Solution Algorithm

Optimal Regulation of Noncompete Contracts

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1 Model

This section collects the equations that characterize the equilibrium and the social optimum.

1.1 Equilibrium: Joint Value Maximization

The joint value functions are linear in productivity z, $J^c(z) = j^c z$ and $J^n(z,\kappa) = j^n(\kappa)z$. For a firm-worker match under a standard contract:

$$\bar{\theta}^c = 1 \tag{1}$$

$$c'(\mu^c) = j^c = \frac{1 - c(\mu^c)}{r - \mu^c}. \tag{2}$$

For a match of a cost type κ under a noncompete contract:

$$\bar{\theta}^n = 1 + \frac{1 - F\big(\bar{\theta}^n\big)}{f\big(\bar{\theta}^n\big)} \tag{3}$$

$$c'(\mu^n(\kappa)) = j^n(\kappa) = \frac{1 - c(\mu^n(\kappa))}{r - \mu^n - \lambda p\big(\bar{\theta}^n - 1\big)\big(1 - F\big(\bar{\theta}^n\big)\big)}. \tag{4}$$

The noncompete duration is $\pi = \frac{1}{r}\log(\theta^n)$.

The cutoff type selecting into noncompete clauses is

$$\bar{\kappa} = j^c \lambda p\big(\bar{\theta}^n - 1\big)\big(1 - F\big(\bar{\theta}^n\big)\big).$$

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The corresponding job-to-job transition rates are

$$\eta^c = \lambda(1 - F(1)) \tag{5}$$
$$\eta^n = \lambda\big[p\big(1 - F\big(\bar{\theta}^n\big)\big) + (1-p)(1 - F(1))\big]. \tag{6}$$

The separation rates adjust the job-to-job transition rates by the exogenous death rate δ.

1.2 Equilibrium: Distribution and Aggregation

In steady state, the conditional densities, $g^c(z)$ and $g^n(z;\kappa)$ follow the KF equations:

$$0 = -\mu^c g^c_z(z) + \frac{1}{2}\sigma^2 g^c_{zz}(z) + \delta[h(z) - g^c(z)] + \lambda\int_1^\infty \left[g^c\left(\frac{z}{\theta}\right) - g^c(z)\right] dF(\theta) \tag{7}$$

$$\begin{aligned} 0 = &-\mu^n(\kappa) z g^n_z(z;\kappa) + \frac{1}{2}\sigma^2 z^2 g^n_{zz}(z;\kappa) + \delta[h(z) - g^n(z,\kappa)] \\ &+ \lambda\left\{p\int_{\bar{\theta}^n}^\infty \left[g^n\left(\frac{z}{\theta};\kappa\right) - g^n(z;\kappa)\right] dF(\theta) + (1-p)\int_1^\infty \left[g^n\left(\frac{z}{\theta};\kappa\right) - g^n(z;\kappa)\right] dF(\theta)\right\}. \end{aligned} \tag{8}$$

The conditional aggregate productivities are

$$Z^c = \frac{\delta \int z dH(z)}{\delta - \mu^c - \lambda\int_1^\infty(\theta - 1)dF(\theta)} \tag{9}$$

$$Z^n(\kappa) = \frac{\delta \int z dH(z)}{\delta - \mu^n(\kappa) - \lambda\big[p\int_{\bar{\theta}^n}^\infty(\theta-1)dF(\theta) + (1-p)\int_1^\infty(\theta-1)dF(\theta)\big]}. \tag{10}$$

The conditional aggregate net output:

$$Y^c = Z^c(1 - c(\mu^c)) \quad \text{and} \quad Y^n(\kappa) = Z^n(\kappa)(1 - c(\mu^n(\kappa)) - \kappa). \tag{11}$$

1.3 Equilibrium: Wage Profiles

The worker's value functions in terms of the wage-productivity ratio $x \equiv \log\big(\frac{w}{z}\big)$: $\forall x \in [0,\bar{x}^c]$,

$$\begin{aligned} (r + \lambda - \mu^c)u^c(x) = {} & e^x - \left(\mu^c + \frac{1}{2}\sigma^2\right)u^c_x(x) + \frac{1}{2}\sigma^2 u^c_{xx}(x) \\ &+ \lambda\left\{F(\underline{\theta}^c(x))u^c(x) + \int_{\underline{\theta}^c(x)}^1 \theta dF(\theta) j^c + (1 - F(1))j^c\right\}; \end{aligned} \tag{12}$$

$\forall x \in [0,\bar{x}^n(\kappa)]$,

$$
\begin{aligned}
(r+\lambda-\mu^n(\kappa))u^n(x,\kappa) &= e^x - \left(\mu^n(\kappa)+\frac{1}{2}\sigma^2\right)u_x^n(x,\kappa)+\frac{1}{2}\sigma^2 u_{xx}^n(x,\kappa) \\
&+ \lambda p\left\{F(\underline{\theta}^n(x,\kappa))u^n(x,\kappa)+\int_{\underline{\theta}^n(x,\kappa)}^{\bar{\theta}^n}\theta dF(\theta)e^{-r\pi}j^n(\kappa)+\big(1-F\big(\bar{\theta}^n\big)\big)j^n(\kappa)\right\} \\
&+ \lambda(1-p)\left\{F(\underline{\theta}^u(x,\kappa))u^n(x,\kappa)+\int_{\underline{\theta}^u(x,\kappa)}^{1}\theta dF(\theta)j^n(\kappa)+(1-F(1))j^n(\kappa)\right\},
\end{aligned}
\tag{13}
$$

with the bidding-threshold conditions:

$$
u^c(x)=j^c\underline{\theta}^c(x) \quad \text{and} \quad u^n(x,\kappa)=j^n(\kappa)\underline{\theta}^u(x,\kappa)=e^{-r\pi}j^n(\kappa)\underline{\theta}^n(x,\kappa); \tag{14}
$$

and the value-matching and smooth-pasting conditions:

$$
u^c(\bar{x}^c)=j^c \quad \text{and} \quad u_x^c(\bar{x}^c)=0; \quad u^n(\bar{x}^n,\kappa)=j^n(\kappa) \quad \text{and} \quad u_x^n(\bar{x}^n(\kappa),\kappa)=0. \tag{15}
$$

The initial wage-productivity ratio x_0 satisfies:

$$
u^c(x_0^c)=\beta j^c \quad \text{and} \quad u^n(x_0^n,\kappa)=\beta j^n(\kappa). \tag{16}
$$

It evolves over tenure t according to: $\forall x \in [0,\bar{x}^c]$.

$$
\begin{aligned}
\psi_t^c(x,t) &= \left(\mu^c+\frac{1}{2}\sigma^2\right)\psi_x^c(x,t)+\frac{1}{2}\sigma^2\psi_{xx}^c(x,t) \\
&+ \lambda\left\{\frac{f(\underline{\theta}^c(x))}{F(1)}\int_{\underline{x}^c}^{x}\psi^c(\tilde{x},t)d\tilde{x}-\left(1-\frac{F(\underline{\theta}^c(x))}{F(1)}\right)\psi^c(x,t)\right\},
\end{aligned}
\tag{17}
$$

$\forall x \in [0,\bar{x}^n(\kappa)]$,

$$
\begin{aligned}
\psi_t^n(x,\kappa,t) &= \left(\mu^n(\kappa)+\frac{1}{2}\sigma^2\right)\psi_x^n(x,\kappa,t)+\frac{1}{2}\sigma^2\psi_{xx}^n(x,\kappa,t) \\
&+ \lambda p\left\{\frac{f(\underline{\theta}^n(x,\kappa))}{F\big(\bar{\theta}^n\big)}\int_{\underline{x}^n(\kappa)}^{x}\psi^n(\tilde{x},\kappa,t)d\tilde{x}-\left(1-\frac{F(\underline{\theta}^n(x,\kappa))}{F\big(\bar{\theta}^n\big)}\right)\psi^n(x,\kappa,t)\right\} \\
&+ \lambda(1-p)\left\{\frac{f(\underline{\theta}^u(x,\kappa))}{F(1)}\int_{\underline{x}^n(\kappa)}^{x}\psi^n(\tilde{x},\kappa,t)d\tilde{x}-\left(1-\frac{F(\underline{\theta}^u(x,\kappa))}{F(1)}\right)\psi^n(x,\kappa,t)\right\}.
\end{aligned}
\tag{18}
$$

Adjusting for the productivity growth, the wage growth is

$$\mathbb{E}[\log(w_t)] - \log(w_0) = \mathbb{E}[x_t] - x_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t. \tag{19}$$

1.4 Social Optimum

The social match values are also linear productivity z, $S^c(z) = s^c z$ and $S^n(z,\kappa) = s^n(\kappa)z$.

$$s^c = \frac{1 - c(\mu^c)}{r - \mu^c - \lambda\int_{\bar{\theta}^c}^{\infty}(\theta - 1)dF(\theta)} \tag{20}$$

$$s^n(\kappa) = \frac{1 - c(\mu^n(\kappa)) - \kappa}{r - \mu^n(\kappa) - \lambda\Big[p\int_{\bar{\theta}^n(\kappa)}^{\infty}(\theta - 1)dF(\theta) + (1-p)\int_{\bar{\theta}^c}^{\infty}(\theta - 1)dF(\theta)\Big]}. \tag{21}$$

The social-optimal poaching threshold:[1]

$$\bar{\theta}^{n*}(\kappa) = 1 + \frac{\varepsilon\Delta}{\varepsilon\Delta + 1}\frac{1 - F\big(\bar{\theta}^{n*}(\kappa)\big)}{f\big(\bar{\theta}^{n*}(\kappa)\big)}. \tag{22}$$

where $\Delta \equiv \frac{\lambda\left[p\int_{\bar{\theta}^n}^{\infty}\left(\theta - \bar{\theta}^n\right)dF(\theta) + (1-p)\int_1^{\infty}(\theta-1)dF(\theta)\right]}{r - \mu^n(\kappa) - \lambda p\left(\bar{\theta}^n - 1\right)\left(1 - F\left(\bar{\theta}^n\right)\right)}\frac{\mu^n(\kappa) - \frac{1}{2}\sigma^2}{r - \mu^n(\kappa) - \lambda p\left(\bar{\theta}^n - 1\right)\left(1 - F\left(\bar{\theta}^n\right)\right)}$. The planner includes a noncompete clause for cost types $\kappa < \bar{\kappa}^*$ where

$$s^n(\bar{\kappa}^*) = s^c. \tag{23}$$

The time-zero welfare is obtained:

$$\mathcal{W}^0 = \iint s(\kappa)z dG(z,\kappa,0) + \frac{\delta}{\rho}\int s(\kappa)d\Phi(\kappa)\int z dH(z). \tag{24}$$

2 Algorithm

The equations above can be solved either in closed form or with equation solvers in Matlab, except for the ones involving wage setting and wage-tenure profiles. I outline the algorithm for solving the wages.[2]

[1] The optimal poaching threshold depends on the contracting cost type κ, after accounting for how the agents' investment incentive $\mu^n(\kappa)$ depends on κ as we alter the poaching threshold. In equation (22) (Proposition 3), I omit this dependency for ease of notation since it is quantitatively negligible.

[2] The algorithm builds on the following notes for solving optimal stopping problems: https://benjaminmoll.com/wp-content/uploads/2020/06/option_simple.pdf and https://benjaminmoll.com/wp-content/uploads/2020/06/hopenhayn.pdf.

2.1 Solve Wage Setting

I illustrate the finite-difference method for the wage setting for workers under standard contracts, which can be easily extended to the case under noncompete contracts.

Before proceeding further, first simplify the HJB equation (12) by adjusting the bidding outcomes:

$$(r-\mu^c)u^c(x) = a(x) - \left(\mu^c + \frac{1}{2}\sigma^2\right)u^c_x(x) + \frac{1}{2}\sigma^2 u^c_{xx}(x), \tag{25}$$

where the expected flow payoff $a(x) \equiv e^x + \lambda\int_{\underline{\theta}^c(x)}^1 (1-F(\theta))d\theta j^c$. Further, to incorporate the value-matching and smooth-pasting conditions, rewrite the equation in terms of an HJB variational inequality (HJBVI) form:

$$\min\left\{(r-\mu^c)u^c(x) - a(x) - \left(\mu^c + \frac{1}{2}\sigma^2\right)u^c_x(x) + \frac{1}{2}\sigma^2 u^c_{xx}(x), u^c(x) - j^c\right\} = 0. \tag{26}$$

- Define a grid for the wage-productivity ratio: $x \in \{x_1, x_2, \cdots, x_I\}$. Use a simple even-spaced Δx grid.

- The initial guess of the worker's value in the k-th iteration $u^c(x) \in \left\{u_1^{c,k}, u_2^{c,k}, \cdots, u_I^{c,k}\right\}$. Solve the bidding threshold $\underline{\theta}^c(x)$ according to

$$\underline{\theta}_i^{c,k} = \frac{u_i^{c,k}}{j^c}.$$

  and calculate the expected flow payoff

$$a_i^k = e^{x_i} + \lambda\int_{\underline{\theta}_i^{c,k}}^1 (1-F(\theta))d\theta j^c.$$

- The first-order and second-order difference approximations: $\forall i < I$,

$$u^c_x(x_i) \approx \frac{u^c_{i+1} - u^c_i}{\Delta x},$$
$$u^c_{xx}(x_i) \approx \frac{u^c_{i+1} - 2u^c_i + u^c_{i-1}}{(\Delta x)^2}.$$

  To solve the updated value $u^{c,k+1}$, the finite-difference approximation of (25): $\forall i < I$,

$$\frac{u_i^{c,k+1} - u_i^{c,k}}{\Delta} + (r-\mu^c)u_i^{c,k+1} = a_i^k - \left(\mu^c + \frac{1}{2}\sigma^2\right)\frac{u_{i+1}^{c,k+1} - u_i^{c,k+1}}{\Delta x} + \frac{1}{2}\sigma^2\frac{u_{i+1}^{c,k+1} - 2u_i^{c,k+1} + u_{i-1}^{c,k+1}}{(\Delta x)^2}.$$

On the last grid, it is guaranteed that $u_x^c(x_I) = 0$ and $u_{xx}^c(x_I) = 0$:

$$\frac{u_I^{c,k+1} - u_I^{c,k}}{\Delta} + (r - \mu)u_I^{n+1} = a_I^k.$$

Collect the terms in the finite-difference approximation above:

$$\frac{u_i^{c,k+1} - u_i^{c,k}}{\Delta} + (r - \mu^c)u_i^{c,k+1} = a_i^k + Xu_{i-1}^{c,k+1} + Yu_i^{c,k+1} + Zu_{i+1}^{c,k+1},$$

where

$$X = \frac{\frac{1}{2}\sigma^2}{(\Delta x)^2}, \quad Y = \frac{\mu^c + \frac{1}{2}\sigma^2}{\Delta x} - \frac{\sigma^2}{(\Delta x)^2}, \quad \text{and} \quad Z = -\frac{\mu^c + \frac{1}{2}\sigma^2}{\Delta x} + \frac{\frac{1}{2}\sigma^2}{(\Delta x)^2}.$$

Express the finite-difference approximation of the HJBVI (26) in matrix form:

$$\min\left\{\frac{1}{\Delta}\left(u^{c,k+1} - u^{c,k}\right) + (r - \mu)u^{c,k+1} - a^k - \mathbf{A}u^{c,k+1}, u^{c,k+1} - j^c\right\} = 0, \tag{27}$$

where

$$a^k = \begin{bmatrix} e^{x_1} + \lambda\int_{\underline{\theta}_1^{c,k}}^1 (1 - F(\theta))d\theta j^c \\ e^{x_2} + \lambda\int_{\underline{\theta}_2^{c,k}}^1 (1 - F(\theta))d\theta j^c \\ \vdots \\ e^{x_I} + \lambda\int_{\underline{\theta}_I^{c,k}}^1 (1 - F(\theta))d\theta j^c \end{bmatrix}, \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} X+Y & Z & & & & \\ X & Y & Z & & & \\ & X & Y & Z & & \\ & & \ddots & \ddots & \ddots & \\ & & & X & Y & Z \\ & & & & X & Y+Z \end{bmatrix}.$$

- Use `LCP.m` solver for linear complementarity problems to solve (27): it solves the updated value $u^{c,k+1}$ and the boundary $\bar{x}^{c,k+1}$. Iterate until the value function converges.

Once the wage-setting problems are solved, the wage-tenure profiles can be obtained by solving forward the wage distribution over tenure according to equations (17) and (18).