SUPPLEMENT TO "TIME LOTTERIES AND STOCHASTIC IMPATIENCE"

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APPENDIX SA: LOCAL RISK ATTITUDES TOWARD TIME LOTTERIES

CONSIDER THE CASE of discrete time and infinite horizon ($T = \mathbb{N}$) and suppose that preferences are represented by

$$V(p) = \mathbb{E}_p\big[D(t)u(x)\big], \tag{SA.1}$$

where $u : X \to \mathbb{R}_+$ is continuous and strictly increasing, and $D : \mathbb{N} \to (0, 1]$ is a strictly decreasing discount function. Notice that (SA.1) generalizes EDU by allowing for nonexponential discounting.

A function D is *discretely convex* if it is convex for all points in its domain, that is,

$$\alpha D(t_1) + (1 - \alpha)D(t_2) \ge D\big(\alpha t_1 + (1 - \alpha)t_2\big)$$

for all $t_1, t_2 \in \mathbb{N}$ and $\alpha \in (0, 1)$ with $\alpha t_1 + (1 - \alpha)t_2 \in \mathbb{N}$. A function D is *discretely concave* if $-D$ is discretely convex.

The following proposition establishes the relationship between attitudes toward time lotteries and the convexity of the discount function.

PROPOSITION SA.1: *Preferences represented by* (SA.1) *are RSTL if and only if D is discretely convex. Moreover, they cannot be RATL.*

PROOF: First, we show that preferences are RSTL (RATL) if and only if D is discretely convex (concave). The value of $\delta_{(x,\bar{t})}$ is

$$V(\delta_{(x,\bar{t})}) = \sum_{\tau \neq \bar{t}} D(\tau)u(c) + D(\bar{t})u(c + x),$$

whereas the value of the time lottery $p = \langle p_x(t), t \rangle_{t \in \mathbb{N}}$ with $\sum_t p_x(t)t = \bar{t}$ is

$$V(p) = \sum_t p_x(t)\left[\sum_{\tau \neq t} D(\tau)u(c) + D(t)u(c + x)\right].$$

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Therefore,

$$V(p) \geq V(\delta_{(x,\bar{t})}) \quad \Longleftrightarrow \quad \left[\sum p_x(t)D(t) - D(\bar{t})\right]\left[u(c+x) - u(c)\right] \geq 0,$$

which, because u is strictly increasing, holds if and only if D is discretely convex.

Next, we show that D cannot be discretely concave. Suppose D is discretely concave, so that

$$D(t) \leq D(1) + (t-1)\left[D(2) - D(1)\right].$$

Taking $t \geq \frac{2D(1)-D(2)}{D(1)-D(2)}$ and using the fact that D is strictly decreasing, we obtain $D(t) < 0$, which contradicts the fact that the discount function is positive.   *Q.E.D.*

The second part of Proposition SA.1 states that discount functions cannot be discretely concave, implying that we cannot have RATL.

In light of Proposition SA.1, we ask whether discounted utility can satisfy a local version of RATL. We say that preferences are *locally risk averse toward time lotteries at time t* if a sure payment at t is preferred to a random payment occurring at either $t - 1$ or at $t + 1$ with equal probabilities, that is,

$$V(\delta_{(x,t)}) \geq V(\langle 0.5, (x, t-1); 0.5, (x, t+1)\rangle)$$

for all $x \in [w, b]$. Similarly, we say that preferences are *locally risk seeking at t* if the reverse inequalities hold.

Our next proposition shows that even this weaker version of RATL is inconsistent with preferences represented by (SA.1). Thus, even if we abandon (global) convexity, it would be of limited help.

PROPOSITION SA.2: *Suppose preferences are represented by* (SA.1). *The set of periods in which preferences are locally RATL is finite*.

PROOF: The sequence $\{D(t)\}$ is monotone and bounded. Thus, by the monotone convergence theorem, it converges to some number, say $\bar{d} \geq 0$. We need to show that the sequence $\{D(t+1) + D(t-1) - 2D(t)\}$ has no negative limit points:

$$\liminf_{t\to\infty} \left(D(t+1) + D(t-1) - 2D(t)\right) \geq 0.$$

Suppose this is not true. Then there exists $\epsilon > 0$ and a subsequence $\{D(t_k)\}$ such that

$$D(t_k + 1) + D(t_k - 1) - 2D(t_k) \leq -\epsilon$$

for all t_k. However, because $D(t_k)$ converges to $\bar{d}$, it follows that $D(t_k + 1) + D(t_k - 1) - 2D(t_k)$ converges to zero. Thus, there exists $\bar{t}_k$ such that for all $t > \bar{t}_k$,

$$-\frac{\epsilon}{2} \leq D(t_k + 1) + D(t_k - 1) - 2D(t_k) \leq \frac{\epsilon}{2},$$

which contradicts the previous inequality.   *Q.E.D.*

APPENDIX SB: PROOFS OF THE RESULTS IN APPENDIX B

SB.1. *Proof of Proposition 5*

First notice that, because preferences are dynamically consistent, there is no loss in taking $t = 3$. To simplify the expressions, it is convenient to let $\lambda \equiv (c + x)/c > 1$ denote the consumption with the prize as a proportion of consumption without it. Using the formula in the text, the utility of the safe lottery equals

$$V_0 = \left[(1 - \beta)c\right]^{\frac{1}{1-\rho}} \left[1 + \beta + \lambda^{1-\rho}\beta^2 + \frac{\beta^3}{1 - \beta}\right]^{\frac{1}{1-\rho}},$$

and the utility of the risky lottery is

$$V_0 = \left[(1-\beta)c\right]^{\frac{1}{1-\rho}} \left\{1 + \beta \left[\frac{\left(\lambda^{1-\rho} + \frac{\beta}{1 - \beta}\right)^{\frac{1-\alpha}{1-\rho}} + \left(1 + \beta + \lambda^{1-\rho}\beta^2 + \frac{\beta^3}{1 - \beta}\right)^{\frac{1-\alpha}{1-\rho}}}{2}\right]^{\frac{1-\rho}{1-\alpha}}\right\}^{\frac{1}{1-\rho}}.$$

Comparing these two expressions, we find that preferences are locally RSTL at t if and only if the following inequality holds:

$$\left\{1 + \beta \left[\frac{\left(\lambda^{1-\rho} + \frac{\beta}{1 - \beta}\right)^{\frac{1-\alpha}{1-\rho}} + \left(1 + \beta + \lambda^{1-\rho}\beta^2 + \frac{\beta^3}{1 - \beta}\right)^{\frac{1-\alpha}{1-\rho}}}{2}\right]^{\frac{1-\rho}{1-\alpha}}\right\}^{\frac{1}{1-\rho}}$$

$$> \left(1 + \beta + \lambda^{1-\rho}\beta^2 + \frac{\beta^3}{1 - \beta}\right)^{\frac{1}{1-\rho}}. \tag{SB.1}$$

To simplify notation, in what follows we will repeatedly use the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) \equiv x^{\frac{1-\alpha}{1-\rho}}$. We will also repeatedly use the following fact:

$$\frac{\lambda^{1-\rho} + \frac{\beta}{1 - \beta} + 1 + \beta + \lambda^{1-\rho}\beta^2 + \frac{\beta^3}{1 - \beta}}{2} \left\{\begin{matrix} > \\ < \end{matrix}\right\} 1 + \lambda^{1-\rho}\beta + \frac{\beta^2}{1 - \beta}$$

$$\iff \quad \rho \left\{\begin{matrix} < \\ > \end{matrix}\right\} 1. \tag{SB.2}$$

We first verify that (SB.1) always holds when $\alpha \le 1$.

LEMMA SB.1: *Let $\alpha \le 1$. Then preferences are RSTL.*

PROOF: There are three cases: (i) $\alpha \le \rho \le 1$, (ii) $\rho < \alpha \le 1$, and (iii) $\alpha \le 1 < \rho$.

Case (i): $\alpha \leq \rho \leq 1$. Since $1 - \rho < 0$, inequality (SB.1) can be written as

$$\frac{\left(\lambda^{1-\rho} + \dfrac{\beta}{1-\beta}\right)^{\frac{1-\alpha}{1-\rho}} + \left(1 + \beta + \lambda^{1-\rho}\beta^2 + \dfrac{\beta^3}{1-\beta}\right)^{\frac{1-\alpha}{1-\rho}}}{2} > \left(1 + \lambda^{1-\rho}\beta + \dfrac{\beta^2}{1-\beta}\right)^{\frac{1-\alpha}{1-\rho}}.$$

Because $\rho < 1$, inequality (SB.2), we also know that

$$\frac{\lambda^{1-\rho} + \dfrac{\beta}{1-\beta} + 1 + \beta + \lambda^{1-\rho}\beta^2 + \dfrac{\beta^3}{1-\beta}}{2} > 1 + \lambda^{1-\rho}\beta + \dfrac{\beta^2}{1-\beta}.$$

The result then follows from Jensen's inequality, since $f(x)$ is increasing and convex when $\alpha, \rho \leq 1$.

Case (ii): $\rho < \alpha \leq 1$. To simplify notation, define $\eta \equiv \frac{1-\alpha}{1-\rho} \in (0, 1)$, where $\eta > 0$ since both α and ρ are lower than 1, and $\eta < 1$ since $\alpha > \rho$. We can rewrite inequality (SB.1) substituting α for η as

$$\frac{\left(\lambda^{1-\rho} + \dfrac{\beta}{1-\beta}\right)^{\eta} + \left(1 + \beta + \lambda^{1-\rho}\beta^2 + \dfrac{\beta^3}{1-\beta}\right)^{\eta}}{2} > \left(1 + \lambda^{1-\rho}\beta + \dfrac{\beta^2}{1-\beta}\right)^{\eta}.$$

Rearrange this condition as

$$T(\eta) \equiv \left(\frac{1}{\dfrac{1}{\lambda^{1-\rho} + \dfrac{\beta}{1-\beta}} + \beta}\right)^{\eta} + \left(\frac{1}{1 + \lambda^{1-\rho}\beta + \dfrac{\beta^2}{1-\beta}} + \beta\right)^{\eta} > 2.$$

It is straightforward to verify that T is a convex function of η. Recall that $\eta \in (0, 1)$. Evaluating at $\eta = 0$, we obtain $T(0) = 2$.

Since T is a convex function of η, it suffices to show that its derivative with respect to η at zero is positive. We claim that this is true. To see this, notice that

$$T'(0) = \ln\left(\frac{\dfrac{1}{1 + \lambda^{1-\rho}\beta + \dfrac{\beta^2}{1-\beta}} + \beta}{\dfrac{1}{\lambda^{1-\rho} + \dfrac{\beta}{1-\beta}} + \beta}\right), \tag{SB.3}$$

which can be shown to be strictly positive for any $\rho < 1$. Thus, $T(\eta) > 2$ for all $\eta \in (0, 1]$, establishing RSTL.

Case (iii): $\alpha \leq 1 < \rho$. Inequality (SB.1) can be simplified as

$$\left[\frac{\left(\lambda^{1-\rho} + \frac{\beta}{1-\beta}\right)^{\frac{1-\alpha}{1-\rho}} + \left(1 + \beta + \lambda^{1-\rho}\beta^2 + \frac{\beta^3}{1-\beta}\right)^{\frac{1-\alpha}{1-\rho}}}{2}\right]^{\frac{1-\rho}{1-\alpha}} < 1 + \lambda^{1-\rho}\beta + \frac{\beta^2}{1-\beta}.$$

Since $\frac{1-\alpha}{1-\rho} < 0$, this holds if

$$\frac{\left(\lambda^{1-\rho} + \frac{\beta}{1-\beta}\right)^{\frac{1-\alpha}{1-\rho}} + \left(1 + \beta + \lambda^{1-\rho}\beta^2 + \frac{\beta^3}{1-\beta}\right)^{\frac{1-\alpha}{1-\rho}}}{2} > \left(1 + \lambda^{1-\rho}\beta + \frac{\beta^2}{1-\beta}\right)^{\frac{1-\alpha}{1-\rho}}. \quad \text{(SB.4)}$$

Notice that $f(x) = x^{\frac{1-\alpha}{1-\rho}}$ is convex since

$$f''(x) = \left(\frac{1-\alpha}{1-\rho}\right)\left(\frac{1-\alpha}{1-\rho} - 1\right)x^{\frac{1-\alpha}{1-\rho}-2} > 0,$$

where we used $\frac{1-\alpha}{1-\rho} < 0$ and $\frac{1-\alpha}{1-\rho} - 1 < 0$. Thus, by Jensen's inequality,

$$\frac{\left(\lambda^{1-\rho} + \frac{\beta}{1-\beta}\right)^{\frac{1-\alpha}{1-\rho}} + \left(1 + \beta + \lambda^{1-\rho}\beta^2 + \frac{\beta^3}{1-\beta}\right)^{\frac{1-\alpha}{1-\rho}}}{2}$$

$$> \left(\frac{\lambda^{1-\rho} + \frac{\beta}{1-\beta} + 1 + \beta + \lambda^{1-\rho}\beta^2 + \frac{\beta^3}{1-\beta}}{2}\right)^{\frac{1-\alpha}{1-\rho}}. \quad \text{(SB.5)}$$

From condition (SB.2), we have

$$\frac{\lambda^{1-\rho} + \frac{\beta}{1-\beta} + 1 + \beta + \lambda^{1-\rho}\beta^2 + \frac{\beta^3}{1-\beta}}{2} < 1 + \lambda^{1-\rho}\beta + \frac{\beta^2}{1-\beta}.$$

Raising to the power of $\frac{1-\alpha}{1-\rho} < 0$, gives

$$\left(\frac{\lambda^{1-\rho} + \frac{\beta}{1-\beta} + 1 + \beta + \lambda^{1-\rho}\beta^2 + \frac{\beta^3}{1-\beta}}{2}\right)^{\frac{1-\alpha}{1-\rho}} > \left(1 + \lambda^{1-\rho}\beta + \frac{\beta^2}{1-\beta}\right)^{\frac{1-\alpha}{1-\rho}}.$$

Substituting in (SB.5), we obtain

$$\frac{\left(\lambda^{1-\rho} + \frac{\beta}{1-\beta}\right)^{\frac{1-\alpha}{1-\rho}} + \left(1 + \beta + \lambda^{1-\rho}\beta^2 + \frac{\beta^3}{1-\beta}\right)^{\frac{1-\alpha}{1-\rho}}}{2} > \left(1 + \lambda^{1-\rho}\beta + \frac{\beta^2}{1-\beta}\right)^{\frac{1-\alpha}{1-\rho}},$$

which is precisely the condition for RSTL (SB.4). *Q.E.D.*

LEMMA SB.2: *Let $\alpha \le \rho$. Then preferences are RSTL.*

PROOF: By Lemma SB.1, the result is immediate when $\alpha \le 1$. Therefore, let $\alpha > 1$ (which, by the statement of the lemma, requires $\rho > 1$).

Rearranging inequality (SB.1), we obtain the following condition for RSTL:

$$\frac{\left(\lambda^{1-\rho} + \frac{\beta}{1-\beta}\right)^{\frac{1-\alpha}{1-\rho}} + \left(1 + \beta + \lambda^{1-\rho}\beta^2 + \frac{\beta^3}{1-\beta}\right)^{\frac{1-\alpha}{1-\rho}}}{2} < \left(1 + \lambda^{1-\rho}\beta + \frac{\beta^2}{1-\beta}\right)^{\frac{1-\alpha}{1-\rho}}. \quad \text{(SB.6)}$$

Moreover, from condition (SB.2), we have

$$\frac{\lambda^{1-\rho} + \frac{\beta}{1-\beta} + 1 + \beta + \lambda^{1-\rho}\beta^2 + \frac{\beta^3}{1-\beta}}{2} < 1 + \lambda^{1-\rho}\beta + \frac{\beta^2}{1-\beta}.$$

Notice that $f(x)$ is increasing when $\alpha, \rho \ge 1$ and it is concave when $\rho \ge \alpha$. Then condition (SB.6) follows by Jensen's inequality. *Q.E.D.*

We are now ready to prove the main result.

First, suppose $\rho < 1$. Let $\gamma \equiv -\frac{1-\alpha}{1-\rho} \in (0, +\infty)$ so we can rewrite inequality (SB.1) in terms of γ and ρ as

$$\frac{1}{\left(\lambda^{1-\rho} + \frac{\beta}{1-\beta}\right)^{\gamma}} + \frac{1}{\left(1 + \beta + \lambda^{1-\rho}\beta^2 + \frac{\beta^3}{1-\beta}\right)^{\gamma}} < \frac{2}{\left(1 + \lambda^{1-\rho}\beta + \frac{\beta^2}{1-\beta}\right)^{\gamma}},$$

which can be simplified as

$$G(\gamma) \equiv \left(\frac{1}{\lambda^{1-\rho} + \frac{\beta}{1-\beta}} + \beta\right)^{\gamma} + \left(\frac{1}{\frac{1}{1 + \lambda^{1-\rho}\beta + \frac{\beta^2}{1-\beta}} + \beta}\right)^{\gamma} < 2.$$

The first term in the expression on the left is convex and decreasing in γ, because the term inside the first brackets is smaller than 1:

$$\rho \le 1 \quad \Longrightarrow \quad \frac{1}{\lambda^{1-\rho} + \frac{\beta}{1-\beta}} + \beta \le 1.$$

The second term is convex and increasing in γ because the term inside the second brackets is greater than 1:

$$\rho \le 1 \quad \Longrightarrow \quad \frac{1}{\frac{1}{1 + \lambda^{1-\rho}\beta + \frac{\beta^2}{1-\beta}} + \beta} \ge 1.$$

Since the sum of convex functions is convex, it follows that G is a convex function of γ.

Evaluating γ at the extremes, we obtain $G(0) = 2$ and $\lim_{\gamma \to \infty} G(\gamma) = +\infty > 2$. Moreover, note that

$$G'(0) = \ln \left(\frac{\dfrac{1}{\lambda^{1-\rho} + \dfrac{\beta}{1-\beta}} + \beta}{\dfrac{1}{1 + \lambda^{1-\rho}\beta + \dfrac{\beta^2}{1-\beta}} + \beta} \right),$$

which, following some algebraic manipulations, can be shown to be strictly negative.

Thus, there exists $\bar{\gamma} > 0$ such that $G(\gamma) > 2$ (RATL) if and only if $\gamma > \bar{\gamma}$. But, since $\gamma = -\frac{1-\alpha}{1-\rho}$ (so that γ is strictly increasing in α), then there exists a finite $\bar{\alpha}_{\rho,\beta,x,c} > \max\{1, \rho\}$ such that we have RATL if $\alpha > \bar{\alpha}_{\rho,\beta,x,c}$ and RSTL if $\alpha < \bar{\alpha}_{\rho,\beta,x,c}$. This concludes the proof for $\rho < 1$.

Now suppose that $\alpha > \rho \geq 1$ (the result is trivial if $\alpha \leq \rho$ from Lemma SB.2). Let $\eta \equiv \frac{1-\alpha}{1-\rho} \geq 1$. Then we have RSTL if and only if

$$\frac{\left(\lambda^{1-\rho} + \dfrac{\beta}{1-\beta}\right)^\eta + \left(1 + \beta + \lambda^{1-\rho}\beta^2 + \dfrac{\beta^3}{1-\beta}\right)^\eta}{2} < \left(1 + \lambda^{1-\rho}\beta + \dfrac{\beta^2}{1-\beta}\right)^\eta.$$

Rearrange this condition as

$$H(\eta) \equiv \left(\frac{1}{\dfrac{1}{\lambda^{1-\rho} + \dfrac{\beta}{1-\beta}} + \beta} \right)^\eta + \left(\frac{1}{1 + \lambda^{1-\rho}\beta + \dfrac{\beta^2}{1-\beta}} + \beta \right)^\eta < 2. \qquad \text{(SB.7)}$$

As before, it can be shown that $H(\eta)$ is convex. Notice that $\lim_{\eta \to \infty} H(\eta) = +\infty > 2$. Moreover, $H(1) < 2$, since

$$\lambda^{1-\rho} < 1 \iff \frac{1}{\dfrac{1}{\lambda^{1-\rho} + \dfrac{\beta}{1-\beta}} + \beta} + \frac{1}{1 + \lambda^{1-\rho}\beta + \dfrac{\beta^2}{1-\beta}} + \beta < 2.$$

Thus, there exists $\bar{\eta} > 0$ such that $H(\eta) > 2$ (RATL) if and only if $\eta > \bar{\eta}$. The result then follows from the fact that η is increasing in α.

To conclude the proof, it remains to be shown that $\lim_{x \searrow 0} \bar{\alpha}_{\rho,\beta,x,c} = +\infty$. Both sides of (SB.1) are equal to $\left(\frac{1}{1-\beta}\right)^{\frac{1}{1-\rho}}$ when $\lambda = 1$. The derivative of the expression on the right (utility of the safe lottery) with respect to λ at $\lambda = 1$ is

$$\left(\frac{1}{1-\beta}\right)^{\frac{\rho}{1-\rho}} \beta^2. \qquad \text{(SB.8)}$$

The derivative of the expression on the left (utility of the risky lottery) with respect to λ at $\lambda = 1$ is

$$\beta \frac{1+\beta^2}{2} \left(\frac{1}{1-\beta} \right)^{\frac{\rho}{1-\rho}}. \tag{SB.9}$$

With some algebraic manipulations, it can be shown that for any $\beta \in (0, 1)$, the term in (SB.8) is lower than the one in (SB.9). $Q.E.D.$

SB.2. *Proof of Proposition 6*

The proof of the proposition will be presented in a series of lemmas. As in the proof of Proposition 5, due to multiplicative separability, it suffices to consider lotteries with background consumption $c = 1$. We start by obtaining a formula for the value of the kind of lotteries considered throughout this proof.

LEMMA SB.3: *In EZ, the value of lottery* $p \equiv \frac{1}{2} \times [1, (x, 2)] + \frac{1}{2} \times [1, (y, t)]$ *is*

$$U(p) = \left\{ (1-\beta) \right.$$
$$\left. + \beta \left[\frac{\left[(1-\beta)(x+1)^{1-\rho} + \beta \right]^{\frac{1-\alpha}{1-\rho}} + \left\{ 1 + (1-\beta) \cdot \beta^{t-2}[(y+1)^{1-\rho} - 1] \right\}^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}.$$

PROOF: For notational simplicity, let $z_1 \equiv 1 + x$ and $z_2 \equiv 1 + y$. We start by calculating the continuation value in period t, which is a constant stream of one in both lotteries: $V_t = 1$. Proceeding backwards, there are two possible states of the world, each with 50% chance: one in which the early prize is paid (in period 2), and one in which the late prize is paid (at $t > 2$).

When the early prize is paid, the individual still gets a constant stream of one for any $t > 2$, so that $V_3 = 1$. Plugging back in the utility at period 2, gives

$$V_2 = \left[(1-\beta) z_1^{1-\rho} + \beta \right]^{\frac{1}{1-\rho}}.$$

When the late prize is paid, we have

$$V_t = \left[(1-\beta) z_2^{1-\rho} + \beta \right]^{\frac{1}{1-\rho}}.$$

We claim that, for any $n = \{1, \ldots, t-2\}$,

$$V_{t-n} = \left[1 - (1-\beta)\beta^n \left(1 - z_2^{1-\rho} \right) \right]^{\frac{1}{1-\rho}},$$

so that, in particular,

$$V_2 = \left[1 - (1-\beta) \cdot \beta^{t-2} \left(1 - z_2^{1-\rho} \right) \right]^{\frac{1}{1-\rho}}.$$

To see this, we proceed inductively. At $t-1$, we have

$$V_{t-1} = \left\{ (1-\beta) + \beta \left[E_t\left(V_t^{1-\alpha}\right) \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

$$= \left\{ (1-\beta) + \beta \left\{ \left[(1-\beta)z_2^{1-\rho} + \beta \right]^{\frac{1-\alpha}{1-\rho}} \right\}^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

$$= \left[(1-\beta)\left(1 + \beta z_2^{1-\rho}\right) + \beta^2 \right]^{\frac{1}{1-\rho}}.$$

Moving back another period, gives

$$V_{t-2} = \left\{ 1 - \beta + \beta \left[(1-\beta)\left(1 + \beta z_2^{1-\rho}\right) + \beta^2 \right] \right\}^{\frac{1}{1-\rho}}$$

$$= \left\{ (1-\beta)\left(1 + \beta + \beta^2 z_2^{1-\rho}\right) + \beta^3 \right\}^{\frac{1}{1-\rho}}.$$

To obtain the induction result, suppose that

$$V_{t-n} = \left\{ (1-\beta)\left(1 + \beta + \cdots + \beta^{n-1} + \beta^n z_2^{1-\rho}\right) + \beta^{n+1} \right\}^{\frac{1}{1-\rho}}.$$

Then

$$V_{t-(n+1)} = \left[(1-\beta) + \beta \left(V_{t-n}^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}} \right]^{\frac{1}{1-\rho}}$$

$$= \left[(1-\beta) + \beta \left[(1-\beta)\left(1 + \beta + \cdots + \beta^n z_2^{1-\rho}\right) + \beta^{n+1} \right] \right]^{\frac{1}{1-\rho}}$$

$$= \left[(1-\beta)\left(1 + \beta + \beta^2 + \cdots + \beta^n + \beta^{n+1} z_2^{1-\rho}\right) + \beta^{n+2} \right]^{\frac{1}{1-\rho}},$$

establishing the induction formula. Using the formula for the geometric progression gives

$$V_{t-n} = \left[(1-\beta)\left(\frac{1-\beta^n}{1-\beta} + \beta^n z_2^{1-\rho} \right) + \beta^{n+1} \right]^{\frac{1}{1-\rho}}$$

$$= \left[1 - \beta^n + \beta^{n+1} + (1-\beta)\left(\beta^n z_2^{1-\rho} \right) \right]^{\frac{1}{1-\rho}}$$

$$= \left[1 + (1-\beta)\beta^n\left(z_2^{1-\rho} - 1 \right) \right]^{\frac{1}{1-\rho}},$$

which establishes our claim.

Since each of the two states happens with probability $\frac{1}{2}$, we have

$$E_1\left[V_2^{1-\alpha}\right] = \frac{\left[(1-\beta)z_1^{1-\rho} + \beta \right]^{\frac{1-\alpha}{1-\rho}} + \left[1 + (1-\beta)\cdot\beta^{t-2}\left(z_2^{1-\rho} - 1\right) \right]^{\frac{1-\alpha}{1-\rho}}}{2}.$$

Therefore, the value from the lottery equals

$$V_1 = \left\{ (1-\beta) + \beta \left\{ \frac{\left[(1-\beta)z_1^{1-\rho} + \beta \right]^{\frac{1-\alpha}{1-\rho}} + \left[1 + (1-\beta)\cdot\beta^{t-2}\left(z_2^{1-\rho} - 1\right) \right]^{\frac{1-\alpha}{1-\rho}}}{2} \right\}^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}},$$

concluding the proof. *Q.E.D.*

Next, we obtain necessary conditions for EZ preferences not to be RSTL. By stationarity, it suffices to compare lotteries in which the early prize is paid at $t = 2$. That is, preferences are RSTL if and only if, for all $x > 0$ and all $\zeta \in \{1, 2, 3, \ldots\}$,

$$\frac{1}{2} \times [1, (x, 2)] + \frac{1}{2} \times [1, (x, 2 + 2\zeta)] \succeq [1, (x, 2 + \zeta)].$$

As in the proof of Proposition 5, let $\lambda \equiv \frac{c+x}{c} = 1 + x > 1$. The value of the safe time lottery is

$$V^S \equiv (1 - \beta)^{\frac{1}{1-\rho}} \cdot \left[\beta^{\zeta+1}(\lambda^{1-\rho} - 1) + \frac{1}{1-\beta} \right]^{\frac{1}{1-\rho}}.$$

Using the formula from Lemma SB.3, we obtain the value of the risky time lottery:

$$V^R = \left\{ (1 - \beta) + \beta \left[\frac{[(1 - \beta)\lambda^{1-\rho} + \beta]^{\frac{1-\alpha}{1-\rho}} + [1 + (1 - \beta) \cdot \beta^{t_2-2}(\lambda^{1-\rho} - 1)]^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

$$= (1 - \beta)^{\frac{1}{1-\rho}} \left\{ 1 + \beta \left[\frac{\left(\frac{1}{1-\beta} + \lambda^{1-\rho} - 1 \right)^{\frac{1-\alpha}{1-\rho}} + \left[\frac{1}{1-\beta} + \beta^{2\zeta}(\lambda^{1-\rho} - 1) \right]^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}.$$

Therefore, preferences are RSTL if and only if

$$\left\{ 1 + \beta \left[\frac{\left(\frac{1}{1-\beta} + \lambda^{1-\rho} - 1 \right)^{\frac{1-\alpha}{1-\rho}} + \left[\frac{1}{1-\beta} + \beta^{2\zeta}(\lambda^{1-\rho} - 1) \right]^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

$$\geq \left[\frac{1}{1-\beta} + \beta^{\zeta+1}(\lambda^{1-\rho} - 1) \right]^{\frac{1}{1-\rho}} \tag{SB.10}$$

for all $\lambda > 1$ and all $\zeta \geq 1$.

To simplify notation, let $f(x) \equiv x^{\frac{1-\alpha}{1-\rho}}$. In the proofs below, we will repeatedly use the following fact:

$$\frac{\left(\frac{1}{1-\beta} + \lambda^{1-\rho} - 1 \right) + \left[\frac{1}{1-\beta} + \beta^{2\zeta}(\lambda^{1-\rho} - 1) \right]}{2} \left\{ \begin{matrix} > \\ < \end{matrix} \right\} \frac{1}{1-\beta} + \beta^{\zeta}(\lambda^{1-\rho} - 1)$$

$$\iff \quad \rho \left\{ \begin{matrix} < \\ > \end{matrix} \right\} 1. \tag{SB.11}$$

We first verify that (SB.10) always holds when $\alpha < 1$.

LEMMA SB.4: *Let $\alpha < 1$. Then preferences are RSTL.*

PROOF: There are three cases: (i) $\alpha \le \rho < 1$, (ii) $\rho < \alpha < 1$, and (iii) $\alpha < 1 < \rho$.
Case (i): $\alpha \le \rho < 1$. Since $\rho < 1$, equation (SB.10) can be written as

$$1 + \beta \left[\frac{\left(\frac{1}{1-\beta} + \lambda^{1-\rho} - 1 \right)^{\frac{1-\alpha}{1-\rho}} + \left[\frac{1}{1-\beta} - \beta^{2\zeta}(\lambda^{1-\rho} - 1) \right]^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}}$$

$$\ge \frac{1}{1-\beta} + \beta^{\zeta+1}(\lambda^{1-\rho} - 1).$$

Algebraic manipulations and the fact that $\frac{1-\rho}{1-\alpha} > 0$ allow us to rewrite this condition as

$$\frac{f\left(\frac{1}{1-\beta} + \lambda^{1-\rho} - 1 \right) + f\left(\frac{1}{1-\beta} - \beta^{2\zeta}(\lambda^{1-\rho} - 1) \right)}{2} \ge f\left(\frac{1}{1-\beta} + \beta^{\zeta}(\lambda^{1-\rho} - 1) \right).$$

Since f is increasing and convex when $\alpha < 1$ and $\rho < 1$, (SB.11) implies that this inequality is true.
Case (ii): $\rho < \alpha < 1$. Use $\rho < 1$ to rewrite equation (SB.10) as

$$\frac{\left(\frac{1}{1-\beta} + \lambda^{1-\rho} - 1 \right)^{\frac{1-\alpha}{1-\rho}} + \left[\frac{1}{1-\beta} + \beta^{2\zeta}(\lambda^{1-\rho} - 1) \right]^{\frac{1-\alpha}{1-\rho}}}{2} \ge \left[\frac{1}{1-\beta} + \beta^{\zeta}(\lambda^{1-\rho} - 1) \right]^{\frac{1-\alpha}{1-\rho}}.$$

Rearrange this condition as

$$\left[\frac{\frac{1}{1-\beta} + \lambda^{1-\rho} - 1}{\frac{1}{1-\beta} + \beta^{\zeta}(\lambda^{1-\rho} - 1)} \right]^{\frac{1-\alpha}{1-\rho}} + \left[\frac{\frac{1}{1-\beta} + \beta^{2\zeta}(\lambda^{1-\rho} - 1)}{\frac{1}{1-\beta} + \beta^{\zeta}(\lambda^{1-\rho} - 1)} \right]^{\frac{1-\alpha}{1-\rho}} \ge 2.$$

To simplify notation, denote $\gamma \equiv \frac{1-\alpha}{1-\rho} \in (0, 1)$, where $\gamma > 0$ since $\alpha < 1$ and $\rho < 1$ and $\gamma < 1$ follows from $\rho < \alpha$. After some algebraic manipulations, this inequality can be written as

$$M(\gamma) \equiv \left[\frac{1}{\frac{1 - \beta^{\zeta}}{1 + (1-\beta)(\lambda^{1-\rho} - 1)} + \beta^{\zeta}} \right]^{\gamma} + \left[\frac{1}{\frac{1}{1 - \beta^{\zeta}} + \beta^{\zeta}\left(\frac{1-\beta}{1 - \beta^{\zeta}} \right)(\lambda^{1-\rho} - 1)} + \beta^{\zeta} \right]^{\gamma} \ge 2.$$

It is straightforward to show that the expression on the left, $M(\gamma)$, is a convex function. Recall that $\gamma \in (0, 1)$. Note that $M(0) = 2$. Since M is convex, it suffices to show that its

derivative with respect to γ at zero is positive. But note that

$$M'(0) = \ln \left[\frac{\dfrac{1}{\dfrac{1}{1-\beta^\zeta} + \beta^\zeta \cdot \left(\dfrac{1-\beta}{1-\beta^\zeta}\right)(\lambda^{1-\rho}-1)} + \beta^\zeta}{\dfrac{1-\beta^\zeta}{1+(1-\beta)(\lambda^{1-\rho}-1)} + \beta^\zeta} \right],$$

which, after some algebraic manipulations, can be shown to be strictly positive for any $\rho < 1$. Thus, $M(\gamma) > 2$ for all $\gamma \in (0, 1]$, establishing that (SB.10) holds.

Case (iii): $\alpha < 1 < \rho$. Since $\rho > 1$, equation (SB.10) becomes

$$\left[\frac{\left(\dfrac{1}{1-\beta} + \lambda^{1-\rho} - 1\right)^{\frac{1-\alpha}{1-\rho}} + \left[\dfrac{1}{1-\beta} + \beta^{2\zeta}(\lambda^{1-\rho}-1)\right]^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \leq \frac{1}{1-\beta} + \beta^\zeta(\lambda^{1-\rho}-1),$$

and, because $\frac{1-\rho}{1-\alpha} < 0$, this inequality holds if and only if

$$\frac{f\left(\dfrac{1}{1-\beta} + \lambda^{1-\rho} - 1\right) + f\left(\dfrac{1}{1-\beta} + \beta^{2\zeta}(\lambda^{1-\rho}-1)\right)}{2}$$
$$\geq f\left(\frac{1}{1-\beta} + \beta^\zeta(\lambda^{1-\rho}-1)\right). \tag{SB.12}$$

Note that $\alpha < 1 < \rho$ implies that f is decreasing and convex. Since f is convex, Jensen's inequality implies

$$\frac{f\left(\dfrac{1}{1-\beta} + \lambda^{1-\rho} - 1\right) + f\left(\dfrac{1}{1-\beta} + \beta^{2\zeta}(\lambda^{1-\rho}-1)\right)}{2}$$
$$> f\left(\frac{\left(\dfrac{1}{1-\beta} + \lambda^{1-\rho} - 1\right) + \left[\dfrac{1}{1-\beta} + \beta^{2\zeta}(\lambda^{1-\rho}-1)\right]}{2} \right).$$

Then, by (SB.11) and the fact that f is decreasing, it follows that

$$\frac{f\left(\dfrac{1}{1-\beta} + \lambda^{1-\rho} - 1\right) + f\left(\dfrac{1}{1-\beta} + \beta^{2\zeta}(\lambda^{1-\rho}-1)\right)}{2} > f\left(\frac{1}{1-\beta} + \beta^\zeta(\lambda^{1-\rho}-1)\right),$$

showing that inequality (SB.12) holds.   *Q.E.D.*

LEMMA SB.5: *Let $\alpha \leq \rho$. Then preferences are RSTL.*

PROOF: By the previous lemma, the result is immediate when $\alpha < 1$. Therefore, let $\alpha > 1$ (which, by the statement of the lemma, requires $\rho > 1$). Using $\rho \geq \alpha > 1$, we can rewrite condition (SB.10) as

$$\frac{f\left(\frac{1}{1-\beta} + \lambda^{1-\rho} - 1\right) + f\left(\frac{1}{1-\beta} + \beta^{2\zeta}(\lambda^{1-\rho} - 1)\right)}{2} \leq f\left(\frac{1}{1-\beta} + \beta^{\zeta}(\lambda^{1-\rho} - 1)\right),$$

which follows from condition (SB.11) and from the fact that f is increasing and concave when $\rho > \alpha > 1$. *Q.E.D.*

Therefore, when either $\alpha < 1$ or $\alpha \leq \rho$, preferences must be RSTL. To show that a violation of RSTL implies a violation of SI, we must consider the remaining cases (where violations of RSTL are possible): $\alpha > 1 > \rho$ and $\alpha \geq \rho > 1$. The next two lemmas consider each of these cases separately.

LEMMA SB.6: *Let $\alpha > 1 > \rho$. Then preferences violate SI.*

PROOF: Consider the lotteries $p_H \equiv \frac{1}{2} \times [1, (H, 2)] + \frac{1}{2} \times [1, (1, 3)]$ and $q_H \equiv \frac{1}{2} \times [1, (1, 2)] + \frac{1}{2} \times [1, (H, 3)]$. To show that preferences violate SI, it suffices to show that $q_H \succ p_H$ for some $H > 1$. By Lemma SB.3, the values of these lotteries are

$$U(p_H) = \left\{ (1 - \beta) \right.$$
$$\left. + \beta\left[\frac{\left[(1-\beta)(H+1)^{1-\rho} + \beta\right]^{\frac{1-\alpha}{1-\rho}} + \left[1 + (1-\beta) \cdot \beta(2^{1-\rho} - 1)\right]^{\frac{1-\alpha}{1-\rho}}}{2}\right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

and

$$U(q_H) = \left\{ (1 - \beta) \right.$$
$$\left. + \beta\left[\frac{\left[(1-\beta)2^{1-\rho} + \beta\right]^{\frac{1-\alpha}{1-\rho}} + \left[1 + (1-\beta) \cdot \beta((H+1)^{1-\rho} - 1)\right]^{\frac{1-\alpha}{1-\rho}}}{2}\right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}.$$

Since $\alpha > 1 > \rho$, we find that $U(q_H) > U(p_H)$ if and only if

$$\left[(1-\beta)2^{1-\rho} + \beta\right]^{\frac{1-\alpha}{1-\rho}} - \left[(1-\beta)(H+1)^{1-\rho} + \beta\right]^{\frac{1-\alpha}{1-\rho}}$$
$$< \left[1 + \beta(1-\beta)\left(2^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}} - \left\{1 + \beta(1-\beta)\left[(H+1)^{1-\rho} - 1\right]\right\}^{\frac{1-\alpha}{1-\rho}}.$$

Because $\frac{1-\alpha}{1-\rho} < 0$, as $H \nearrow +\infty$, the LHS converges to $[(1-\beta)2^{1-\rho} + \beta]^{\frac{1-\alpha}{1-\rho}}$, whereas the RHS converges to $[1 + \beta(1-\beta)(2^{1-\rho} - 1)]^{\frac{1-\alpha}{1-\rho}}$. Thus, there exists $\bar{H}$ such that this inequality holds for all $H > \bar{H}$ if

$$\left[(1-\beta)2^{1-\rho} + \beta\right]^{\frac{1-\alpha}{1-\rho}} < \left[1 + \beta(1-\beta)\left(2^{1-\rho} - 1\right)\right]^{\frac{1-\alpha}{1-\rho}}.$$

Use the fact that $\frac{1-\alpha}{1-\rho} < 0$ to rewrite this inequality as

$$(1-\beta)2^{1-\rho} + \beta > 1 + \beta(1-\beta)(2^{1-\rho} - 1) \quad \Longleftrightarrow \quad (2^{1-\rho} - 1)(1-\beta) > 0,$$

which is always true since $\rho < 1$.   Q.E.D.

LEMMA SB.7: *Let $\alpha > \rho > 1$. If $\frac{\rho-1}{\alpha-\rho} < 1 - \frac{\ln[1-(1-\beta)\beta]}{\ln\beta}$, then preferences violate SI.*

PROOF: We claim that there exist H and L with $H > L > 0$ such that

$$\frac{1}{2} \times [1, (H, 2)] + \frac{1}{2} \times [1, (L, 3)] \prec \frac{1}{2} \times [1, (L, 2)] + \frac{1}{2} \times [1, (H, 2+3)] \quad \text{(SB.13)}$$

if and only if

$$\frac{\rho - 1}{\alpha - \rho} < 1 - \frac{\ln[1 - (1-\beta)\beta]}{\ln\beta}. \quad \text{(SB.14)}$$

For each fixed z_H and z_L, consider the following lotteries:

$$p_{H,L} \equiv \frac{1}{2} \times [1, (z_H - 1, 2)] + \frac{1}{2} \times [1, (z_L - 1, 3)]$$

and

$$q_{H,L} \equiv \frac{1}{2} \times [1, (z_L - 1, 2)] + \frac{1}{2} \times [1, (z_H - 1, 3)].$$

By Lemma SB.3, the values of lotteries $p_{H,L}$ and $q_{H,L}$ are

$$U(p_{H,L}) = \left\{ (1-\beta) + \beta \left[\frac{\left[(1-\beta)z_H^{1-\rho} + \beta\right]^{\frac{1-\alpha}{1-\rho}} + \left[1 + \beta(1-\beta)(z_L^{1-\rho} - 1)\right]^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}},$$

$$U(q_{H,L}) = \left\{ (1-\beta) + \beta \left[\frac{\left[(1-\beta)z_L^{1-\rho} + \beta\right]^{\frac{1-\alpha}{1-\rho}} + \left[1 + \beta(1-\beta)(z_H^{1-\rho} - 1)\right]^{\frac{1-\alpha}{1-\rho}}}{2} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}.$$

For notational simplicity, let $\mu_H \equiv 1 - z_H^{1-\rho}$ and $\mu_L \equiv 1 - z_L^{1-\rho}$ and note that $0 < \mu_L < \mu_H < 1$ (since $1 < z_L < z_H$ and $\rho > 1$). Using the fact that $\frac{1}{1-\rho} < 0$ and $\frac{1-\alpha}{1-\rho} > 0$, it follows that $U(q_{H,L}) > U(p_{H,L})$ if and only if

$$\left[(1-\beta)(1 - \mu_L) + \beta\right]^{\frac{1-\alpha}{1-\rho}} + \left[1 - \beta(1-\beta)\mu_H\right]^{\frac{1-\alpha}{1-\rho}}$$

$$< \left[(1-\beta)(1 - \mu_H) + \beta\right]^{\frac{1-\alpha}{1-\rho}} + \left[1 - \beta(1-\beta)\mu_L\right]^{\frac{1-\alpha}{1-\rho}}.$$

Let $\phi(\mu) \equiv [(1-\beta)(1-\mu) + \beta]^{\frac{1-\alpha}{1-\rho}} - [1 - \beta(1-\beta)\mu]^{\frac{1-\alpha}{1-\rho}}$, and note that, by the previous inequality, there exists $z_H > z_L > 1$ such that $U(q_{H,L}) > U(p_{H,L})$ if and only if $\phi(\mu_H) > \phi(\mu_L)$ for some μ_H and μ_L with $0 < \mu_L < \mu_H < 1$. That is, $U(q_{H,L}) > U(p_{H,L})$ for some $z_H > z_L > 1$ if and only if $\phi(\cdot)$ is not weakly decreasing in the interval $(0, 1)$, which, because $\phi(\cdot)$ is differentiable, is true if and only if $\phi'(\mu) > 0$ for some μ.

Differentiating $\phi(\cdot)$, gives

$$\phi'(\mu) = \left(\frac{1-\alpha}{1-\rho}\right)(1-\beta)\left\{\beta\left[1 - \beta(1-\beta)\mu\right]^{\frac{1-\alpha}{1-\rho} - 1} - \left[(1-\beta)(1-\mu) + \beta\right]^{\frac{1-\alpha}{1-\rho} - 1}\right\},$$

so that $\phi'(\mu) > 0$ if and only if

$$\beta\big[1 - \beta(1-\beta)\mu\big]^{\frac{1-\alpha}{1-\rho}-1} > \big[(1-\beta)(1-\mu) + \beta\big]^{\frac{1-\alpha}{1-\rho}-1}.$$

Notice that the terms inside the brackets are positive (because $\mu \in (0, 1)$), so we can simplify this condition as

$$\mu > \frac{1}{1-\beta} \cdot \frac{1 - \beta^{\frac{\rho-1}{\alpha-\rho}}}{1 - \beta^{\frac{\rho-1}{\alpha-\rho}+1}}$$

for some $\mu \in (0, 1)$. But this is true if and only if the inequality holds for $\mu = 1$:

$$1 > \frac{1}{1-\beta} \cdot \frac{1 - \beta^{\frac{\rho-1}{\alpha-\rho}}}{1 - \beta^{\frac{\rho-1}{\alpha-\rho}+1}},$$

which can be rearranged as

$$\beta^{\frac{\rho-1}{\alpha-\rho}} > \frac{\beta}{1 - (1-\beta)\beta}.$$

Taking logs of both sides gives the following necessary and sufficient condition for (SB.13):

$$\frac{\rho-1}{\alpha-\rho}\ln\beta > \ln\beta - \ln\big[1 - (1-\beta)\beta\big],$$

which, because $\ln\beta < 0$, can be rearranged as

$$\frac{\rho-1}{\alpha-\rho} < 1 - \frac{\ln\big[1 - (1-\beta)\beta\big]}{\ln\beta},$$

which is condition (SB.14). *Q.E.D.*

LEMMA SB.8: *Let $\alpha \geq \rho > 1$. Preferences are RSTL if and only if*

$$\frac{\left(\frac{1}{1-\beta} - y\right)^{\frac{1-\alpha}{1-\rho}} + \left(\frac{1}{1-\beta} - \beta^{2\zeta}y\right)^{\frac{1-\alpha}{1-\rho}}}{2} \leq \left(\frac{1}{1-\beta} - \beta^{\zeta}y\right)^{\frac{1-\alpha}{1-\rho}} \tag{SB.15}$$

for all $y \in (0, 1)$ and all $\zeta \in \{1, 2, 3, \ldots\}$.

PROOF: Let $\gamma \equiv \frac{1-\alpha}{1-\rho} > 0$. By (SB.10) (and the fact that $\rho > 1$), preferences are RSTL if and only if

$$\frac{\left(\frac{1}{1-\beta} + \lambda^{1-\rho} - 1\right)^{\gamma} + \left[\frac{1}{1-\beta} + \beta^{2\zeta}\big(\lambda^{1-\rho} - 1\big)\right]^{\gamma}}{2} \leq \left[\frac{1}{1-\beta} + \beta^{\zeta}\big(\lambda^{1-\rho} - 1\big)\right]^{\gamma}$$

for all $\lambda > 1$ and all $\zeta \geq 1$. Let $y \equiv 1 - \lambda^{1-\rho}$ and notice that $y \in (0, 1)$ (since $\lambda \in (1, +\infty)$ and $\rho > 1$). Thus, we can rewrite the RSTL condition as

$$\frac{\left(\frac{1}{1-\beta} - y\right)^{\gamma} + \left(\frac{1}{1-\beta} - \beta^{2\zeta} y\right)^{\gamma}}{2} \leq \left(\frac{1}{1-\beta} - \beta^{\zeta} y\right)^{\gamma}$$

for all $y \in (0, 1)$. *Q.E.D.*

LEMMA SB.9: *Let $\alpha \geq \rho > 1$ and suppose preferences violate RSTL. Then preferences violate SI.*

PROOF: Let $\gamma \equiv \frac{1-\alpha}{1-\rho} > 1$. We claim that for each fixed y, β, and ζ, there exists a threshold $\bar{\gamma}_{y,\beta,\zeta}$ such that preferences are RSTL if and only if $\gamma \geq \bar{\gamma}_{y,\beta,\zeta}$, which, by the previous lemma, is equivalent to

$$\frac{\left(\frac{1}{1-\beta} - y\right)^{\gamma} + \left(\frac{1}{1-\beta} - \beta^{2\zeta} y\right)^{\gamma}}{2} > \left(\frac{1}{1-\beta} - \beta^{\zeta} y\right)^{\gamma} \quad \Longleftrightarrow \quad \gamma < \bar{\gamma}_{y,\beta,\zeta}. \quad \text{(SB.16)}$$

To see this, rearrange (SB.16) as

$$R(\gamma) \equiv \left(\frac{\frac{1}{1-\beta} - y}{\frac{1}{1-\beta} - \beta^{\zeta} y}\right)^{\gamma} + \left(\frac{\frac{1}{1-\beta} - \beta^{2\zeta} y}{\frac{1}{1-\beta} - \beta^{\zeta} y}\right)^{\gamma} > 2. \quad \text{(SB.17)}$$

Notice first that R is a convex function of γ, since

$$R''(\gamma) = \left(\frac{\frac{1}{1-\beta} - y}{\frac{1}{1-\beta} - \beta^{\zeta} y}\right)^{\gamma} \cdot \left[\ln\left(\frac{\frac{1}{1-\beta} - y}{\frac{1}{1-\beta} - \beta^{\zeta} y}\right)\right]^2$$

$$+ \left(\frac{\frac{1}{1-\beta} - \beta^{2\zeta} y}{\frac{1}{1-\beta} - \beta^{\zeta} y}\right)^{\gamma} \cdot \left[\ln\left(\frac{\frac{1}{1-\beta} - \beta^{2\zeta} y}{\frac{1}{1-\beta} - \beta^{\zeta} y}\right)\right]^2$$

$$> 0.$$

Algebraic manipulations establish that (SB.17) fails for $\gamma = 1$. Moreover, (SB.17) is always true for γ large enough, since $\lim_{\gamma \to \infty} R(\gamma) = +\infty > 2$. Therefore, there exists a unique $\bar{\gamma}_{\beta,y,\zeta} > 1$ such that the inequality holds if and only if $\gamma > \bar{\gamma}_{\beta,y,\zeta}$.

Recall from Lemma SB.7 that preferences violate SI if

$$\frac{\rho - 1}{\alpha - \rho} < 1 - \frac{\ln[1 - (1 - \beta)\beta]}{\ln \beta}.$$

Since $\gamma - 1 = \frac{\alpha - \rho}{\rho - 1}$, this condition can be written as

$$\frac{1}{\gamma - 1} < 1 - \frac{\ln\left[1 - (1 - \beta)\beta\right]}{\ln \beta},$$

which can be further simplified as

$$\gamma > \frac{\ln\left[1 - (1 - \beta)\beta\right] - 2\ln\beta}{\ln\left[1 - (1 - \beta)\beta\right] - \ln\beta}.$$

Therefore, preferences violate RSTL if and only if $\gamma \geq \bar{\gamma}_{\beta,y,\zeta}$, whereas they violate SI if $\gamma \geq \frac{\ln[1-(1-\beta)\beta]-2\ln\beta}{\ln[1-(1-\beta)\beta]-\ln\beta}$. To conclude the proof, it suffices to show that the cutoff for RSTL violations is higher than the (sufficient) cutoff for SI violations:

$$\bar{\gamma}_{\beta,y,\zeta} \geq \frac{\ln\left[1 - (1 - \beta)\beta\right] - 2\ln\beta}{\ln\left[1 - (1 - \beta)\beta\right] - \ln\beta}.$$

Recall that $\bar{\gamma}_{\beta,y,\zeta}$ solves

$$R(\gamma) = \left(\frac{\frac{1}{1-\beta} - y}{\frac{1}{1-\beta} - \beta^{\zeta}y}\right)^{\gamma} + \left(\frac{\frac{1}{1-\beta} - \beta^{2\zeta}y}{\frac{1}{1-\beta} - \beta^{\zeta}y}\right)^{\gamma} = 2.$$

Recall that R is convex, $R(1) < 2$ and $R(\infty) > 2$. Thus, we need to show that

$$R\left(\frac{\ln\left[1 - (1 - \beta)\beta\right] - 2\ln\beta}{\ln\left[1 - (1 - \beta)\beta\right] - \ln\beta}\right) < 2.$$

Note that

$$\frac{\ln\left[1 - (1 - \beta)\beta\right] - 2\ln\beta}{\ln\left[1 - (1 - \beta)\beta\right] - \ln\beta} < 2.$$

So, it suffices to show that

$$\left(\frac{\frac{1}{1-\beta} - y}{\frac{1}{1-\beta} - \beta^{\zeta}y}\right)^{2} + \left(\frac{\frac{1}{1-\beta} - \beta^{2\zeta}y}{\frac{1}{1-\beta} - \beta^{\zeta}y}\right)^{2} < 2$$

for all for all $y \in (0, 1)$ and all ζ, β. Rearrange this expression as

$$\frac{\left(\frac{1}{1-\beta} - y\right)^{2} + \left(\frac{1}{1-\beta} - \beta^{2\zeta}y\right)^{2}}{2} < \left(\frac{1}{1-\beta} - \beta^{\zeta}y\right)^{2}.$$

With some algebraic manipulations, this inequality can be rewritten as

$$\left(1 + \beta^{\zeta}\right)^{2} < \frac{2}{1 - \beta}$$

for all $y \in (0, 1)$ and all $\zeta = 1, 2, 3 \ldots$. Since the LHS is decreasing in ζ (because $\beta < 1$), it suffices to verify this condition at $\zeta = 1$, where we have

$$(1 + \beta)^2 < \frac{2}{1 - \beta} \quad \Longleftrightarrow \quad 0 < 1 - \beta + \beta^2 + \beta^3.$$

Let $\xi(\beta) \equiv 1 - \beta + \beta^2 + \beta^3$ and notice that

$$\xi'(\beta) = -1 + 2\beta + 3\beta^2,$$

which has roots $\beta = -1$ and $\beta = \frac{1}{3}$. Moreover, ξ is convex at $\beta \in [0, 1]$ since $\xi''(\beta) = 2 + 6\beta > 0$. Therefore, $\xi'(\beta) < 0$ for $\beta \in [0, \frac{1}{3})$ and $\xi'(\beta) > 0$ for $\beta \in (\frac{1}{3}, 1]$, showing that ξ has a minimum at $\beta = \frac{1}{3}$:

$$\xi(\beta) \geq \xi\left(\frac{1}{3}\right) = 1 - \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 > 0,$$

concluding the proof. $Q.E.D.$

Combining the results from the lemmas above, it follows that preferences that satisfy SI must be RSTL, concluding the proof of the proposition. To see this, recall that:

• When $\alpha < 1$, preferences are always RSTL, regardless of whether they satisfy SI (Lemma SB.4).

• When $\rho \geq \alpha$, preferences are always RSTL, regardless of whether they satisfy SI (Lemma SB.5).

• When $\alpha > 1 > \rho$, SI never holds (Lemma SB.6).

• When $\alpha > \rho > 1$, SI holds if γ is below a threshold that is lower than the threshold for RSTL ($\frac{\ln[1-(1-\beta)\beta]-2\ln\beta}{\ln[1-(1-\beta)\beta]-\ln\beta} < \bar{\gamma}_{\beta,y,\zeta}$), so that SI implies RSTL (Lemma SB.9). $Q.E.D.$