Econometrica Supplementary Material

SUPPLEMENT TO "THE CONVERSE ENVELOPE THEOREM"
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S.1. THE FAILURE OF THE STANDARD IMPLEMENTABILITY ARGUMENT

WHEN THE AGENT'S PREFERENCES have the quasilinear form $f(y, p, t) = h(y, t) - p$, a standard argument establishes the implementability of increasing allocations without resort to the converse envelope theorem. I first outline the argument, then show how it fails absent quasilinearity, necessitating my alternative approach based on the converse envelope theorem.

Fix an increasing allocation $Y : [0, 1] \to \mathcal{Y}$. Choose a P so that (Y, P) satisfies the envelope formula.[1] We then have for any $r, t \in [0, 1]$ that

$$\begin{aligned} & f\big(Y(t), P(t), t\big) - f\big(Y(r), P(r), t\big) \\ &\quad = \big[V_{Y,P}(t) - V_{Y,P}(r)\big] - \big[f\big(Y(r), P(r), t\big) - f\big(Y(r), P(r), r\big)\big] \\ &\quad = \int_r^t \big[f_3\big(Y(s), P(s), s\big) - f_3\big(Y(r), P(r), s\big)\big]\,\mathrm{d}s \end{aligned}$$

by the envelope formula and Lebesgue's fundamental theorem of calculus.

For quasilinear preferences, $f_3(y, p, s)$ does not vary with p, and f is single-crossing iff $y \mapsto f_3(y, 0, s)$ is increasing for every $s \in [0, 1]$.[2] Since Y is also increasing, this implies that the above integrand is nonnegative, which (since $r, t \in [0, 1]$ were arbitrary) shows that (Y, P) is incentive-compatible.

These properties of quasilinearity are very special, however. In general, single-crossing has nothing directly to say about the type derivative f_3, and so cannot be used to sign the integrand. The standard argument thus fails.

The argument may of course be salvaged by replacing single-crossing with the brute assumption that the integrand is nonnegative. But this assumption lacks a choice interpretation, being a restriction on the *type* derivative f_3 of the utility representation f. A theorem with such a hypothesis would have no economic meaning. (By contrast, single-crossing has a straightforward choice interpretation, described in the text.)

S.2. SOME REGULAR OUTCOME SPACES (§4.2)

PROPOSITION S.1: *The following partially ordered sets are regular*:

(a) $\mathbf{R}^n$ *equipped with the usual* (*product*) *order*: $(y_1, \dots, y_n) \lesssim (y_1', \dots, y_n')$ *iff* $y_i \le y_i'$ *for every* $i \in \{1, \dots, n\}$.

(b) *The space* ℓ^1 *of summable sequences equipped with the product order*: $(y_i)_{i\in\mathbf{N}} \lesssim (y_i')_{i\in\mathbf{N}}$ *iff* $y_i \le y_i'$ *for every* $i \in \mathbf{N}$.

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[1] In the quasilinear case, such a P is given explicitly by $P(t) := h(Y(t), t) - \int_0^t h_2(Y(s), s)\,\mathrm{d}s$, obviating the need to invoke the existence lemma in Appendix B.1.1.

[2] This is easily shown, and does not depend on exactly how "single-crossing" is formalized.

(c) *For any measure space* $(\Omega, \mathcal{F}, \mu)$, *the space* $\mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ *of (equivalence classes of* μ*-a.e. equal)* μ*-integrable functions* $\Omega \to \mathbf{R}$, *equipped with the partial order* $\lesssim$ *defined by* $y \lesssim y'$ *iff* $y \leq y'$ μ*-a.e.*
(*Special case*: *for any probability space, the space of finite-expectation random variables, ordered by "a.s. smaller."*)
(d) *For any finite set* Ω *and probability* $\mu_0 \in \Delta(\Omega)$, *the space of mean-*μ_0 *Borel probability measures on* $\Delta(\Omega)$, *equipped with the Blackwell informativeness order defined in* §4.4.[3]
(e) *The open intervals of* $(0, 1)$ (*including* $\emptyset$), *ordered by set inclusion* $\subseteq$.

We will use the following sufficient condition for chain-separability.

LEMMA S.1: *If there is a strictly increasing function* $\mathcal{Y} \to \mathbf{R}$, *then* $\mathcal{Y}$ *is chain-separable*.

(The converse is false: there are chain-separable spaces that admit no strictly increasing real-valued function.)

PROOF: Suppose that $\phi : \mathcal{Y} \to \mathbf{R}$ is a strictly increasing function, and let $Y \subseteq \mathcal{Y}$ be a chain; we will show that Y has a countable order-dense subset. By inspection, the restriction $\phi|_Y$ of ϕ to Y is an order-embedding of Y into $\mathbf{R}$; thus Y is order-isomorphic to a subset of $\mathbf{R}$ (namely $\phi(Y)$). The order-isomorphs of subsets of $\mathbf{R}$ are precisely those chains that have a countable order-dense subsets (see, e.g., Theorem 24 in Birkhoff (1967, p. 200)); thus Y has a countable order-dense subset. *Q.E.D.*

PROOF OF PROPOSITION S.1(a)–(c): $\mathbf{R}^n$ is exactly $\mathcal{L}^1(\{1, \ldots, n\}, 2^{\{1,\ldots,n\}}, c)$ where c is the counting measure; similarly, ℓ^1 is $\mathcal{L}^1(\mathbf{N}, 2^{\mathbf{N}}, c)$. It therefore suffices to establish (c).

So fix a measure space $(\Omega, \mathcal{F}, \mu)$, and let $\mathcal{Y} := \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ be ordered by "μ-a.e. smaller." $\mathcal{Y}$ is order-dense-in-itself since if $y \leq y''$ μ-a.e. and $y \neq y''$ on a set of positive μ-measure, then $y' := (y + y'')/2$ lives in $\mathcal{Y}$ and satisfies $y \leq y' \leq y''$ μ-a.e. and $y \neq y' \neq y''$ on a set of positive μ-measure.

For countable-chain completeness, take any countable chain $Y \subseteq \mathcal{Y}$, and suppose that it has a lower bound $y \in \mathcal{Y}$; we will show that Y has an infimum. (The argument for upper bounds is symmetric.) Define $y_\star : \Omega \to \mathbf{R}$ by $y_\star(\omega) := \inf_{y \in Y} y(\omega)$ for each $\omega \in \Omega$; it is well-defined (i.e., it maps into $\mathbf{R}$, with the possible exception of a μ-null set) since Y has a lower bound. Clearly $y' \leq y_\star \leq y''$ μ-a.e. for any lower bound y' of Y and any $y'' \in Y$, so it remains only to show that $y_\star$ lives in $\mathcal{Y}$, meaning that it is measurable and that its integral is finite. Measurability obtains since Y is countable (e.g., Proposition 2.7 in Folland (1999)). As for the integral, since $y \leq y_\star \leq y_0$ μ-a.e. and y and y_0 are integrable (live in $\mathcal{Y}$), we have

$$-\infty < \int_\Omega y \, \mathrm{d}\mu \leq \int_\Omega y_\star \, \mathrm{d}\mu \leq \int_\Omega y_0 \, \mathrm{d}\mu < +\infty.$$

For chain-separability, define $\phi : \mathcal{Y} \to \mathbf{R}$ by $\phi(y) := \int_\Omega y \, \mathrm{d}\mu$ for each $y \in \mathcal{Y}$. ϕ is strictly increasing: if $y \leq y'$ μ-a.e. and $y \neq y'$ on a set of positive μ-measure, then $\phi(y) < \phi(y')$. Chain-separability follows by Lemma S.1. *Q.E.D.*

[3]A proof that this is a partial order (in particular, antisymmetric) may be found in Müller (1997, Theorem 5.2).

PROOF OF PROPOSITION S.1(d): Fix a finite set Ω and a probability $\mu_0 \in \Delta(\Omega)$, and let $\mathcal{Y}$ be the space of Borel probability measures with mean μ_0, equipped with the Blackwell informativeness order $\lesssim$. $\mathcal{Y}$ is order-dense-in-itself because if $y, y'' \in \mathcal{Y}$ satisfy $\int_{\Delta(\Omega)} v \, \mathrm{d}y \leq \int_{\Delta(\Omega)} v \, \mathrm{d}y''$ for every continuous and convex $v : \Delta(\Omega) \to \mathbf{R}$, with the inequality strict for some $v = \widehat{v}$, then $y' := (y + y'')/2$ also lives in $\mathcal{Y}$ and satisfies $\int_{\Delta(\Omega)} v \, \mathrm{d}y \leq \int_{\Delta(\Omega)} v \, \mathrm{d}y' \leq \int_{\Delta(\Omega)} v \, \mathrm{d}y''$ for every continuous and convex $v : \Delta(\Omega) \to \mathbf{R}$, with both inequalities strict for $v = \widehat{v}$.

For countable chain-completeness, let $Y \subseteq \mathcal{Y}$ be a countable chain with an upper bound in $\mathcal{Y}$; we will show that it has a supremum. (The argument for infima is analogous.) This is trivial if Y has a maximum element, so suppose not. Then there is a strictly increasing sequence $(y_n)_{n \in \mathbf{N}}$ in Y that has no upper bound in Y. This sequence is trivially tight since $\Delta(\Omega)$ is a compact metric space, so has a weakly convergent subsequence $(y_{n_k})_{k \in \mathbf{N}}$ by Prokhorov's theorem;[4] call the limit $y^\star$. Then by the monotone convergence theorem for real numbers and the definition of weak convergence, we have for every for every continuous (hence bounded) and convex $v : \Delta(\Omega) \to \mathbf{R}$ that

$$\sup_{y \in Y} \int_{\Delta(\Omega)} v \, \mathrm{d}y = \lim_{k \to \infty} \int_{\Delta(\Omega)} v \, \mathrm{d}y_{n_k} = \int_{\Delta(\Omega)} v \, \mathrm{d}y^\star,$$

which is to say that $y^\star$ is the supremum of Y.

For chain-separability, it suffices by Lemma S.1 to identify a strictly increasing function $\mathcal{Y} \to \mathbf{R}$. Let v be any strictly convex function $\Delta(\Omega) \to \mathbf{R}$,[5] and define $\phi : \mathcal{Y} \to \mathbf{R}$ by $\phi(y) := \int_{\Delta(\Omega)} v \, \mathrm{d}y$. Take $y < y'$ in $\mathcal{Y}$; we must show that $\phi(y) < \phi(y')$. By a standard embedding theorem (e.g., Theorem 7.A.1 in Shaked and Shanthikumar (2007)), there exists a probability space on which there are random vectors X, X' with respective laws y, y' such that $\mathbf{E}(X'|X) = X$ a.s. and $X \neq X'$ with positive probability. Thus

$$\phi(y') = \mathbf{E}\big(v(X')\big) = \mathbf{E}\big(\mathbf{E}\big[v(X')|X\big]\big) > \mathbf{E}\big(v\big(\mathbf{E}\big[X'|X\big]\big)\big) = \mathbf{E}\big(v(X)\big) = \phi(y)$$

by Jensen's inequality. *Q.E.D.*

PROOF OF PROPOSITION S.1(e): Write $\mathcal{Y}$ for the open intervals of $(0,1)$. $\mathcal{Y}$ is order-dense-in-itself since if $(a, b) \subsetneq (a'', b'')$ then $(a', b') := ([a + a'']/2, [b + b'']/2)$ is an open interval (lives in $\mathcal{Y}$) and satisfies $(a, b) \subsetneq (a', b') \subsetneq (a'', b'')$.

For countable chain-completeness, we must show that every countable chain has an infimum and supremum. So take a countable chain $Y \subseteq \mathcal{Y}$, define $y^\star := \bigcup_{y \in Y} y$, and let $y_\star$ be the interior of $\bigcap_{y \in Y} y$. Both are open intervals, so live in $\mathcal{Y}$. Clearly $y \subseteq y^\star \subseteq y^+$ for any $y \in Y$ and any set y^+ containing every member of Y, so $y^\star$ is the supremum of Y. Similarly, $y_\star \subseteq \bigcap_{y' \in Y} y' \subseteq y$ for any $y \in Y$, and $y_- \subseteq y_\star$ for any *open* set y_- contained in every member of Y since $y_\star$ is by definition the $\subseteq$-largest open set contained in $\bigcap_{y \in Y} y$.

For chain-separability, define $\phi : \mathcal{Y} \to \mathbf{R}$ by $\phi((a, b)) := b - a$. It is clearly strictly increasing, giving us chain-separability by Lemma S.1. *Q.E.D.*

S.3. PROOF OF THE APPROXIMATION LEMMA (APPENDIX B.1.2)

Let $Y : [0, 1] \to \mathcal{Y}$ be increasing. Then $Y([0, 1])$ is a chain. The result is trivial if $Y([0, 1])$ is a singleton, so suppose not.

[4] For example, Theorem 5.1 in Billingsley (1999).

[5] For example, the $\mathcal{L}^2$ norm $\|\cdot\|_2$, which is strictly convex on $\Delta(\Omega)$ by Minkowski's inequality.

We will first show (steps 1–3) that $Y([0,1])$ may be embedded in a chain $\mathcal{C} \subseteq \mathcal{Y}$ with $\inf \mathcal{C} = Y(0)$ and $\sup \mathcal{C} = Y(1)$ that is order-dense-in-itself, order-complete, and order-separable. We will then argue (step 4) that this chain $\mathcal{C}$ is order-isomorphic and homeomorphic to the unit interval, allowing us to treat Y as a function $[0,1] \to [0,1]$.

Step 1: *construction of* $\mathcal{C}$. Write $\lesssim$ for the partial order on $\mathcal{Y}$. Define $\mathcal{Y}'$ to be the set of all outcomes $y' \in \mathcal{Y}$ that are $\lesssim$-comparable to every $y \in Y([0,1])$ and that satisfy $Y(0) \lesssim y' \lesssim Y(1)$.

We claim that $\mathcal{Y}'$ is order-dense-in-itself. Suppose to the contrary that there are $y < y''$ in $\mathcal{Y}'$ for which no $y' \in \mathcal{Y}'$ satisfies $y < y' < y''$. Observe that by definition of $\mathcal{Y}'$, any $x \in Y([0,1])$ must be comparable to both y and y'', so that

$$\left\{x \in Y\big([0,1]\big) : x \lesssim y \text{ or } y'' \lesssim x\right\} = Y\big([0,1]\big).$$

Since it is order-dense-in-itself, the grand space $\mathcal{Y}$ does contain an outcome y' such that $y < y' < y''$. Since $\lesssim$ is transitive (being a partial order), it follows that y' is comparable to every element of

$$\left\{x \in \mathcal{Y} : x \lesssim y \text{ or } y'' \lesssim x\right\} \supseteq \left\{x \in Y\big([0,1]\big) : x \lesssim y \text{ or } y'' \lesssim x\right\} = Y\big([0,1]\big).$$

But then y' lies in $\mathcal{Y}'$ by definition of the latter—a contradiction.

Clearly $Y(1)$ is an upper bound of any chain in $\mathcal{Y}'$. It follows by the Hausdorff maximality principle (which is equivalent to the Axiom of Choice) that there is a chain $\mathcal{C} \subseteq \mathcal{Y}'$ that is maximal with respect to set inclusion. (That is, $\mathcal{C} \cup \{y\}$ fails to be a chain for every $y \in \mathcal{Y}' \setminus \mathcal{C}$.)

Step 2: *easy properties of* $\mathcal{C}$. By definition of $\mathcal{Y}'$, any maximal chain in $\mathcal{Y}'$ (in particular, $\mathcal{C}$) contains $Y([0,1])$ and has infimum $Y(0)$ and supremum $Y(1)$.

To see that $\mathcal{C}$ is order-dense-in-itself, assume toward a contradiction that there are $c < c''$ for which no $c' \in \mathcal{C}$ satisfies $c < c' < c''$, so that (since $\mathcal{C}$ is a chain)

$$\left\{c' \in \mathcal{C} : c' \lesssim c\right\} \cup \left\{c' \in \mathcal{C} : c'' \lesssim c'\right\} = \mathcal{C}.$$

Because $\mathcal{Y}'$ is order-dense-in-itself, there is a $y' \in \mathcal{Y}' \setminus \mathcal{C}$ with $c < y' < c''$. It follows by transitivity of $\lesssim$ that y' is comparable to every element of

$$\left\{c' \in \mathcal{C} : c' \lesssim c\right\} \cup \left\{c' \in \mathcal{C} : c'' \lesssim c'\right\} = \mathcal{C}.$$

But then $\mathcal{C} \cup \{y'\}$ is a chain in $\mathcal{Y}'$, contradicting the maximality of $\mathcal{C}$.

To establish that $\mathcal{C}$ is order-separable, we must find a countable order-dense subset of $\mathcal{C}$. Because the grand space $\mathcal{Y}$ is chain-separable, it contains a countable set $\mathcal{K}$ that is order-dense in $\mathcal{C}$. Since $\mathcal{C}$ is a chain contained in

$$\left\{y \in \mathcal{Y} : Y(0) \lesssim y \lesssim Y(1)\right\},$$

we may assume without loss of generality that every $k \in \mathcal{K}$ satisfies $Y(0) \lesssim k \lesssim Y(1)$ and is comparable to every element of $\mathcal{C}$. It follows that $\mathcal{K}$ is contained in $\mathcal{Y}'$ (by definition of the latter). We claim that $\mathcal{K}$ is contained in $\mathcal{C}$. Suppose to the contrary that there is a $k \in \mathcal{K}$ that does not lie in $\mathcal{C}$; then $\mathcal{C} \cup \{k\}$ is a chain in $\mathcal{Y}'$, which is absurd since $\mathcal{C}$ is maximal.

Step 3: *order-completeness of* $\mathcal{C}$. Since every subset of $\mathcal{C}$ has a lower and an upper bound (namely $Y(0)$ and $Y(1)$, respectively), what must be shown is that every subset of the chain $\mathcal{C}$ has an infimum and a supremum in $\mathcal{C}$. To that end, take any subset $\mathcal{C}'$ of $\mathcal{C}$, necessarily a chain.

We will first (step 3(a)) show that if $\inf \mathcal{C}'$ exists in $\mathcal{Y}$, then it must lie in $\mathcal{C}$. We will then (step 3(b)) construct a countable chain $\mathcal{C}''' \subseteq \mathcal{C}'$, for which $\inf \mathcal{C}'''$ exists in $\mathcal{Y}$ by countable-chain completeness of $\mathcal{Y}$, and show that it is also the infimum in $\mathcal{Y}$ of $\mathcal{C}'$. We omit the analogous arguments for $\sup \mathcal{C}'$.

Step 3(a): $\inf \mathcal{C}' \in \mathcal{C}$ *if the former exists in* $\mathcal{Y}$. Suppose that $\inf \mathcal{C}'$ exists in $\mathcal{Y}$. We claim that it lies in $\mathcal{Y}'$, meaning that $Y(0) \lesssim \inf \mathcal{C}' \lesssim Y(1)$ and that $\inf \mathcal{C}'$ is comparable to every $y \in Y([0,1])$. The former condition is clearly satisfied. For the latter, since $\inf \mathcal{C}'$ is a lower bound of $\mathcal{C}'$, transitivity of $\lesssim$ ensures that it is comparable to every $y \in Y([0,1])$ such that $c' \lesssim y$ for some $c' \in \mathcal{C}'$. To see that $\inf \mathcal{C}'$ is also comparable to every $y \in Y([0,1])$ with $y < c'$ for every $c' \in \mathcal{C}'$, note that any such y is a lower bound of $\mathcal{C}'$. Since $\inf \mathcal{C}'$ is the *greatest* lower bound, we must have $y \lesssim \inf \mathcal{C}'$, showing that $\inf \mathcal{C}'$ is comparable to y.

Now to show that $\inf \mathcal{C}'$ lies in $\mathcal{C}$, decompose the chain $\mathcal{C}$ as

$$\begin{aligned} \mathcal{C} &= \{c \in \mathcal{C} : c \lesssim c' \text{ for every } c' \in \mathcal{C}'\} \cup \{c \in \mathcal{C} : c' < c \text{ for some } c' \in \mathcal{C}'\} \\ &= \{c \in \mathcal{C} : c \lesssim \inf \mathcal{C}'\} \cup \{c \in \mathcal{C} : \inf \mathcal{C}' < c\}. \end{aligned}$$

Clearly $\inf \mathcal{C}'$ is comparable to every element of $\mathcal{C}$, and we showed that it lies in $\mathcal{Y}'$. Thus $\mathcal{C} \cup \{\inf \mathcal{C}'\}$ is a chain in $\mathcal{Y}'$, which by maximality of $\mathcal{C}$ requires that $\inf \mathcal{C}' \in \mathcal{C}$.

Step 3(b): $\inf \mathcal{C}'$ *exists in* $\mathcal{Y}$. By essentially the same construction as we used to embed $Y([0,1])$ in $\mathcal{Y}'$ in step 1, $\mathcal{C}'$ may be embedded in a chain $\mathcal{C}'' \subseteq \mathcal{C}$ that is order-dense-in-itself such that for every $c'' \in \mathcal{C}''$, we have $c'_- \lesssim c'' \lesssim c'_+$ for some $c'_-, c'_+ \in \mathcal{C}'$. By order-separability of $\mathcal{C}$, $\mathcal{C}''$ has a countable order-dense subset $\mathcal{C}'''$, necessarily a chain. By countable chain-completeness of $\mathcal{Y}$, $\inf \mathcal{C}'''$ exists in $\mathcal{Y}$. We will show that it is the greatest lower bound of $\mathcal{C}'$.

Observe that $\inf \mathcal{C}'''$ is a lower bound of $\mathcal{C}''$ since $\mathcal{C}'''$ is order-dense in $\mathcal{C}''$. There can be no greater lower bound of $\mathcal{C}''$ since $\mathcal{C}''' \subseteq \mathcal{C}''$. Thus $\inf \mathcal{C}''$ exists in $\mathcal{Y}$ and equals $\inf \mathcal{C}'''$.

Since $\inf \mathcal{C}''$ is a lower bound of $\mathcal{C}'' \supseteq \mathcal{C}'$, it is a lower bound of $\mathcal{C}'$. On the other hand, by construction of $\mathcal{C}''$, we may find for every $c'' \in \mathcal{C}''$ a $c' \in \mathcal{C}'$ such that $c' \lesssim c''$, so there cannot be a greater lower bound of $\mathcal{C}'$. Thus $\inf \mathcal{C}''$ is the greatest lower bound of $\mathcal{C}'$ in $\mathcal{Y}$.

Step 4: *identification of* $\mathcal{C}$ *with* $[0,1]$. Since $\mathcal{C}$ is an order-separable chain, it is order-isomorphic to a subset $\mathcal{S}$ of $\mathbf{R}$ (see, e.g., Theorem 24 in Birkhoff (1967, p. 200)). It follows that $\mathcal{C}$ with the order topology is homeomorphic to $\mathcal{S}$ with its order topology.

The set $\mathcal{S}$ is dense in an interval $\mathcal{S}' \supseteq \mathcal{S}$ since $\mathcal{S}$ is order-dense-in-itself (because $\mathcal{C}$ is). The interval $\mathcal{S}'$ must be closed and bounded since it contains its infimum and supremum (because $\mathcal{C}$ contains $Y(0)$ and $Y(1)$). Since $\mathcal{S}$ is order-complete (because $\mathcal{C}$ is), it must coincide with its closure, so that $\mathcal{S}' = \mathcal{S}$. Finally, $\mathcal{S}$ is a proper interval since $\mathcal{C}$ is neither empty nor a singleton. In sum, we may identify $\mathcal{C}$ with a closed and bounded proper interval of $\mathbf{R}$—without loss of generality, the unit interval $[0,1]$.

We may therefore treat Y as an increasing function $[0,1] \to [0,1]$. With this simplification, it is straightforward to construct a sequence $(Y_n)_{n\in\mathbf{N}}$ with the desired properties; we omit the details. *Q.E.D.*

S.4. PREFERENCE REGULARITY IN SELLING INFORMATION (§4.4)

In this Appendix, we show that the joint continuity part of preference regularity is satisfied in §4.4. We require two lemmata.

LEMMA S.2: *Let $\mathcal{Y}$ be the set of Borel probability distributions with mean μ_0, equipped with the Blackwell informativeness order (as in §4.4). Give $\mathcal{Y}$ the order topology, and let $\mathcal{C} \subseteq \mathcal{Y}$ be a chain. If a sequence $(y_n)_{n\in\mathbf{N}}$ in $\mathcal{C}$ converges to $y \in \mathcal{C}$ in the relative topology on $\mathcal{C}$, then*

$$\sup_{\substack{v^+,v^-:\Delta(\Omega)\to\mathbf{R} \\ \text{continuous convex} \\ \text{s.t. } |v^+-v^-|\leq 1}} \left| \int_{\Delta(\Omega)} \left(v^+ - v^-\right) \mathrm{d}(y_n - y) \right| \to 0 \quad \text{as } n \to \infty.$$

COROLLARY S.1: *Under the same hypotheses,*

$$\sup_{\substack{v:\Delta(\Omega)\to[-1,1] \\ \text{continuous convex}}} \left| \int_{\Delta(\Omega)} v \, \mathrm{d}(y_n - y) \right| \to 0 \quad \text{as } n \to \infty.$$

PROOF OF LEMMA S.2: Define $d : \mathcal{Y} \times \mathcal{Y} \to \mathbf{R}_+$ by

$$d(y, y') := \sup_{\substack{v^+,v^-:\Delta(\Omega)\to\mathbf{R} \\ \text{continuous convex} \\ \text{s.t. } |v^+-v^-|\leq 1}} \left| \int_{\Delta(\Omega)} \left(v^+ - v^-\right) \mathrm{d}(y - y') \right|.$$

(d is in fact a metric on $\mathcal{Y}$.) Let $(y_n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{C}$ that converges to some $y \in \mathcal{C}$ in the relative topology on $\mathcal{C}$ inherited from the order topology on $\mathcal{Y}$; we will show that $d(y_n, y)$ vanishes as $n \to \infty$.

Let $B_\varepsilon := \{y' \in \mathcal{Y} : d(y, y') < \varepsilon\}$ denote the open d-ball of radius $\varepsilon > 0$ around y. Call $I \subseteq \mathcal{Y}$ an *open order interval* iff either (1) $I = \{y' \in \mathcal{Y} : y' < y^+\}$ for some $y^+ \in \mathcal{Y}$, or (2) $I = \{y' \in \mathcal{Y} : y^- < y'\}$ for some $y^- \in \mathcal{Y}$, or (3) $I = \{y' \in \mathcal{Y} : y^- < y' < y^+\}$ for some $y^- < y^+$ in $\mathcal{Y}$. Open order intervals are obviously open in the order topology on $\mathcal{Y}$.

It suffices to show that for every $\varepsilon > 0$, there is an open order interval $I_\varepsilon \subseteq \mathcal{Y}$ such that $y \in I_\varepsilon \subseteq B_\varepsilon$. For then given any $\varepsilon > 0$, we know that y_n lies in $I_\varepsilon \cap \mathcal{C} \subseteq B_\varepsilon$ for all sufficiently large $n \in \mathbf{N}$ because (in the relative topology on $\mathcal{C}$) $I_\varepsilon \cap \mathcal{C}$ is an open set containing y and $y_n \to y$. And this clearly implies that $d(y_n, y)$ vanishes as $n \to \infty$.

So fix an $\varepsilon > 0$; we will construct an open order interval $I \subseteq \mathcal{Y}$ such that $y \in I \subseteq B_\varepsilon$. There are three cases.

Case 1: $y' < y$ *for no* $y' \in \mathcal{Y}$. Let $y^{++} \in \mathcal{Y}$ be such that $y < y^{++}$. Define

$$y^+ := (1 - \varepsilon/2)y + (\varepsilon/2)y^{++} \in \mathcal{Y} \quad \text{and} \quad I := \left\{y' \in \mathcal{Y} : y' < y^+\right\}.$$

We have $y < y^+$, and thus $y \in I$ since

$$\int_{\Delta(\Omega)} v \, \mathrm{d}\left(y^+ - y\right) = \frac{\varepsilon}{2} \int_{\Delta(\Omega)} v \, \mathrm{d}\left(y^{++} - y\right)$$

is weakly (strictly) positive for every (some) continuous and convex $v : \Delta(\Omega) \to \mathbf{R}$ by $y < y^{++}$. To establish that $I \subseteq B_\varepsilon$, it suffices to show that $d(y, y^+) < \varepsilon$, and this holds because

$$d\left(y, y^+\right) = \frac{\varepsilon}{2} \sup_{\substack{v^+,v^-:\Delta(\Omega)\to\mathbf{R} \\ \text{continuous convex} \\ \text{s.t. } |v^+-v^-|\leq 1}} \left| \int_{\Delta(\Omega)} \left(v^+ - v^-\right) \mathrm{d}\left(y - y'\right) \right| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Case 2: *$y < y'$ for no $y' \in \mathcal{Y}$*. This case is analogous to the first: choose a $y^{--} \in \mathcal{Y}$ such that $y^{--} < y$, and let

$$y^- := (1-\varepsilon/2)y + (\varepsilon/2)y^{--} \quad \text{and} \quad I := \{y' \in \mathcal{Y} : y^- < y'\}.$$

The same arguments as in Case 1 yield $y \in I \subseteq B_\varepsilon$.

Case 3: *$y' < y < y''$ for some $y', y'' \in \mathcal{Y}$*. Define y^+ as in Case 1 and y^- as in Case 2, and let $I := \{y' \in \mathcal{Y} : y^- < y' < y^+\}$. We have $y \in I \subseteq B_\varepsilon$ by the same arguments as in Cases 1 and 2. *Q.E.D.*

LEMMA S.3: *For any continuous function $c : \Delta(\Omega) \to \mathbf{R}$ and any $\varepsilon > 0$, there are continuous convex $w^+, w^- : \Delta(\Omega) \to \mathbf{R}$ such that $w := w^+ - w^-$ satisfies $\sup_{\mu \in \Delta(\Omega)} |c(\mu) - w(\mu)| < \varepsilon$.*

PROOF: Write $\mathcal{W}$ for the space of functions $\Delta(\Omega) \to \mathbf{R}$ that can be written as the difference of continuous convex functions. Since the sum of convex functions is convex, $\mathcal{W}$ is a vector space. It is furthermore closed under pointwise multiplication (Hartman (1959, p. 708)), and thus an algebra. Clearly $\mathcal{W}$ contains the constant functions, and it separates points in the sense that for any distinct $\mu, \mu' \in \Delta(\Omega)$ there is a $w \in \mathcal{W}$ with $w(\mu) \neq w(\mu')$. It follows by the Stone–Weierstrass theorem[6] that $\mathcal{W}$ is dense in the space of continuous functions $\Delta(\Omega) \to \mathbf{R}$ when the latter has the sup metric. *Q.E.D.*

With the lemmata in hand, we can verify the continuity hypothesis.

PROPOSITION S.2: *Consider the setting in §4.4. Let $\mathcal{C} \subseteq \mathcal{Y}$ be a chain, and equip it with the relative topology inherited from the order topology on $\mathcal{Y}$. Then f is (jointly) continuous on $\mathcal{C} \times \mathbf{R} \times [0, 1]$.*

PROOF: Fix a chain $\mathcal{C} \subseteq \mathcal{Y}$, and equip it with the relative topology on $\mathcal{C}$ induced by the order topology on $\mathcal{Y}$. Define $h : \mathcal{C} \times [0, 1] \to \mathbf{R}$ by $h(y, t) := \int_{\Delta(\Omega)} V(\mu, t) y(\mathrm{d}\mu)$, so that $f(y, p, t) = g(h(y, t), p)$. Since g is jointly continuous, we need only show that h is jointly continuous.

It suffices to prove that $h(\cdot, 0)$ is continuous and that $\{h_2(\cdot, t)\}_{t \in [0,1]}$ is equicontinuous.[7] To see why, take (y, t) and (y', t') in $\mathcal{C} \times [0, 1]$ with (wlog) $t \leq t'$, and apply Lebesgue's fundamental theorem of calculus to obtain

$$\begin{aligned} |h(y', t') - h(y, t)| &= \left| h(y', 0) + \int_0^{t'} h_2(y', s)\,\mathrm{d}s - h(y, 0) - \int_0^t h_2(y, s)\,\mathrm{d}s \right| \\ &\leq |h(y', 0) - h(y, 0)| + \int_0^t |h_2(y', s) - h_2(y, s)|\,\mathrm{d}s + \int_t^{t'} |h_2(y', s)|\,\mathrm{d}s. \end{aligned}$$

Given continuity of $h(\cdot, 0)$ (equicontinuity of $\{h_2(\cdot, s)\}_{s \in [0,1]}$), the first term (second term) can be made arbitrarily small by taking y and y' sufficiently close (formally, choosing y' in

[6]See, for example, Folland (1999, Theorem 4.45).

[7]A detail: equicontinuity is a property of functions on a *uniformisable* topological space. To see that $\mathcal{C}$ is uniformisable, we need only convince ourselves that the relative topology on $\mathcal{C}$ inherited from the order topology on $\mathcal{Y}$ is completely regular. This topology is obviously finer than the order topology on $\mathcal{C}$, so it suffices to show that the latter is completely regular. And that is (a consequence of) a standard result; see, for example, Cater (2006).

a neighborhood of y that is small in the sense of set inclusion). By boundedness of h_2, the third term can similarly be made small by choosing t and t' close.

So, take a sequence $(y_n)_{n\in\mathbf{N}}$ in $\mathcal{C}$ converging to some $y \in \mathcal{C}$; we must show that

$$\left|h(y_n, 0) - h(y, 0)\right| \quad \text{and} \quad \sup_{t\in[0,1]}\left|h_2(y_n, t) - h_2(y, t)\right|$$

both vanish as $n \to \infty$. The former is easy: since $V(\cdot, 0)$ is continuous (hence bounded) and convex, we have

$$\begin{aligned}\left|h(y_n, 0) - h(y, 0)\right| &= \left|\int_{\Delta(\Omega)} V(\cdot, 0)\,\mathrm{d}(y_n - y)\right| \\ &\le \Big(\sup_{\mu\in\Delta(\Omega)}\left|V(\mu, 0)\right|\Big) \times \sup_{\substack{v:\Delta(\Omega)\to[-1,1]\\ \text{continuous convex}}}\left|\int_{\Delta(\Omega)} v\,\mathrm{d}(y_n - y)\right|\end{aligned}$$

for every $n \in \mathbf{N}$, and the right-hand side vanishes as $n \to \infty$ by Corollary S.1.

For the latter, fix an $\varepsilon > 0$; we seek an $N \in \mathbf{N}$ such that

$$\left|h_2(y_n, t) - h_2(y, t)\right| < \varepsilon \quad \text{for all } t \in [0, 1] \text{ and } n \ge N.$$

For each $t \in [0, 1]$, since $V_2(\cdot, t)$ is continuous, Lemma S.3 permits us to choose continuous and convex functions $w_t^+, w_t^- : \Delta(\Omega) \to \mathbf{R}$ such that $w_t := w_t^+ - w_t^-$ is uniformly $\varepsilon/3$-close to $V_2(\cdot, t)$. Write K for the constant bounding V_2, and observe that $\{w_t\}_{t\in[0,1]}$ is uniformly bounded by $K' := K + \varepsilon/3$. By Lemma S.2, there is an $N \in \mathbf{N}$ such that

$$\sup_{\substack{v^+,v^-:\Delta(\Omega)\to\mathbf{R}\\ \text{continuous convex}\\ \text{s.t. } |v^+-v^-|\le 1}}\left|\int_{\Delta(\Omega)} \left(v^+ - v^-\right)\mathrm{d}(y_n - y)\right| < \varepsilon/3K' \quad \text{for all } n \ge N,$$

and thus

$$\sup_{t\in[0,1]}\left|\int_{\Delta(\Omega)} w_t\,\mathrm{d}(y_n - y)\right| \le K' \times \varepsilon/3K' = \varepsilon/3 \quad \text{for } n \ge N.$$

Hence for every $t \in [0, 1]$ and $n \ge N$, we have

$$\begin{aligned}\left|h_2(y_n, t) - h_2(y, t)\right| &= \left|\int_{\Delta(\Omega)} V_2(\cdot, t)\,\mathrm{d}(y_n - y)\right| \\ &\le \left|\int_{\Delta(\Omega)} w_t\,\mathrm{d}(y_n - y)\right| + \left|\int_{\Delta(\Omega)} \left[V_2(\cdot, t) - w_t\right]\mathrm{d}(y_n - y)\right| \\ &\le \left|\int_{\Delta(\Omega)} w_t\,\mathrm{d}(y_n - y)\right| + 2\sup_{\mu\in\Delta(\Omega)}\left|V_2(\mu, t) - w_t(\mu)\right| \\ &\le \varepsilon/3 + 2\varepsilon/3 = \varepsilon,\end{aligned}$$

as desired. *Q.E.D.*

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