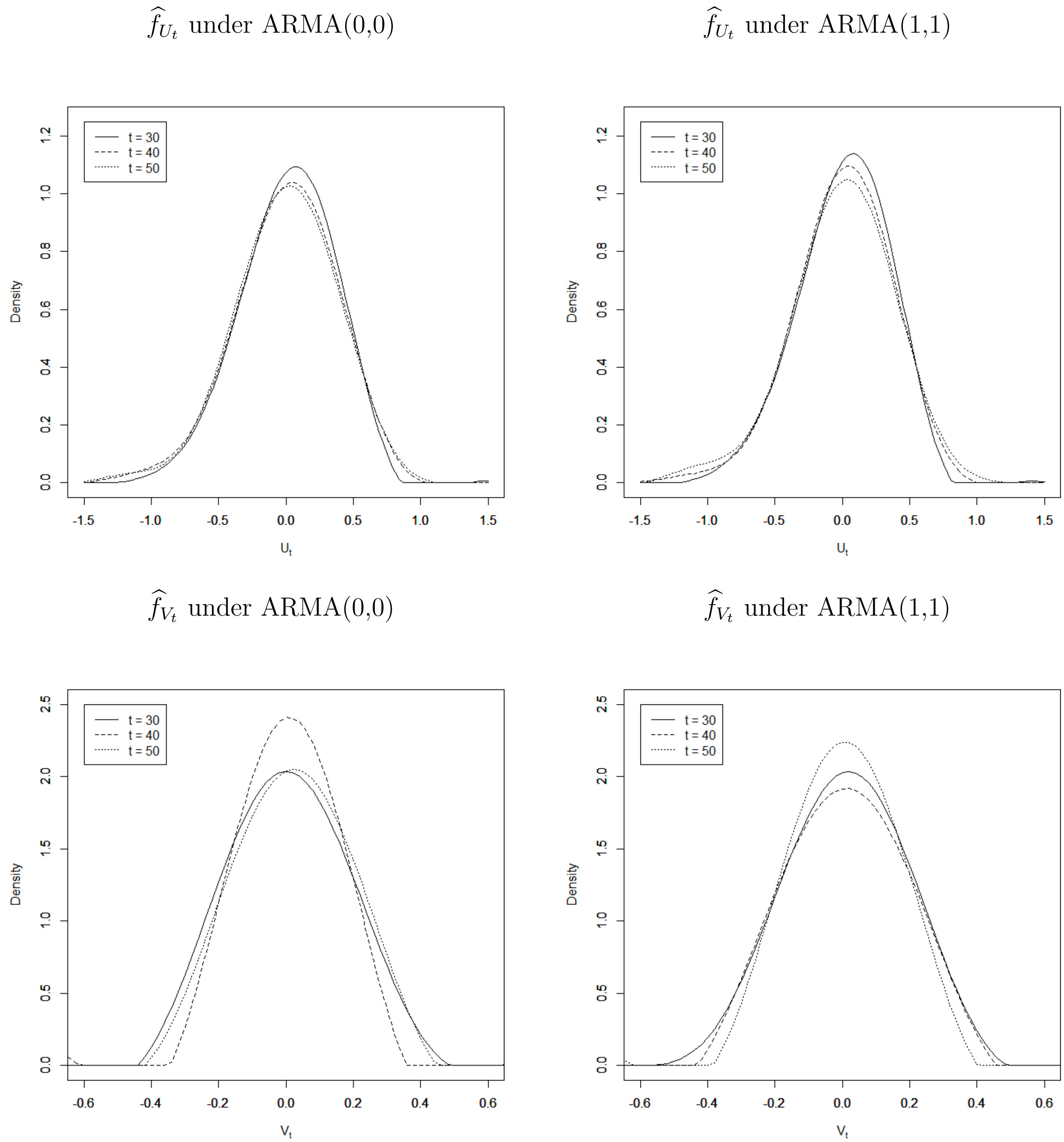


Figure 8: Nonparametric estimates of the marginal densities of the permanent earnings (top row) and the transitory earnings (bottom row) under each of the ARMA(0,0) specification (left column) and the ARMA(1,1) specification (right column). The sample consists of individuals with strong labor force attachment.

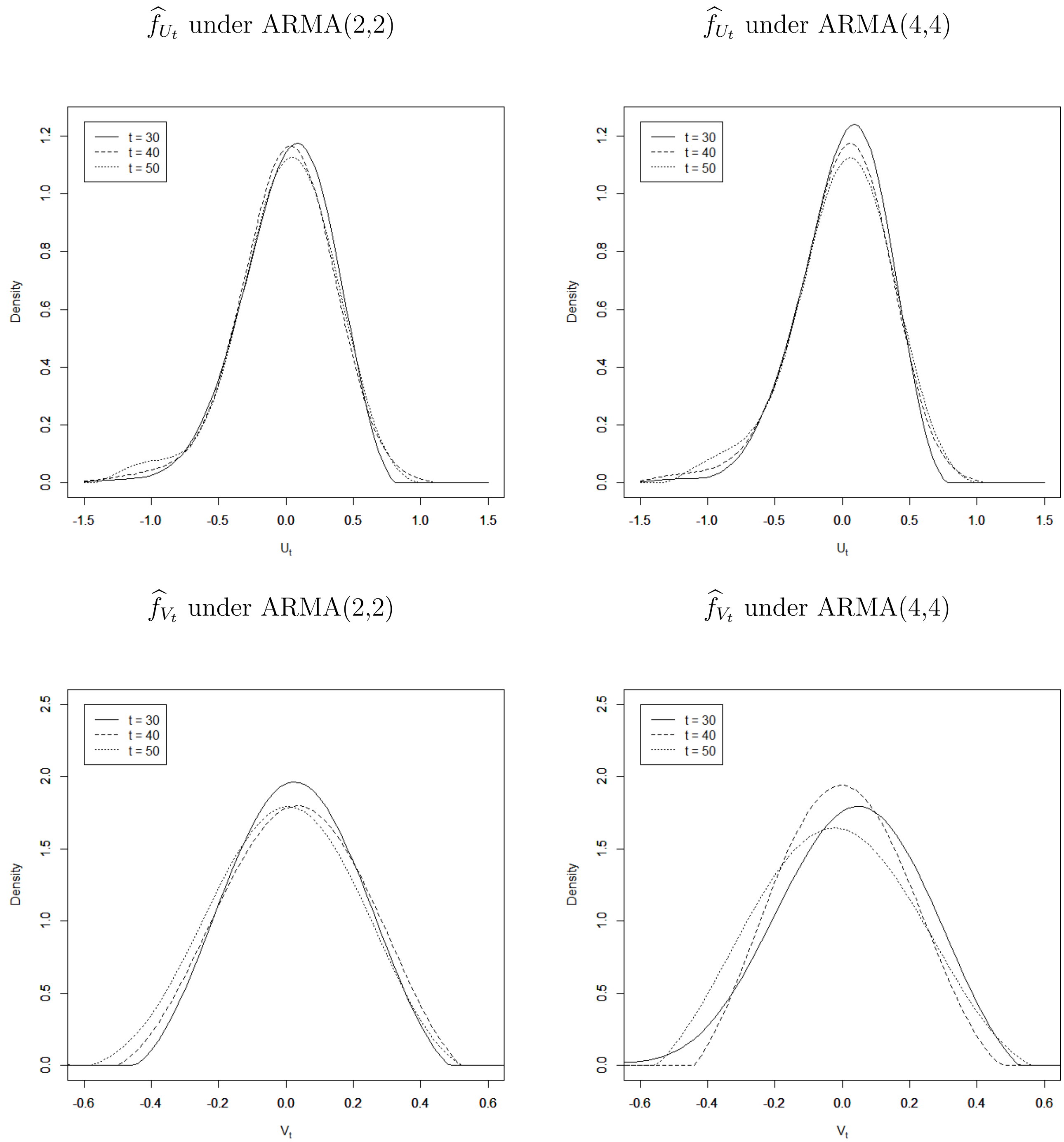


Figure 9: Nonparametric estimates of the marginal densities of the permanent earnings (top row) and the transitory earnings (bottom row) under each of the ARMA(2,2) specification (left column) and the ARMA(4,4) specification (right column). The sample consists of individuals with strong labor force attachment.

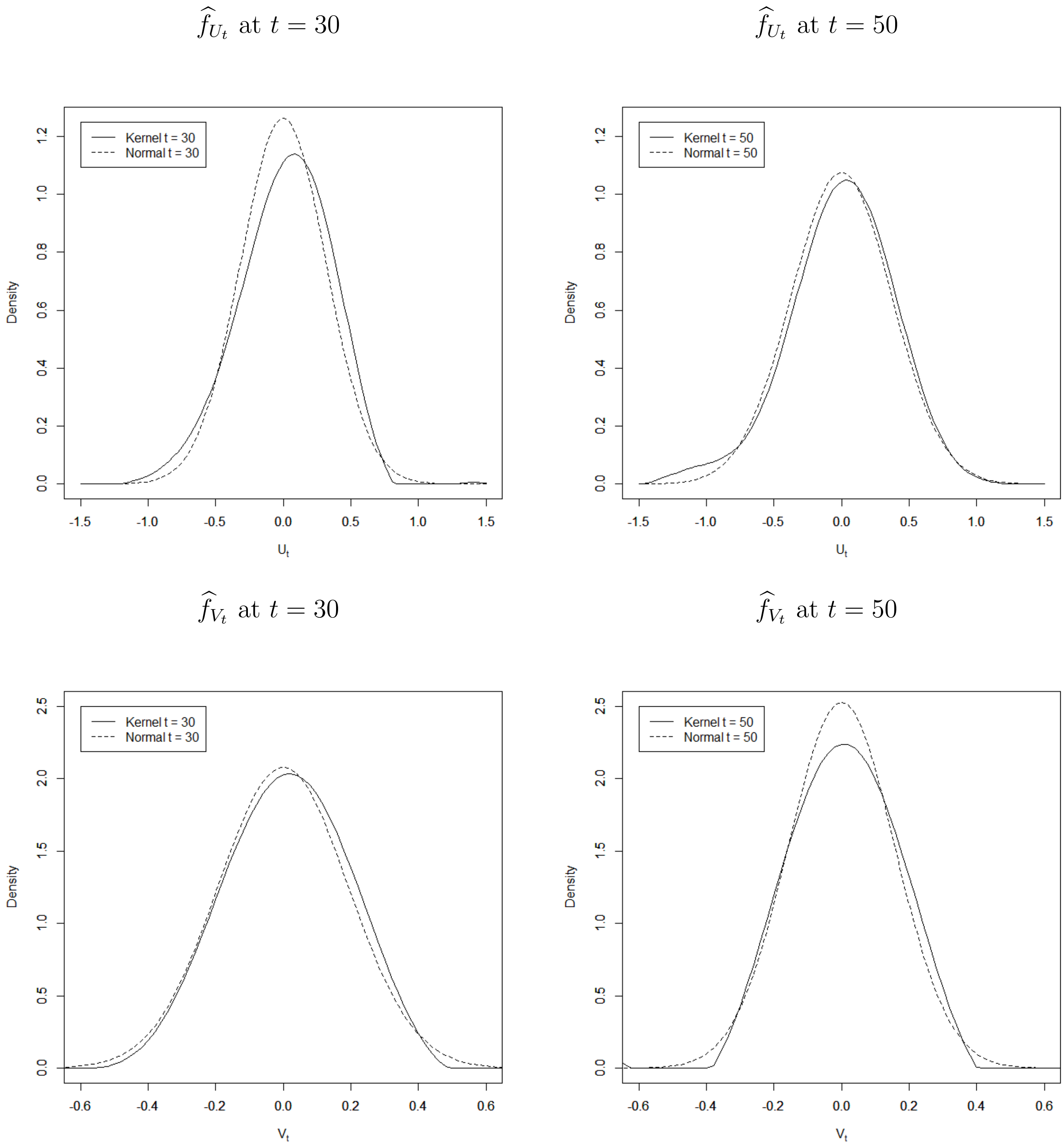


Figure 10: Nonparametric estimates of the marginal densities of the permanent earnings (top row) and the transitory earnings (bottom row) for $t = 30$ (left column) and $t = 50$ (right column) under the ARMA(1,1) specification with Gaussian references (dashed curves). The sample consists of individuals with strong labor force attachment.

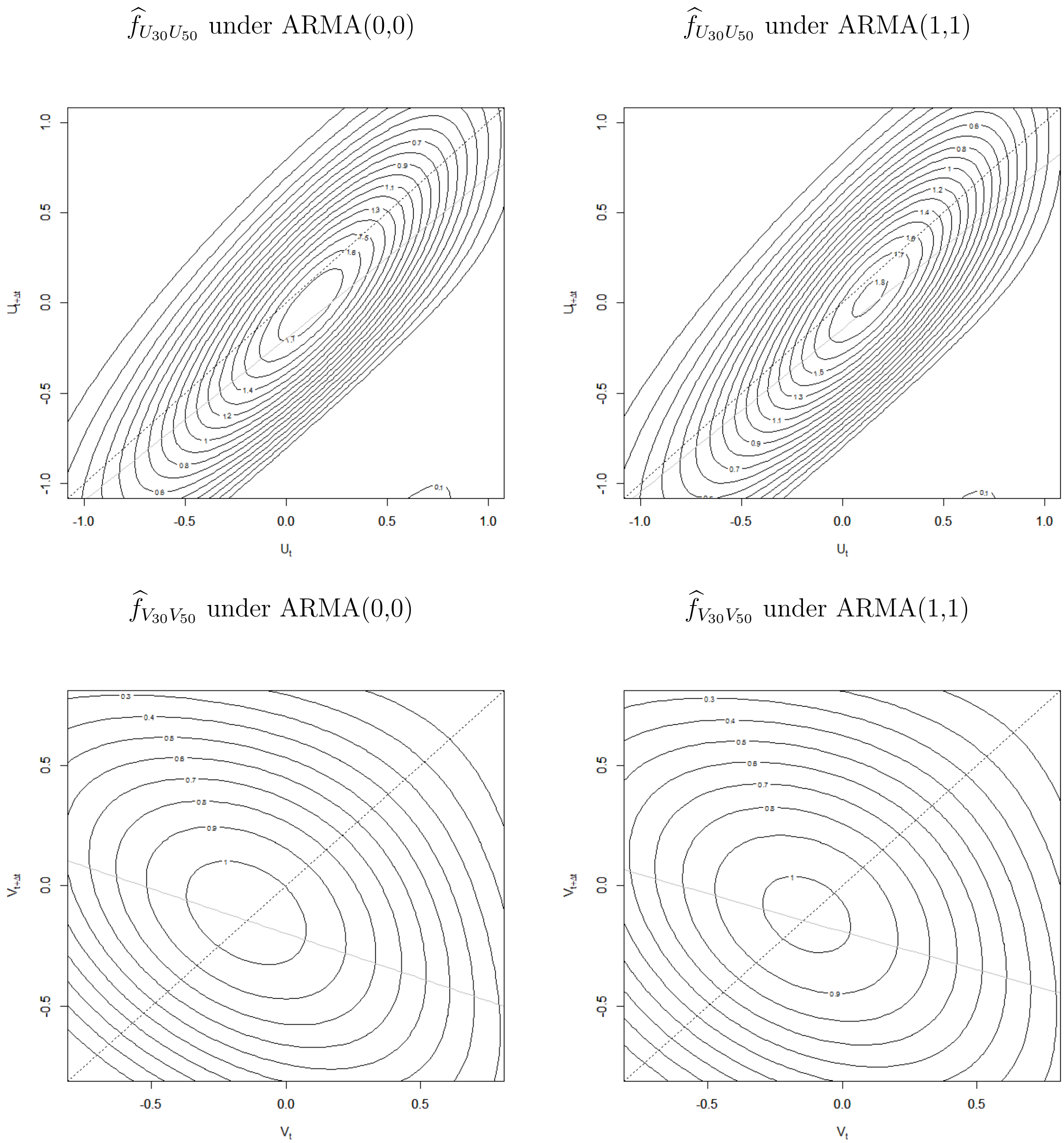


Figure 11: Nonparametric estimates of the long-run joint densities of the permanent earnings (top row) and the transitory earnings (bottom row) under each of the ARMA(0,0) specification (left column) and the ARMA(1,1) specification (right column). The sample consists of individuals with strong labor force attachment.

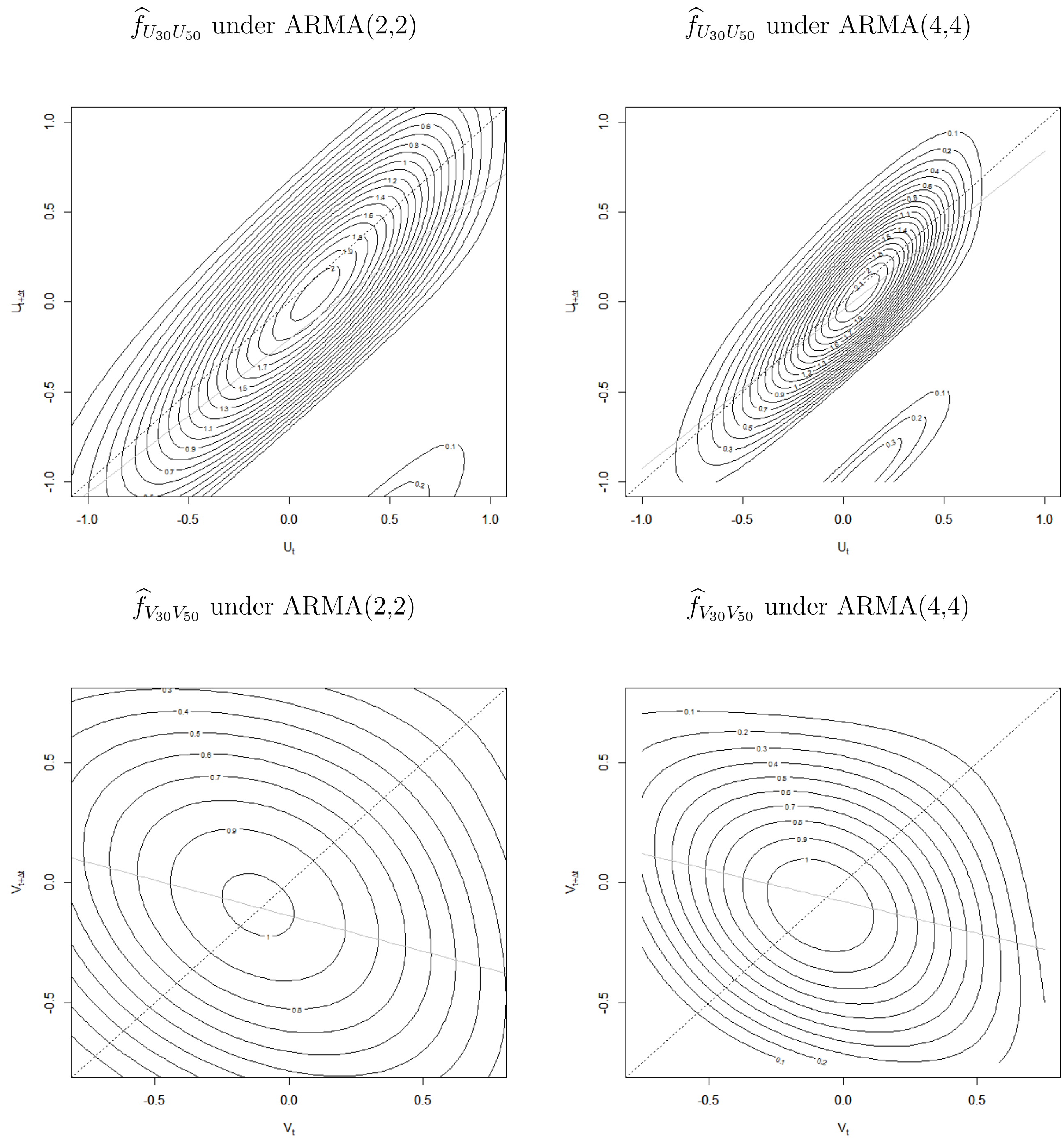
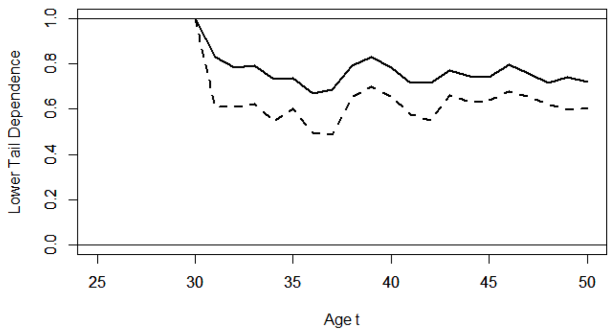
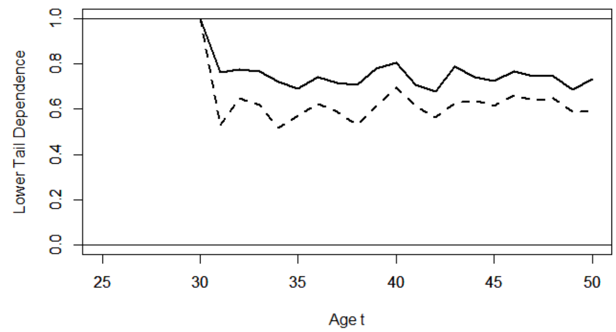


Figure 12: Nonparametric estimates of the long-run joint densities of the permanent earnings (top row) and the transitory earnings (bottom row) under each of the ARMA(2,2) specification (left column) and the ARMA(4,4) specification (right column). The sample consists of individuals with strong labor force attachment.

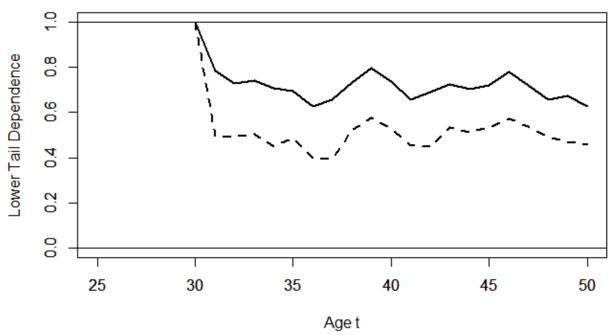
$\lambda^l_{30,t}(0.10)$ under ARMA(1,1)



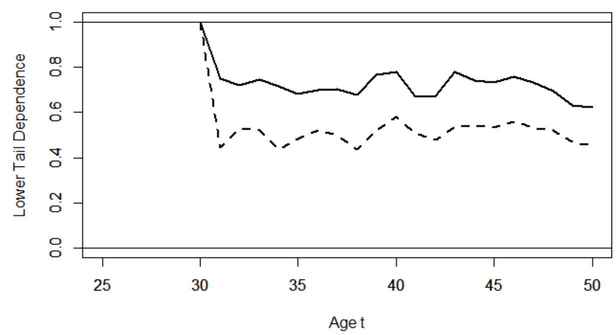
$\lambda^l_{30,t}(0.10)$ under ARMA(2,2)



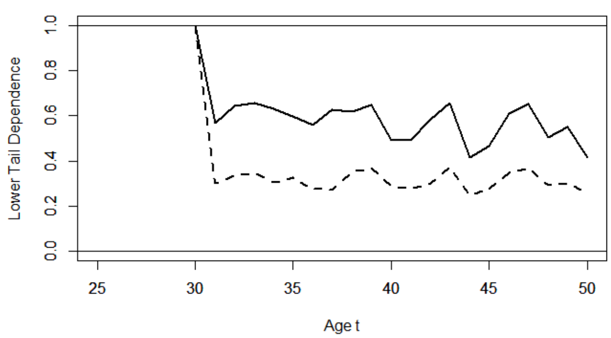
$\lambda^l_{30,t}(0.05)$ under ARMA(1,1)



$\lambda^l_{30,t}(0.05)$ under ARMA(2,2)



$\lambda^l_{30,t}(0.01)$ under ARMA(1,1)



$\lambda^l_{30,t}(0.01)$ under ARMA(2,2)

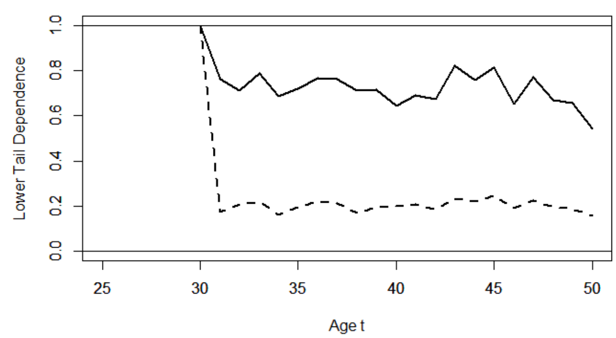


Figure 13: Trajectories of the lower tail dependence measure $\lambda^l_{30,t}(q) = P(U_t \leq F^{-1}_{U_t}(q)|U_{30} \leq F^{-1}_{U_{30}}(q))$ of permanent earnings following the event of permanent earnings less than or equal to the q-th quantile at age 30 for $q \in \{0.10, 0.05, 0.01\}$. The sample consists of individuals with strong labor force attachment. The solid lines represent the trajectories under our semiparametric model, while the dashed lines represent those under the bivariate normal distribution. The results are displayed under each of the ARMA(1,1) and ARMA(2,2) specifications with time-varying coefficients.

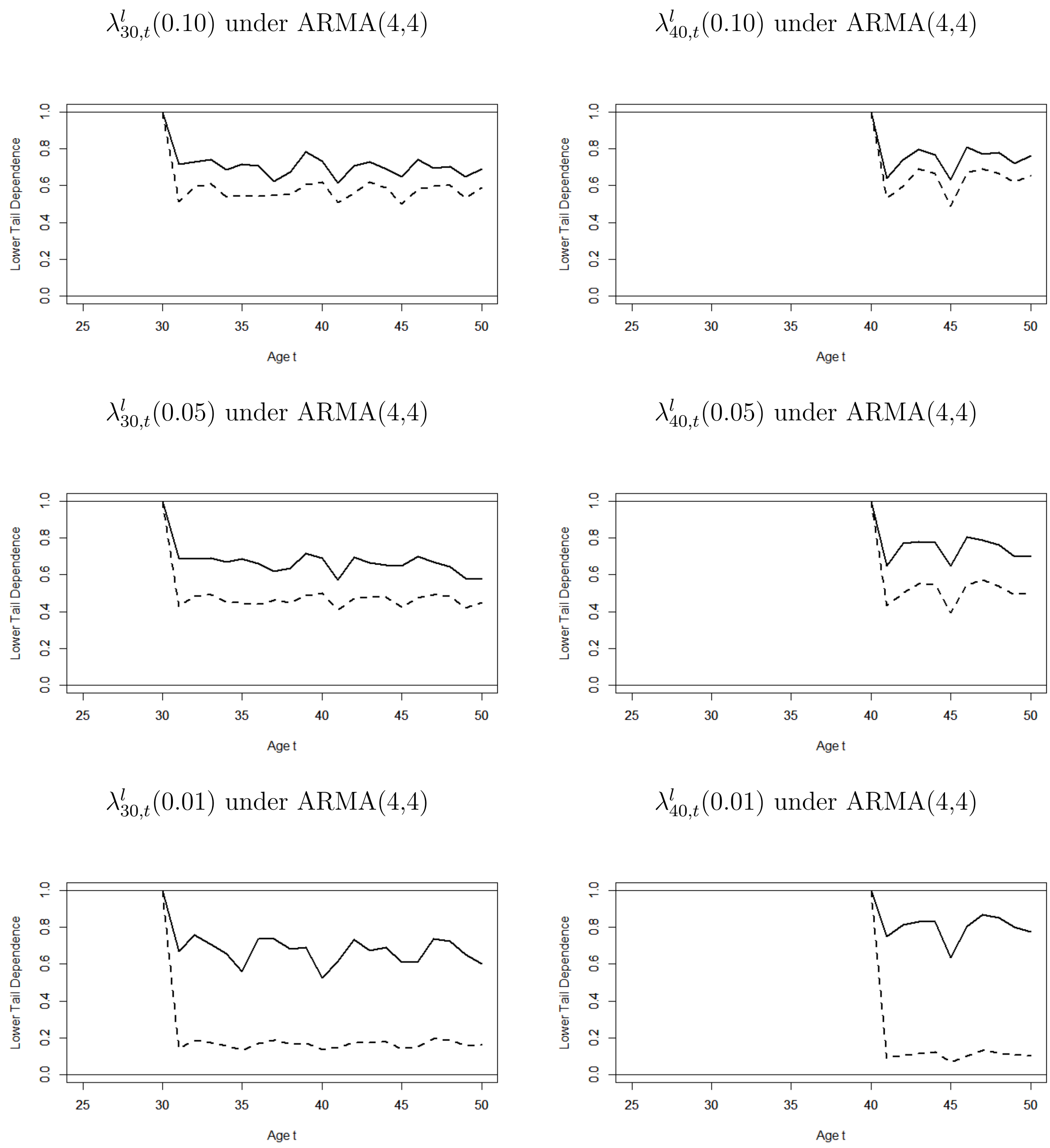


Figure 14: Trajectories of the lower tail dependence measures $\lambda^l_{30,t}(q) = P(U_t \leq F^{-1}_{U_t}(q) | U_{30} \leq F^{-1}_{U_{30}}(q))$ and $\lambda^l_{40,t}(q) = P(U_t \leq F^{-1}_{U_t}(q) | U_{40} \leq F^{-1}_{U_{40}}(q))$ of permanent earnings following the event of permanent earnings less than or equal to the q-th quantile at age 30 and 40, respectively, for $q \in \{0.10, 0.05, 0.01\}$. The sample consists of individuals with strong labor force attachment. The solid lines represent the trajectories under our semiparametric model, while the dashed lines represent those under the bivariate normal distribution. The results are displayed under the ARMA(4,4) specification with time-varying coefficients.

Married					
ARMA(0,0)	Mean	SD	Skewness	Kurtosis	H_0 : Kurtosis$\leqslant$3
U_{30}	0.000	0.331	-0.490	2.867	p-value = 0.687
	(0.014)	(0.010)	(0.107)	(0.273)	
U_{40}	0.000	0.382	-0.353	3.277	p-value = 0.180
	(0.016)	(0.012)	(0.116)	(0.303)	
U_{50}	0.000	0.389	-0.319	3.383	p-value = 0.221
	(0.018)	(0.014)	(0.151)	(0.499)	
V_{30}	-0.000	0.210	-2.354	17.840	p-value = 0.005
	(0.010)	(0.018)	(0.736)	(5.744)	
V_{40}	-0.000	0.165	-2.867	18.320	p-value = 0.012
	(0.010)	(0.017)	(0.951)	(6.626)	
V_{50}	-0.000	0.219	-3.616	26.217	p-value = 0.002
	(0.013)	(0.026)	(0.876)	(7.879)	

Table 1: Estimated distributional indices under the ARMA(0,0) model for the sub-sample of married individuals. The indices include the mean, the standard deviation, the skewness, and the kurtosis. The numbers in parentheses indicate the standard errors of the respective estimates. The last column shows the p-value of the one-sided test of the null hypothesis that kurtosis is less than equal to three, against the alternative hypothesis that it is greater than three.

Married: Time-Varying Coefficients

ARMA(1,1)	Mean	SD	Skewness	Kurtosis	H_0 : Kurtosis$\leqslant$3
U_{30}	-0.000	0.313	-0.339	1.792	p-value = 0.941
	(0.017)	(0.011)	(0.161)	(0.773)	
U_{40}	0.000	0.357	-0.460	3.430	p-value = 0.114
	(0.018)	(0.014)	(0.159)	(0.356)	
U_{50}	-0.000	0.371	-0.379	3.052	p-value = 0.462
	(0.022)	(0.017)	(0.166)	(0.548)	
V_{30}	-0.000	0.228	-2.706	18.568	p-value = 0.003
	(0.014)	(0.023)	(0.740)	(5.597)	
V_{40}	-0.000	0.192	-1.047	7.624	p-value = 0.032
	(0.013)	(0.020)	(0.842)	(5.211)	
V_{50}	0.000	0.208	-3.158	25.745	p-value = 0.038
	(0.018)	(0.032)	(1.453)	(12.827)	

Married: Time-Constant Coefficients

ARMA(1,1)	Mean	SD	Skewness	Kurtosis	H_0 : Kurtosis$\leqslant$3
U_{30}	-0.000	0.309	-0.289	1.464	p-value = 0.971
	(0.018)	(0.013)	(0.205)	(1.083)	
U_{40}	0.000	0.356	-0.465	3.443	p-value = 0.116
	(0.017)	(0.014)	(0.163)	(0.370)	
U_{50}	-0.000	0.371	-0.350	3.027	p-value = 0.480
	(0.020)	(0.016)	(0.161)	(0.534)	
V_{30}	-0.000	0.234	-2.665	17.933	p-value = 0.005
	(0.016)	(0.025)	(0.734)	(5.512)	
V_{40}	-0.000	0.193	-1.014	7.393	p-value = 0.060
	(0.013)	(0.018)	(0.822)	(2.833)	
V_{50}	0.000	0.208	-3.320	25.962	p-value = 0.019
	(0.015)	(0.030)	(1.232)	(11.053)	

Table 2: Estimated distributional indices under the ARMA(1,1) model for the sub-sample of married individuals. The indices include the mean, the standard deviation, the skewness, and the kurtosis. The numbers in parentheses indicate the standard errors of the respective estimates. The last column shows the p-value of the one-sided test of the null hypothesis that kurtosis is less than equal to three, against the alternative hypothesis that it is greater than three.

Married: Time-Varying Coefficients

ARMA(2,2)	Mean	SD	Skewness	Kurtosis	H_0 : Kurtosis$\leqslant$3
U_{30}	-0.000	0.306	-0.519	2.828	p-value = 0.651
	(0.016)	(0.011)	(0.140)	(0.445)	
U_{40}	0.000	0.326	-0.260	3.296	p-value = 0.206
	(0.018)	(0.013)	(0.158)	(0.362)	
U_{50}	-0.000	0.335	-0.380	2.827	p-value = 0.556
	(0.022)	(0.018)	(0.255)	(1.218)	
V_{30}	-0.000	0.223	-2.064	13.689	p-value = 0.027
	(0.014)	(0.019)	(0.685)	(5.543)	
V_{40}	-0.000	0.208	-1.360	6.638	p-value = 0.017
	(0.014)	(0.016)	(0.424)	(1.712)	
V_{50}	0.000	0.244	-1.808	15.764	p-value = 0.044
	(0.020)	(0.029)	(1.076)	(7.494)	

Married: Time-Invariant Coefficients

ARMA(2,2)	Mean	SD	Skewness	Kurtosis	H_0 : Kurtosis$\leqslant$3
U_{30}	-0.000	0.303	-0.528	2.840	p-value = 0.630
	(0.017)	(0.012)	(0.149)	(0.479)	
U_{40}	0.000	0.325	-0.258	3.297	p-value = 0.213
	(0.018)	(0.013)	(0.159)	(0.374)	
U_{50}	-0.000	0.340	-0.353	2.897	p-value = 0.539
	(0.021)	(0.017)	(0.227)	(1.041)	
V_{30}	-0.000	0.226	-1.974	12.910	p-value = 0.031
	(0.015)	(0.019)	(0.676)	(5.314)	
V_{40}	-0.000	0.209	-1.351	6.565	p-value = 0.016
	(0.014)	(0.016)	(0.425)	(1.658)	
V_{50}	0.000	0.236	-2.024	17.348	p-value = 0.037
	(0.017)	(0.028)	(1.106)	(8.053)	

Table 3: Estimated distributional indices under the ARMA(2,2) model for the sub-sample of married individuals. The indices include the mean, the standard deviation, the skewness, and the kurtosis. The numbers in parentheses indicate the standard errors of the respective estimates. The last column shows the p-value of the one-sided test of the null hypothesis that kurtosis is less than equal to three, against the alternative hypothesis that it is greater than three.

Married: Time-Varying Coefficients

ARMA(3,3)	Mean	SD	Skewness	Kurtosis	H_0 : Kurtosis$\leqslant$3
U_{30}	-0.000	0.290	-0.343	2.196	p-value = 0.915
	(0.017)	(0.011)	(0.150)	(0.587)	
U_{40}	-0.000	0.312	0.052	2.821	p-value = 0.685
	(0.019)	(0.013)	(0.168)	(0.370)	
U_{50}	-0.000	0.330	-0.292	2.654	p-value = 0.679
	(0.023)	(0.017)	(0.212)	(0.746)	
V_{30}	-0.000	0.221	-2.205	15.593	p-value = 0.038
	(0.015)	(0.022)	(0.840)	(7.075)	
V_{40}	0.000	0.223	-1.875	8.198	p-value = 0.004
	(0.016)	(0.019)	(0.421)	(1.928)	
V_{50}	-0.000	0.248	-1.948	16.164	p-value = 0.026
	(0.020)	(0.030)	(1.009)	(6.758)	

Married: Time-Invariant Coefficients

ARMA(3,3)	Mean	SD	Skewness	Kurtosis	H_0 : Kurtosis$\leqslant$3
U_{30}	-0.000	0.288	-0.335	2.118	p-value = 0.907
	(0.018)	(0.012)	(0.160)	(0.667)	
U_{40}	-0.000	0.312	0.062	2.804	p-value = 0.703
	(0.019)	(0.013)	(0.168)	(0.368)	
U_{50}	-0.000	0.333	-0.277	2.686	p-value = 0.667
	(0.022)	(0.016)	(0.202)	(0.728)	
V_{30}	-0.000	0.224	-2.153	15.073	p-value = 0.040
	(0.016)	(0.022)	(0.833)	(6.897)	
V_{40}	0.000	0.224	-1.878	8.174	p-value = 0.004
	(0.017)	(0.019)	(0.419)	(1.936)	
V_{50}	-0.000	0.244	-2.076	17.140	p-value = 0.024
	(0.019)	(0.029)	(1.038)	(7.166)	

Table 4: Estimated distributional indices under the ARMA(3,3) model for the sub-sample of married individuals. The indices include the mean, the standard deviation, the skewness, and the kurtosis. The numbers in parentheses indicate the standard errors of the respective estimates. The last column shows the p-value of the one-sided test of the null hypothesis that kurtosis is less than equal to three, against the alternative hypothesis that it is greater than three.

Married: Time-Varying Coefficients

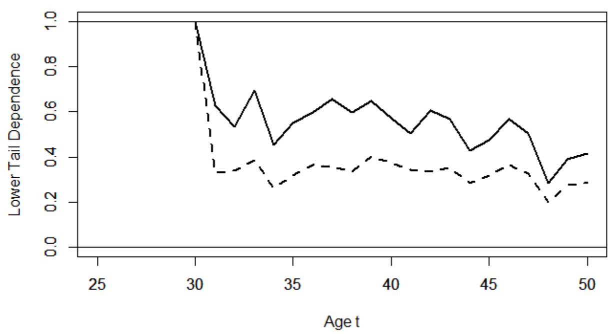
ARMA(4,4)	Mean	SD	Skewness	Kurtosis	H_0 : Kurtosis$\leqslant$3
U_{30}	0.000	0.283	-0.363	2.50	p-value = 0.875
	(0.019)	(0.013)	(0.157)	(0.435)	
U_{40}	-0.000	0.314	-0.364	3.233	p-value = 0.335
	(0.020)	(0.015)	(0.195)	(0.544)	
U_{50}	0.000	0.347	-0.291	2.394	p-value = 0.921
	(0.027)	(0.017)	(0.163)	(0.429)	
V_{30}	-0.000	0.210	-0.917	4.926	p-value = 0.144
	(0.016)	(0.016)	(0.414)	(1.814)	
V_{40}	0.000	0.203	-0.834	6.239	p-value = 0.074
	(0.017)	(0.019)	(0.601)	(2.237)	
V_{50}	-0.000	0.228	-2.732	24.345	p-value = 0.079
	(0.023)	(0.036)	(1.642)	(15.0917)	

Married: Time-Invariant Coefficients

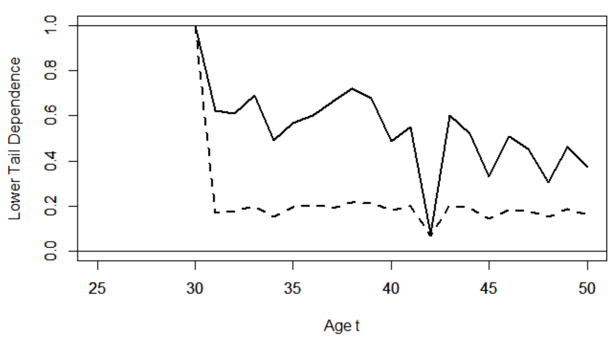
ARMA(4,4)	Mean	SD	Skewness	Kurtosis	H_0 : Kurtosis$\leqslant$3
U_{30}	0.000	0.280	-0.367	2.490	p-value = 0.862
	(0.019)	(0.013)	(0.168)	(0.467)	
U_{40}	-0.000	0.314	-0.366	3.233	p-value = 0.334
	(0.020)	(0.015)	(0.194)	(0.545)	
U_{50}	0.000	0.349	-0.275	2.4378	p-value = 0.883
	(0.025)	(0.016)	(0.158)	(0.473)	
V_{30}	-0.000	0.214	-0.883	4.743	p-value = 0.162
	(0.017)	(0.017)	(0.415)	(1.768)	
V_{40}	0.000	0.203	-0.825	6.218	p-value = 0.076
	(0.017)	(0.019)	(0.615)	(2.247)	
V_{50}	-0.000	0.226	-2.840	24.757	p-value = 0.035
	(0.021)	(0.035)	(1.494)	(11.985)	

Table 5: Estimated distributional indices under the ARMA(4,4) model for the sub-sample of married individuals. The indices include the mean, the standard deviation, the skewness, and the kurtosis. The numbers in parentheses indicate the standard errors of the respective estimates. The last column shows the p-value of the one-sided test of the null hypothesis that kurtosis is less than equal to three, against the alternative hypothesis that it is greater than three.

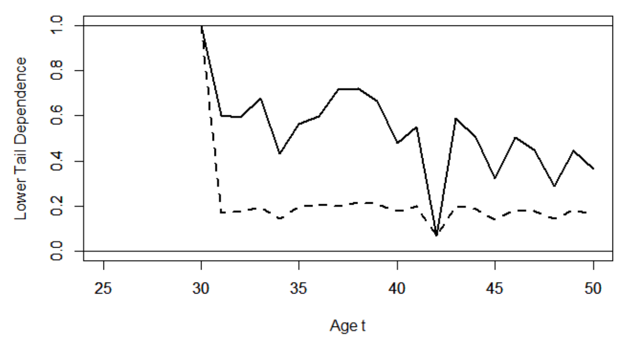
ARMA(0,0)



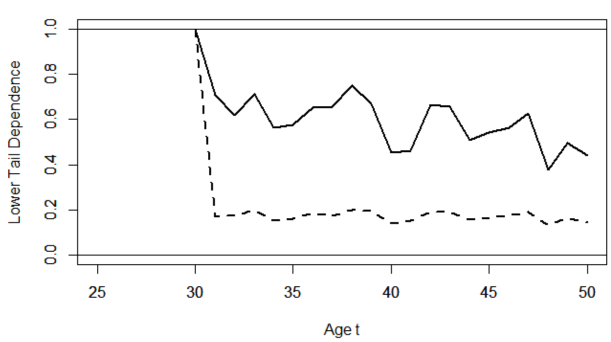
ARMA(1,1)



ARMA(1,1) with Constant Coefficients



ARMA(2,2)



ARMA(2,2) with Constant Coefficients

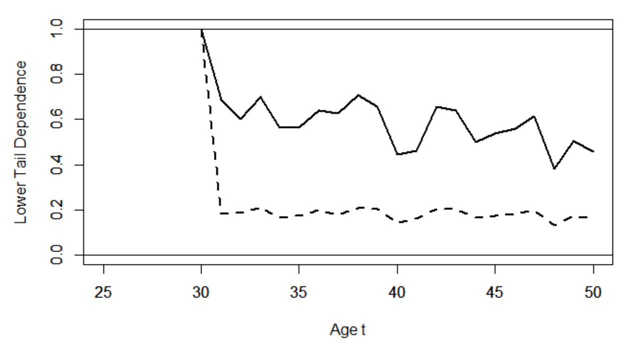


Figure 15: Trajectories of the lower tail dependence measure $\lambda^l_{30,t}(0.01) = P(U_t \leq F^{-1}_{U_t}(0.01)|U_{30} \leq F^{-1}_{U_{30}}(0.01))$ of permanent earnings following the event of permanent earnings less than or equal to the 1 percentile at age 30. The sample consists of married individuals. The solid lines represent the trajectories under our semiparametric model, while the dashed lines represent those under the bivariate normal distribution. The results are displayed under each of the ARMA(0,0), ARMA(1,1), and ARMA(2,2) specifications with time-varying coefficients and time-invariant coefficients.

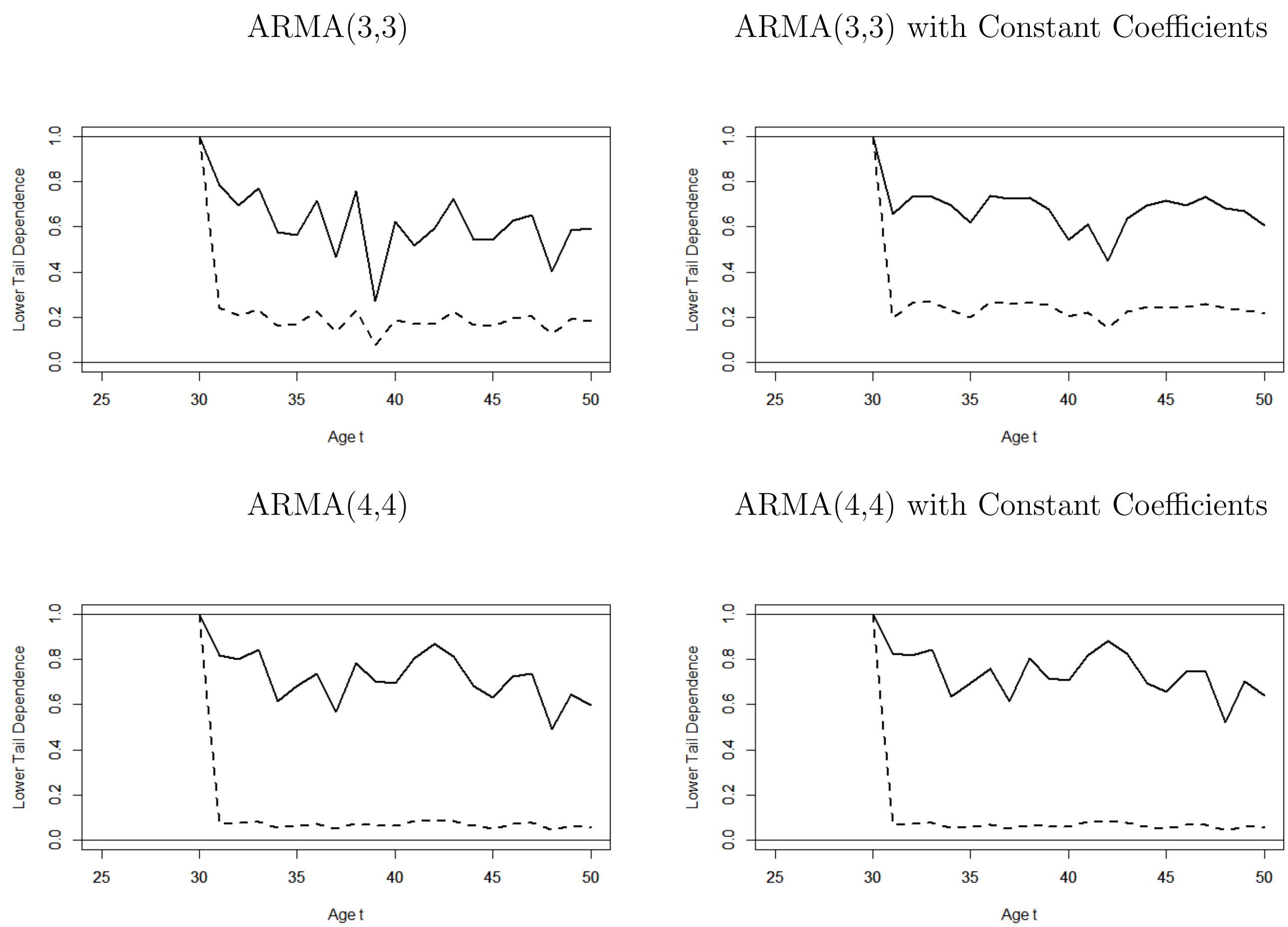


Figure 16: Trajectories of the lower tail dependence measure $\lambda^{l}_{30,t}(0.01) = P(U_t \leq F^{-1}_{U_t}(0.01)|U_{30} \leq F^{-1}_{U_{30}}(0.01))$ of permanent earnings following the event of permanent earnings less than or equal to the 1 percentile at age 30. The sample consists of married individuals. The solid lines represent the trajectories under our semiparametric model, while the dashed lines represent those under the bivariate normal distribution. The results are displayed under each of the ARMA(3,3) and ARMA(4,4) specifications with time-varying coefficients and time-invariant coefficients.