Skiba points and heteroclinic bifurcations, with applications to the shallow lake system

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Abstract

Techniques from dynamical systems, specifically from bifurcation theory, are used to investigate the occurrence of Skiba points in one state, one co–state control systems, for which the effect of the control has a definite direction. A Skiba point is an initial state for which two different optimal solutions of the control problem exist. It is found that the parameter region for which Skiba points occur is bounded by heteroclinic bifurcation manifolds. A local criterion is given that ensures the existence of Skiba points in systems with small discount rates. The analysis is applied to the shallow lake system investigated by Mäler, Xepapapedas and de Zeeuw (2000). For this system, it is shown that for any given parameter value, there is at most one Skiba point.

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1 Introduction

This paper investigates the connection between heteroclinic bifurcations and the occurrence of Skiba points in one–dimensional optimal control problems. That is, optimal controls $u(t)$ are sought, minimising a cost functional

$$\mathcal{C} = \int_0^\infty g(x, u, \lambda) \mathrm{e}^{-\rho t} \, \mathrm{d}t, \tag{1}$$

under the condition that x and u satisfy for all t the state equation

$$\dot{x} = f(x, u, \lambda). \tag{2}$$

Here $x \in X \subset \mathbb{R}$ denotes the state variable, $u \in U \subset \mathbb{R}$ the control, and $\lambda \in \Lambda \subset \mathbb{R}^q$ a q–dimensional parameter. The state space X, the control space U, and the parameter space Λ are all assumed to be open sets. In this article, only the following class of state equations is investigated: it is assumed that the function f satisfies

$$\frac{\partial f}{\partial u}(x, u, \lambda) \neq 0 \tag{3}$$

for all $(x, u, \lambda) \in X \times U \times \Lambda$. Heuristically, this means that the control has a uni–directional effect.

It is well known that in this kind of system, there might exist so–called indifference or Skiba states, for which there are two distinct optimal controls with equal total cost. The central result of this article is that for this kind of system (modulo technical conditions outlined below), the set of parameters for which the control problem has Skiba points is bounded by so–called heteroclinic bifurcation curves of the associated state–co-state system.

Skiba points occur naturally in control problems if there are several equilibria. The concept was introduced by Skiba (1978) for optimal grow paths of economies with a convex–concave production technology, and subsequently indifference points have been found by Dechert and Nishimura (1983) and in many other places, see Deissinger *et al.* (2001) and references therein.

Relation to bifurcation theory. Determining the existence of a point for which there are two equally costly solutions is typically a global problem: the cost functionals of two trajectories have to be computed. Now, typically a system depends on several parameters. Imagine the case that there is a set of parameters for which Skiba points exist, and another for which there are no Skiba points. There is a qualitative difference between these type of systems. Going from one set to the other hence entails a qualitative change, or, in technical terms, a bifurcation. It will be argued below that the simplest bifurcations that can occur for the one–state, one–control systems studied in this article are saddle–node and heteroclinic bifurcations. The former bifurcation is local, the latter global. Hence it may be expected that heteroclinic bifurcations are in some way connected to the occurrence of Skiba points. As announced, the main result of this article is that this is indeed the case.

The significance of this result is that now bifurcation theory can be used to determine the regions in parameter space for which Skiba points do or do not exist. For the shallow lake family of systems, such a diagram is given in figure 1.

Main application. The Shallow lake family (see Brock and Starret, 1999, Dechert and Brock, 2000 and Mälar, Xepapadeas and de Zeeuw, 2000) shall serve as the main application of the general theory throughout the present article, since it has been the initial motivation for this study. Indeed, the first half of this article gives an analysis of the Shallow lake system in terms of the general results obtained in the second half.

Biological model. The dynamics of pollution or eutrophication of shallow lakes gives rise to a simple optimal control problem which nevertheless has quite interesting features. For the general economic and ecological background of the model studied here, the reader is referred to Mälar, Xepapadeas and de Zeeuw (2000), where the following model equation is introduced:

$$\dot{x} = f(x,u) = u - bx + \frac{x^2}{x^2+1}, \qquad x(0) = x_0. \tag{4}$$

Here $x(t) \geq 0$ is proportional to the amount of phosphorus in a shallow lake, $u(t) \geq 0$ to the input of more phosphorus (due to farming); $b \geq 0$ is proportional to the rate of loss of phosphorus due to sedimentation, outflow and sequestration in other biomass. The term x^2/x^2+1 models the biological production of phosphorus in the lake. These remarks are only intended to give an indication of the meaning of the variables; for more detailed information and references, see Mälar, Xepapadeas and de Zeeuw (2000). Of course, there is much more to shallow lakes than the extremely simple model (4): see for instance Scheffer (1998).

In case there is more than one agent releasing phosphorus into the lake, the load $u(t)$ will be the total sum of the individual loads $u_i(t)$:

$$u(t) = \sum_{i=1}^{n} u_i(t), \tag{5}$$

where n denotes the number of agents.

Welfare. The *welfare function*, that is, the benefits to be reaped from the use of the lake, is modelled as:

$$\log u - cx^2.$$

Farmers use artificial fertilisers to grow crops; these fertilisers contain phosphorus, which in the end gets washed into the lake. The term $\log u$ models the benefits to farmers arising from using u units of phosphorus. The lake is also used by fishers and tourists, who are interested in a clean lake. The term $-cx^2$ represents the cost of pollution: here $c \geq 0$ is an economic parameter. Future benefits are discounted by a factor $\mathrm{e}^{-\rho t}$, with $\rho \geq 0$ the *discount factor*.

If $u(t)$ is given and assumed to be continuous, except maybe for a set of isolated jump points, the solution $x : \mathbb{R}_+ \to \mathbb{R}_+$ is determined, and it will be continuous. The *total benefits* of the lake are then given by the *welfare* or *benefit functional* $\mathcal{B}[x,u]$. It takes the functions x and u as its arguments, and it is given by:

$$\mathcal{B}[x,u] = \int_0^\infty \left(\log u - cx^2\right) \mathrm{e}^{-\rho t}\, \mathrm{d}t. \tag{6}$$

The optimal control problem. Note that the dynamics of x are determined by equation (4), once the function $u(t)$ has been chosen. The optimal control problem of a social planner is to find the optimal dumping level u such that $\mathcal{B}$ is maximised. Equivalently, one can determine u to minimise the *cost functional*:

$$\mathcal{C}[x,u] = \int_0^\infty g(x,u)\mathrm{e}^{-\rho t}\, \mathrm{d}t = \int_0^\infty \left(-\log u + cx^2\right) \mathrm{e}^{-\rho t}\, \mathrm{d}t. \tag{7}$$

Dynamic game. If several agricultural agents are using the lake – say, several countries border on the lake – every one of them has a separate cost functional $\mathcal{C}_i[x,u]$ to maximise. However, since the state of the lake is influenced by the actions of the other agents, these have to be taken into account as well. This defines a dynamic game. Now every agent has to choose a strategy: the game is said to be in Nash equilibrium if every agent's strategy, given the opponents', is optimal. Because of the symmetry of the context, and because asymmetric equilibria are much harder to find, attention is restricted to symmetric equilibria, where every agent chooses the same strategy. In Mälar, Xepapadeas and de Zeeuw (2000) it is shown that in the context of an n–agent game, an optimal strategy $u_i(t)$ is equal to an optimal control of the one agent problem with the parameter c replaced by c/n. Hence it suffices to restrict attention to the one agent case.

Optimal solutions. The Pontryagin maximum principle yields a system of two differential equations for the state–co-state equations, whose phase curves correspond to potential optimal trajectories of the system. The actual optimal solution is selected from the continuum of solutions $(x(t), u(t))$ that satisfy $x(0) = x_0$, are defined for all $t \in [0, \infty)$, and satisfy a transversality condition of the form:

$$\lim_{t\to\infty} u(t)\mathrm{e}^{\rho t} = \infty; \tag{8}$$

hence, the control $u(t)$ should remain bounded away from 0 as $t \to \infty$, or it should approach 0 not too quickly.

For positive ρ, the state–co-state system has positive divergence, which excludes the possibility of phase curves that form closed loops. By the Poincaré–Bendixon theorem (see Anosov *et al.*, 1988), the phase portrait then consists only of equilibria and non–intersecting curves. Phase curves that are bounded as $t \to \infty$ are then necessarily stable manifolds of hyperbolic saddle equilibria, or centre manifolds of non–hyperbolic (bifurcating) equilibria.

Skiba points. It turns out that for certain sets of parameters there are initial states $x = x_*$, such that there are optimal controls $u_1(t)$ and $u_2(t)$, and corresponding $x_1(t)$, $x_2(t)$, ($x_1(0) = x_2(0) = x_*$), such that

$$C[x_1, u_1] = C[x_2, u_2]. \tag{9}$$

The initial state x_* is then called an *indifference* or *Skiba* point.

The first part of the article sketches the main ideas in the case of the shallow lake system. Specifically, for parameter (b, c) ranging over an 'interesting' part of the parameter plane (and for ρ fixed), the structure of the system is determined. The results are summarised in figure 1.

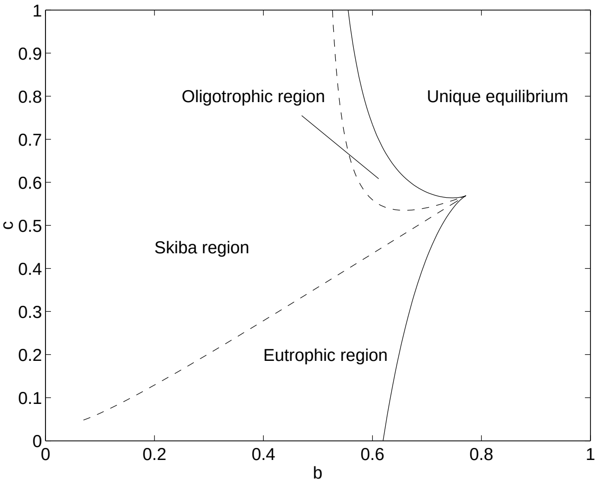

Figure 1: Parameter space of the shallow lake system. *Sketch of the (b, c)–parameter space of the shallow lake system. In the bifurcation diagram, full lines are so–called saddle node bifurcation curves, separating the region of parameters for which there is a unique equilibrium in the system from the region of multiple equilibria. In the oligotrophic region, solutions tending to the 'clean lake' equilibrium are optimal; in the eutrophic region, solutions tending to the 'polluted lake' equilibrium are optimal. Broken lines indicate so–called heteroclinic bifurcation curves. As is shown in the text, these curves bound the Skiba region for which indifference points exist in the shallow lake optimal control problem.*

Results for the shallow lake system. Figure 1 contains a lot of information on the shallow lake system. The parameter b characterises the physical properties of a lake, and may be seen as given. For such a b, there is the following sequence for increasing values of c (see also figure 4): unique equilibrium, which is a polluted lake; several equilibria, but the optimal usage leads to a polluted state; several equilibria, and the final state of the lake depends on its initial state; several equilibria, and the optimal usage leads to a clean lake; unique equilibrium, which is a clean lake. As can be seen from the figure, not all of these have to occur, depending on the value of b.

Moreover, recall the remark on the dynamic game above: if n agents use the lake, the system can be seen as a single agent system, with the value of c replaced by c/n. The figure then shows – as was to be expected intuitively – that the final condition of the lake worsens as n increases, and that if n is large enough, the optimal solution will inevitably lead to a polluted lake.

Finally, note that the existence of a Skiba point has an important consequence for a social planner: since at a Skiba initial state the two optimal solutions are equivalent in economic terms, the decision which one to choose has to be based on other criteria. For instance, in the case of a shallow lake, environmental protection agencies might argue that it is *morally* good to opt for a clean environment.

General theory. The second part of the article gives a general analysis of the occurrence of Skiba points for systems with one phase variable and one control, for which the effect of the control variable has a definite direction. That is, if

$$\dot{x} = f(x,u) \tag{10}$$

is the state equation, the effect of the control is said to have a definite direction if f is continuous and if

$$\frac{\partial f}{\partial u} \neq 0, \tag{11}$$

for any x and u. It is shown that for such a system the parameter region where Skiba points occur is bounded by heteroclinic bifurcations. Moreover, it is shown that if the state co-state system has for $\rho = 0$ a cusp bifurcation point, then for small $\rho > 0$, there exist Skiba points in the system.

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2 Shallow lakes

This section describes the shallow lake optimal control problem, and illustrates in this context some of the ideas of the general theory developped in the later sections.

2.1 The differential equations

To analyse the shallow lake problem, the current value Hamiltonian is formed (see for instance Brock and Malliaris, 1989):

$$H(p,x,u) = -g(x,u) + pf(x,u) = \log u - cx^2 + p\left(u - bx + \frac{x^2}{x^2+1}\right). \tag{12}$$

Here an additional variable $p \in \mathbb{R}$ is introduced, the *co-state*. If $u : [0,\infty) \to \mathbb{R}$ is an optimal solution, Pontryagin's necessary conditions state that $p(t)$, $x(t)$, and $u(t)$ are such that for each t, $u = u(t)$ maximises the function $h(u) = H(p(t), x(t), u)$, and that if $u(t)$ is continuous at t, then:

$$\begin{aligned} \dot{x} = F_1 = {} & \frac{\partial H}{\partial p}, \\ \dot{p} = F_2 = {} & -\frac{\partial H}{\partial x} + \rho p. \end{aligned} \tag{13}$$

Here the right–hand side is evaluated at $(p(t), x(t), u(t))$. That $u = u(t)$ maximises $h(u)$ implies that

$$\frac{\partial H}{\partial u} = \frac{1}{u} + p = 0. \tag{14}$$

Using this relation, equation (13) reads as:

$$\begin{aligned} \dot{x} &= -\frac{1}{p} - bx + \frac{x^2}{x^2+1}, \\ \dot{p} &= \left(\rho + b - \frac{2x}{(x^2+1)^2}\right) p + 2cx, \end{aligned} \tag{15}$$

However, it is often more intuitive to work in state–control variables (x, u). Since equation (14) gives an invertible relation between u and p (as long as both variables are strictly positive), it is possible to switch between (p, x) and (x, u) coordinates. In the latter coordinate system, the equations take the form

$$\begin{aligned} \dot{x} &= u - bx + \frac{x^2}{x^2+1}, \\ \dot{u} &= -\left(\rho + b - \frac{2x}{(x^2+1)^2}\right) u + 2cxu^2. \end{aligned} \tag{16}$$

Both systems will collectively be called the *shallow lake system.*

2.2 Absence of limit cycles

In this subsection, it is shown that there are no limit cycles in a system of the form (13); more generally, there are no invariant bounded regions in the phase space of this system.

Note that system (13) is a parametrised family of autonomous differential equations in the plane. By the Poincaré–Bendixon theorem, the only possible limit sets of a trajectory are equilibria and limit cycles. However, from equation (13) it follows that in (p, x)–coordinates, the vector field F has constant positive divergence ρ, since

$$\operatorname{div} F = \frac{\partial}{\partial x}\frac{\partial H}{\partial p} + \frac{\partial}{\partial p}\left(\rho p - \frac{\partial H}{\partial x}\right) = \rho. \tag{17}$$

Let Φ_t be the flow mapping of the ordinary differential equation (13). It is defined as follows: let $(p(t), x(t))$ be the solution of the equation for given initial condition (p_0, x_0). Then

$$\Phi_t(p_0, x_0) = (p(t), x(t)). \tag{18}$$

A set A_0 of initial conditions is by Φ_t mapped to the set

$$A(t) = \Phi_t A_0 = \left\{(p, x) \, : \, (p, x) = \Phi_t(p_0, x_0) \text{ for some } (p_0, x_0) \in A_0\right\}. \tag{19}$$

We have for the area of $A(t)$ that

$$\frac{\mathrm{d}\ \mathrm{area}\,(A)}{\mathrm{d}t} = \int_{A_0} \operatorname{div} F \; \mathrm{d}S = \rho \int_{A_0} \mathrm{d}S = \rho \ \mathrm{area}\,(A_0), \tag{20}$$

where $\mathrm{d}S$ denotes the standard surface element on $\mathbb{R}^2$. Note that this rules out the existence of invariant bounded regions in the system, since such a region R would have $\mathrm{area}\,(\Phi_t R) = \mathrm{area}\,(R) > 0$ for all t, contradicting the result above. In particular, limit cycles, homoclinic loops and heteroclinic cycles are forbidden, since for all of them, the system would have invariant regions of positive area.

In particular, all bifurcations involving one of these structures, like Hopf or homoclinic bifurcations, are forbidden in the present system. This leaves only saddle–node and heteroclinic bifurcations.

Since all these statements are topological, they are independent of the coordinate system chosen. In particular, they hold for (x, u)–coordinates as well.

2.3 Dynamical analysis of the system

Here and in the next subsection, we look at which kind of local and global qualitative changes (or *bifurcations*) of the system as parameters are varied.

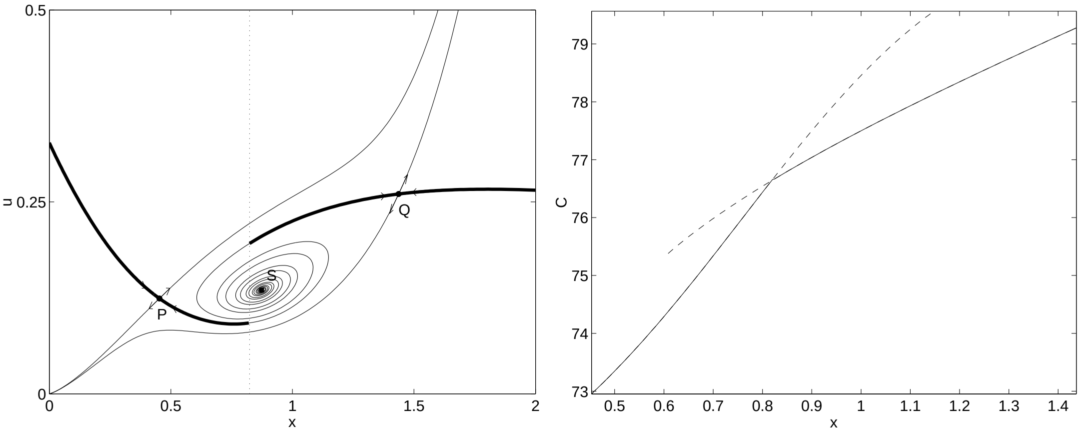


Figure 2: Phase space and cost functions of the shallow lake problem in the Skiba case. *The plot on the left shows phase curves of system (16) for parameter values* $(b, c, \rho) = (0.65, 0.5, 0.03)$*; the optimal solution trajectories are marked by thick lines. The direction of the phase flow is indicated by arrows. The candidates for optimality are the stable manifolds of the saddles* P *and* Q*. In the right–hand picture, the costs of a solution starting on either of these manifolds are given by broken graphs. The lowest value giving the eventual lowest costs is indicated by a full graph. It is found that there is a certain state* x_* *for which the costs for either of the two solutions are equal. This is the Skiba point.*

Figure 2 shows phase space trajectories of the system in (x, u)–coordinates, as well as the location of the three equilibria of the system, two saddles (P and Q), and one source (S), for some given parameter values. Indicated are those curves that correspond to optimal solutions, and the indifference point that exists in this system. Also, the total cost of an optimal solution is plotted as a function of the initial state of the system, showing a discontinuity of the first derivative at the Skiba point (compare Dechert and Nishimura, 1983).

Note that all equilibria are in the set ξ that given as $\{f(x, u) = 0\}$. In the present case, ξ is equal to the graph of the function

$$\Xi : x \mapsto bx - \frac{x^2}{x^2+1}, \tag{21}$$

in (x, u)–space. Note that because of this, equilibria can be ordered by their x coordinates.

Saddle node bifurcations. Saddle node bifurcations — where two separate equilibria coalesce and vanish as parameters cross a saddle node bifurcation manifold — can, by the ordering property, only occur between neighbouring equilibria on the graph of Ξ.

For instance, if b and ρ in the above example are kept fixed, but c is allowed to vary, two saddle–node bifurcations are found. For a certain value of c close to $c = 0.6$, S and Q vanish in a saddle node, while close to $c = 0.2$, S and P undergo a saddle node bifurcation. Compare figure 4.

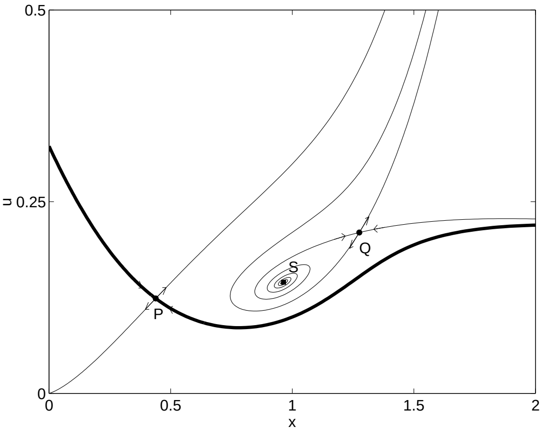


Figure 3: Phase space of the shallow lake system in the oligotrophic case. *Phase curves of system (16) for parameter values* $(b, c, \rho) = (0.65, 0.55, 0.03)$. *Legend as in figure 2.*

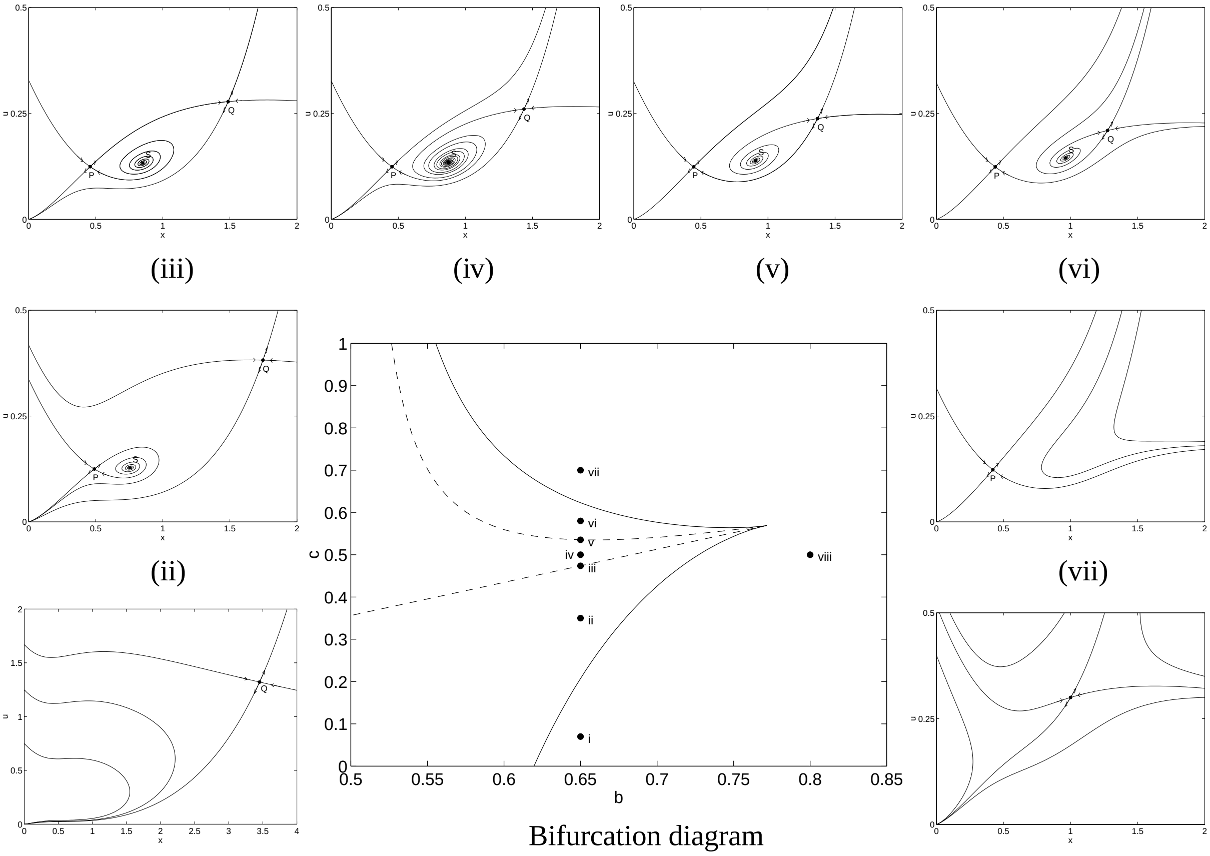
Heteroclinic bifurcations. Figure 3 is made for parameters close to the saddle node bifurcation of S and Q. Note that the phase flow in figures 2 and 3 are qualitatively different: in figure 2, the right–hand stable manifold of P spirals to the unstable equilibrium as $t \to -\infty$, while in the figure 3 it goes off to infinity.

The intermediate situation between these two cases would be that the stable manifold of P approaches Q as $t \to -\infty$, and hence coincides with Q's left hand unstable manifold. This is called a *heteroclinic connection* between P and Q: an orbit that connects two equilibria, lying in the stable manifold of one, and in the unstable manifold of the other. As parameters are varied, for most of them the heteroclinic connection breaks. The set of parameter values for which the system has such a connection is generically a codimension one manifold in parameter space: it is called the set of *heteroclinic bifurcations.*

For instance, if (b, ρ) are kept fixed at $(0.65, 0.03)$, heteroclinic connections are found for c close to 0.53523 and 0.47379. Note that these points are inside the interval defined by the saddle node parameter values, since in its complement, the system has only one equilibrium and heteroclinic orbits are impossible. In the next section, it will be seen that for parameter values in between the two heteroclinic bifurcation values, there are Skiba points in the system, and that this is a general phenomenon.

2.4 The bifurcation diagram

In the examples above, only one parameter has been varied, and only codimension one bifurcations (saddle node and heteroclinic) have been found. To explore the system further, a two–parameter bifurcation diagram



(iii) (iv) (v) (vi)

(ii) (vii)

(i) (viii)

Figure 4: Bifurcation and phase diagrams for the shallow lake system. *Sketch of the* (b, c)*–parameter space of system (16) for* $\rho = 0.03$*, and of the phase space for selected parameter values. In the bifurcation diagram, full lines denote saddle node bifurcations, broken lines heteroclinic bifurcations. The phase space plots (i)–(vii) are drawn for* $b = 0.65$*, and, respectively, (i)* $c = 0.07$*, (ii)* $c = 0.35$*, (iii)* $c = 0.47379$*, (iv)* $c = 0.5$*, (v)* $c = 0.53523$*, (vi)* $c = 0.55$*, and (vii)* $c = 0.7$*. Plot (viii) is drawn for parameters* $(b, c) = (0.8, 0.5)$*. In plots (ii)–(vi), where there are two saddle equilibria, only stable and unstable manifolds of these saddles are drawn. In the other plots, some additional solution curves are shown to indicate the structure of the phase flow. The parameter values for plots (iii) and (v) are heteroclinic bifurcation points, precise up to all given digits. Note finally that the flows in (i), (vii) and (viii) are topologically the same.*

is given in figure 4. Recall that in two dimensions, codimension one manifolds are one–dimensional manifolds, or curves, and codimension two manifolds zero–dimensional manifolds, or points. Hence the sets of saddle node and heteroclinic bifurcations are smooth curves, while the codimension two cusp bifurcation in the diagram occurs as an isolated point.

The results of the previous subsection are illustrated in figure 4, where $\rho = 0.03$. In particular, for typical parameters on the line $b = 0.65$, plots of the phase diagram are given. Note that it looks as though the heteroclinic curves end in the codimension two cusp point. In fact, this is not quite the case, as will be discussed in section 4.

Note that on the right of the curve of saddle node bifurcations, there is only a single equilibrium in the system; on the left, there are three. As will be shown in the next section, for the region on the left of the heteroclinic bifurcation curves, Skiba points exist. The region that is bounded by both heteroclinic and saddle node curves can be divided in an upper and a lower component, where the eutrophic solution or the oligotrophic solution, respectively, is always optimal. In the phase diagrams, the oligotrophic and eutrophic equilibria are consistently labelled P and Q respectively. It will follow from the general theory, developed in the next section, that in diagram (b) of figure 4, the optimal solution to the control problem is given by the stable manifold of Q, the eutrophic equilibrium, whereas in diagram (d), the stable manifold of P is optimal.

3 General theory

This section investigates general systems with one state and one control variable, but depending on several parameters. For a large subclass of these systems, it is shown that the region of parameters for which a Skiba point exists is bounded by codimension one heteroclinic bifurcation manifolds (in the case of the shallow lake, these are the heteroclinic bifurcation curves illustrated in figure 1). This is the main result of the present article.

The next subsection describes the class of control problems to which our theory can be applied; it is shown that the shallow lake problem does indeed fit into this class. In subsections 3.2 to 3.4, some lemmas needed subsequently are derived or quoted. Finally, subsections 3.5 and 3.6 show how heteroclinic bifurcations are related to the boundary of the region of Skiba parameters. The treatment is largely self–contained.

3.1 Assumptions

Here the class of optimal control problems is specified to which the main result, to be formulated below, is applicable. For the sake of convenience, partial derivatives are denoted either by

$$\frac{\partial f}{\partial x}, \quad \text{or by} \quad f_x. \tag{22}$$

The following assumptions are made.

The state equation is of the form:

$$\dot{x} = f(x, u, \lambda), \tag{23}$$

with $x \in X$, $u \in U$, $\lambda \in \Lambda$, where the state space X and the control region U are open subsets of $\mathbb{R}$, and where the parameter space Λ is an open subset of $\mathbb{R}^q$.

All functions encountered in the data of the problem are infinitely differentiable (*i.e.* smooth).

The partial derivative f_u with respect to the control u of the function f is assumed to be non-vanishing for all values of (x, u, λ). For the sake of definiteness, the slightly stronger restriction

$$f_u(x, u, \lambda) > 0, \tag{24}$$

is imposed. It is straightforward to change the arguments in this section for the case $f_u < 0$.

The cost functional of the problem is of the form:

$$\mathcal{C}[x, u] = \int_0^\infty g(x(t), u(t), \lambda) \mathrm{e}^{-\rho t} \, \mathrm{d}t, \tag{25}$$

where $\rho > 0$ is one of the components of the multi–dimensional parameter λ.

The Hamiltonian $H(p, x, u, \lambda)$ of the problem

$$H = -g + pf, \tag{26}$$

is assumed, for (p, x, λ) fixed in $\mathbb{R} \times X \times \Lambda$, to attain its maximum u_* in the interior of U. Moreover, at that point, the second derivative of H is assumed to be strictly negative:

$$H_{uu}(p, x, u_*, \lambda) < 0. \tag{27}$$

Remarks. Intuitively, the condition that $f_u > 0$ expresses the fact that the effect of the control variable has a definite direction. In the present formulation, it always increases the rate of change of the state variable. The last condition, equation (27), ensures that the variable u can always be smoothly expressed in p (see lemma 1).

3.2 Consequences of the maximum principle

This subsection shows that as a consequence of the Pontryagin maximum principle and the above assumptions, there is a one–to–one correspondence between u and p. In other words, expressing the dynamics in state–costate variables is equivalent to expressing them in state–control variables. This is convenient for theoretical reasons.

In order not to overburden the notation unnecessarily, the parameter λ is dropped in this subsection and the next. It is straightforward to incorporate again.

Pontryagin's maximum principle establishes the following necessary relations between $p(t)$, $x(t)$, and the optimising control $u(t)$ for all $t \geq 0$:

$$\begin{aligned} \dot{x} &= H_p &&= f, \\ \dot{p} &= -H_x + \rho p, \end{aligned} \tag{28}$$

and, using equation (27)

$$H_u = -g_u + p f_u = 0, \tag{29}$$

$$H_{uu} = -g_{uu} + p f_{uu} < 0. \tag{30}$$

Relation (29) serves to express p in terms of x and u:

$$p = \frac{g_u}{f_u}. \tag{31}$$

Using this, and the inequality $f_u > 0$ (equation (24)), inequality (30) rewrites as

$$0 > f_{uu} g_u - f_u g_{uu}, \tag{32}$$

In fact, the following lemma holds true.

Lemma 1 *For every fixed x, the function*

$$\varphi(u; x) = \frac{g_u(x,u)}{f_u(x,u)} \tag{33}$$

is differentiable and strictly monotone as a function of u. Hence it has a differentiable inverse $\varphi^{-1}(p; x)$.

Proof
Fixing x and differentiation relation (31) with respect to u yields

$$\varphi_u = \frac{g_{uu} f_u - g_u f_{uu}}{f_u^2} = \frac{g_{uu} - p f_{uu}}{f_u}, \tag{34}$$

where relation (31) has been used in the second equality. From (30), it follows that the numerator is positive; the denominator is nonzero by (24), and hence $\varphi_u \neq 0$ everywhere. ∎

As remarked, this lemma allows us to switch between (p, x) and (x, u) representations of an optimal trajectory.

3.3 Comparing costs

This subsection describes how the total cost of a possible optimal trajectory depends on the trajectory's initial point. In particular, if only two trajectories are compared, there is at most one point where the costs of the two trajectories are equal.

The following lemma is taken from Skiba (1978) (proposition 2).

Lemma 2 *If $\gamma(t) = (x(t), u(t))$ is an optimal trajectory, starting at $\gamma(0) = (x_0, u_0)$, then the cost functional $\mathcal{C}[\gamma]$ equals*

$$\mathcal{C}[\gamma] = -\frac{1}{\rho} H(p_0, x_0, u_0), \tag{35}$$

where $p_0 = \varphi(u_0; x_0)$.

For $\rho > 0$, the right–hand side of (35) defines a function $C(x_0, u_0)$, which will be called the *cost function.* Note the familiar fact that the hard part of the optimal control problem is to determine whether (x_0, u_0) is the initial point of an optimal trajectory. If that is the case, the total cost of that trajectory is given by lemma 2.

Remark that C is defined for all points, not necessarily starting points of optimal trajectories. The next two lemmas express how C depends on its variables.

Lemma 3 *The signs of f and C_u are opposite.*

Proof
Combining (26), (31) and (35) yields

$$\rho\, C(x,u) = g - \frac{g_u}{f_u} f, \tag{36}$$

whenever (x,u) are the starting values of an optimal control. Differentiation with respect to u yields:

$$\rho\, C_u = g_u - \frac{g_{uu} f_u - g_u f_{uu}}{f_u^2} f - \frac{g_u}{f_u} f_u = \frac{f_{uu} g_u - f_u g_{uu}}{f_u^2} f. \tag{37}$$

Inequality (32) now implies the lemma. ∎

The next lemma is a direct corollary of lemma 3.

Lemma 4 *If for given x and λ, the values $u_1 < u_2$ are candidates for starting values of an optimal control, and if they are such that for all $u \in (u_1, u_2)$ we have that $f(x,u) > 0$, then*

$$C(x, u_2) < C(x, u_1). \tag{38}$$

Note that this is equivalent to proposition 5 of Brock and Starrett (1999).

Equation (37) gives an expression for the variation of C with u; consider now the variation of C along solution curves of (28). Note that if $f(x_0, u_0) \neq 0$, there is a smooth function $\tilde{u}(x)$ such that in a small enough neighbourhood of (x_0, u_0), an orbit of the differential equation (28) can be parametrised by $\tilde{\gamma}(x) = \Big(\varphi(\tilde{u}(x); x), x\Big) = (\tilde{p}(x), x)$. Obviously, this might not respect the time parametrisation of the solution trajectory.

Lemma 5 *Assume that $f(x_0, u_0) \neq 0$. Then*

$$\frac{\mathrm{d}C}{\mathrm{d}x} = -p, \tag{39}$$

where the derivative is evaluated at (x_0, u_0) along a trajectory of (28).

Proof
Assume that there is a neighbourhood V of t_0, such that $x(t) \in U$ for all $t \in V$, and that on V, the function $x(t)$ is invertible with inverse $t(x)$. Then:

$$\rho \frac{\mathrm{d}C}{\mathrm{d}x} = -\frac{\mathrm{d}H}{\mathrm{d}x} = -\frac{\mathrm{d}H}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = -\left(H_x \dot{x} + H_p \dot{p}\right)\frac{1}{\dot{x}} = -\left(H_x H_p + H_p(-H_x + \rho p)\right)\frac{1}{H_p} = -\rho\, p \tag{40}$$

Since $\rho > 0$, the lemma follows. ∎

Consider now the situation that two trajectories γ_1 and γ_2 are given, both parametrised by x in some interval of X. That is, $\gamma_j = (p_j(x), x, u_j(x))$ for $j = 1, 2$. Assume that these trajectories are the only candidates for the optimal solution. To compare the respective costs of these trajectories for the same initial state x, let $\Delta(x)$ be

$$\Delta(x) = C(x, u_2(x)) - C(x, u_1(x)). \tag{41}$$

If $\Delta(x) = 0$, there are two optimal solutions starting at x, and from the point of view of minimising costs, a decision maker is indifferent about choosing either of the two.

Lemma 6 *Let there be two solution trajectories γ_1 and γ_2 of (28), candidates for the optimal solution. Assume that there is an interval $[a, b] \subset X$ such that both trajectories can be parametrised by x over that interval. Moreover, let $\Delta(a)\Delta(b) < 0$, where $\Delta(x)$ is given by (41). Then there exists a* unique $x_* \in [a, b]$ *such that*

$$\Delta(x_*) = 0. \tag{42}$$

The indifference point x_* of the lemma is called a *Skiba point.*

Proof
Existence of x_* is clear from the intermediate value theorem. For uniqueness, note that lemma 5 implies that:

$$\frac{\mathrm{d}\Delta}{\mathrm{d}x} = -\left(p_2(x) - p_1(x)\right). \tag{43}$$

The right–hand side vanishes only if $p_2(x) = p_1(x)$. However, because of the theorem of existence and uniqueness of solutions of differential equations, this can only happen if γ_1 and γ_2 coincide. Since $\Delta(a) \neq 0$, that is ruled out. It follows that Δ is strictly monotone (increasing or decreasing as the case may be) as a function of x. ∎

3.4 Dynamical considerations

After analysing the behaviour of the cost function C, this section investigates the dynamical properties of the system (28) in either (p, x) or (x, u)–coordinates. In particular, it is shown that the system has a simple structure: as time goes to infinity, orbits tend either to an equilibrium, or they leave every compact set. Moreover, there are no bounded regions that are invariant under the flow. This puts a restriction on the type of bifurcations that can occur in the system. Finally, the structure of the set of equilibria is discussed.

3.4.1 Limit sets

Lemma 7 *In a system of the form (28), with $\rho > 0$, equilibria are the only possible limit sets of trajectories. Moreover, non–degenerate equilibria are either saddles or sources.*

Proof
Recall that the Poincaré–Bendixon theorem (see Anosov *et al.* (1988), p. 29) states that the limit set of a trajectory of a planar flow is either an equilibrium, a cycle (a closed curve), or a union $\mathcal{U}$ of equilibria and phase curves. These latter phase curves have as their limit sets equilibria of the union $\mathcal{U}$. We have to show that the latter two possibilities cannot occur.

The right–hand side of (28) can be viewed as a vector field $F = (F_1, F_2) = (H_p, -H_x + \rho p)$. The divergence $\operatorname{div} F$ of F equals:

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial p} = H_{px} + \rho - H_{xp} = \rho > 0. \tag{44}$$

It follows that no bounded region of the phase space can be invariant under the phase flow. To see this, let S be a bounded invariant region. Note that the boundary ∂S then consists entirely of phase curves. Green's theorem (Spivak, 1965, page 134) yields for this case that

$$\rho \operatorname{area}(S) = \int\!\!\int_S \operatorname{div} F \, \mathrm{d}x \, \mathrm{d}p = \int_{\partial S} \begin{pmatrix} -F_2 \\ F_1 \end{pmatrix} \cdot \, \mathbf{ds} = 0. \tag{45}$$

Here, the first equality follows from $\operatorname{div} F = \rho$, the second is Green's theorem, and the third is a consequence of the fact that ∂S consists of phase curves and that the vector $F^\perp = (-F_2, F_1)$ is orthogonal to tangent vectors to these phase curves, since they are proportional to $F = (F_1, F_2)$. It follows that $\operatorname{area}(S) = 0$. In particular, limit cycles are ruled out, as well as homoclinic loops or heteroclinic cycles.

To exclude the third type of limit sets, assume there is a phase curve γ having a limit set of that type. Choose a point y on one of the phase curves in the union $\mathcal{U}$ such that $F(y) \neq 0$. Let Σ be a transversal to the flow at y, that is a curve through y such that $F(y)$ is not tangent to Σ at y. Since y is a limit point of γ, there is a monotone increasing sequence $\{t_i\}_{i=1}^\infty$ such that $\gamma(t_i) \in \Sigma$, $\gamma(t) \notin \Sigma$ for $t \in (t_i, t_{i+1})$, and $\gamma(t_i) \to y$.

Consider the region S_i formed by $\gamma_i = \gamma([t_i, t_{i+1}])$ and the part Σ_i of Σ connecting $\gamma(t_i)$ to $\gamma(t_{i+1})$. We apply Green's theorem to obtain

$$\rho \operatorname{area}(S_i) = \int\!\!\int_{S_i} \operatorname{div} F \, \mathrm{d}x \, \mathrm{d}p = \int_{\partial S_i} F^\perp \cdot \, \mathbf{ds}, \tag{46}$$

where ∂S_i denotes the boundary of S_i. Note that the left–hand side of this equality is equal to $\rho \operatorname{area}(S_i)$. Moreover $\operatorname{area}(S_{i+1}) > \operatorname{area}(S_i)$, since the flow has positive divergence.

The right–hand side of (46) splits into:

$$\int_{\partial S_i} F^\perp \cdot \, \mathbf{ds}, = \int_{\gamma_i} F^\perp \cdot \, \mathbf{ds} + \int_{\Sigma_i} F^\perp \cdot \, \mathbf{ds}. \tag{47}$$

As above, the first term vanishes since γ_i is a phase curve. Let K be a compact neighbourhood of y such that for all $i > i_0$, $\gamma(t_i) \in K$, and let $M = \sup_{z \in K} |F(z)|$. The second term can then be estimated by

$$\left| \int_{\Sigma_i} F^\perp \cdot \, \mathbf{ds} \right| \leq M |\Sigma_i|, \tag{48}$$

where $|\Sigma_i|$ denotes the length of Σ_i. Hence, it follows from (46) that

$$0 < \rho \ \operatorname{area}(S_{i_0}) < \rho \ \operatorname{area}(S_i) \leq M |\Sigma_i|. \tag{49}$$

Taking limits, we obtain a contradiction since $M|\Sigma_i| \to 0$ as $i \to \infty$.

Hence, equilibria are the only limit sets possible. At an equilibrium P, the linearisation of the vector field is determined by $A = DF(P)$. Since the sum of the eigenvalues μ_1, μ_2 of A is equal to the trace of A, it follows that

$$\mu_1 + \mu_2 = \mathrm{tr}(A) = \frac{\partial F_1}{\partial x}(P) + \frac{\partial F_2}{\partial p}(P) = \operatorname{div} F(P) = \rho > 0. \tag{50}$$

But then at least one of the eigenvalues has to be positive, and the equilibrium cannot be a sink. ∎

The following lemma is obtained as a corollary of the proof of the preceding lemma.

Lemma 8 *In a parametrised family of differential equations of the form (28), with $\rho > 0$, the only generic codimension one bifurcations that can possibly occur are saddle node and heteroclinic bifurcations.*

Proof
In the proof of the previous lemma, it has been remarked that neither limit cycles nor homoclinic loops can occur. Hence both Hopf and homoclinic bifurcations are forbidden. The only generic codimension one bifurcations left are those mentioned in the lemma. ∎

Remark. Since the conclusions of the previous two lemmas are topological, they are preserved under changes of variables. In particular, they also hold for the formulation of the system in (x, u)–coordinates.

3.4.2 Equilibria

In the remainder of this subsection, a closer look is taken at the location of the equilibria of the system in (x, u)–coordinates, that is of (28) after the change from (p, x) to (x, u)–coordinates:

$$\begin{cases} \dot{x} = F_1(x, u, \lambda) = f(x, u, \lambda) \\ \dot{u} = F_2(x, u, \lambda) \end{cases} \tag{51}$$

For $\lambda \in \Lambda$ fixed, let ξ and η be given as

$$\begin{cases} \xi \ = \{(x, u) \ : \ F_1(x, u, \lambda) = 0\} \\ \eta \ = \{(x, u) \ : \ F_2(x, u, \lambda) = 0\}, \end{cases} \tag{52}$$

The assumption $\frac{\partial F_1}{\partial u} = f_u > 0$ implies that ξ is actually the graph of a function. By the implicit function theorem, there is an open subset A of $X \times \Lambda$, and a smooth function $v : A \to U$, such that ξ equals the graph of v; that is, such that

$$F_1(x, v(x, \lambda), \lambda) = 0. \tag{53}$$

In the following, it will be assumed that $A = X \times \Lambda$ to avoid troublesome technicalities.

Note that because of this property, equilibria of (51) can be ordered by their x–coordinates. So it is assumed that all n equilibria are labelled

$$P_1 = (x_1, u_1), P_2 = (x_2, u_2), \cdots, P_n = (x_n, u_n), \tag{54}$$

such that $x_j < x_k$ if $j < k$.

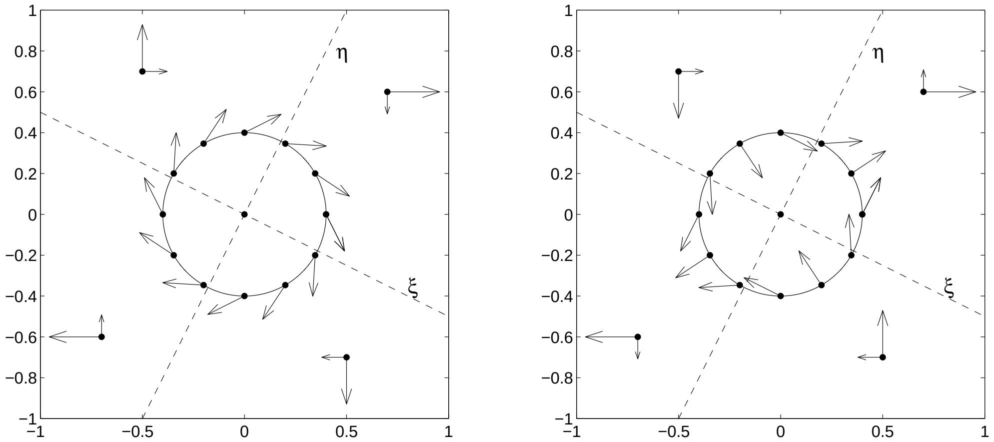


Figure 5: Difference between positive and negative index. *The phase diagram of two vector fields at an equilibrium are shown, respectively $f(x, y) = (x + 2y, 2x - y)$ (left), and $f(x, y) = (x + 2y, -2x + y)$ (right). Four conventional x and y–components of the vector field are given in both pictures, as well as the curves ξ and η where $\dot{x} = 0$ and $\dot{\eta} = 0$ respectively. Moreover, the vector field is given for points on a circle around the equilibrium, showing that it makes a full counter–clockwise turn (left) and a full clockwise turn (right) respectively, as the circle is followed in the counter–clockwise direction. The number of full counter–clockwise turns of the vector field is called the* index *of the equilibrium. Clockwise turns are counted with a minus sign. The equilibrium in the left picture has hence index $+1$, while in the right picture, the index is -1.*

Index. For the following discussion, please see figure 5. Let $V \subset \mathbb{R}^2$ be an open set, $X : V \to \mathbb{R}^2$ a continuous vector field, and $p_0 \in V$ an isolated equilibrium of X, i.e. $X(p_0) = 0$. Assume that $\varepsilon > 0$ is such that p_0 is the only equilibrium contained in the closed ε–ball $\overline{B(p_0, \varepsilon)}$ around p_0. Then $\gamma : [0, 2\pi] \to V$ given by $\gamma(t) = p_0 + (\varepsilon \cos t, \varepsilon \sin t)$ parametrises a circle of radius ε around p_0.

Note that there exist *continuous* functions $\rho, \vartheta : [0, 2\pi] \to \mathbb{R}$ such that

$$X(\gamma(t)) = (\rho(t) \cos \vartheta(t), \rho(t) \sin \vartheta(t)). \tag{55}$$

Since γ is closed, it follows that $X(\gamma(1)) = X(\gamma(0))$, and hence that

$$\vartheta(1) = \vartheta(0) + 2\pi k. \tag{56}$$

The number k is called the *Poincaré index* of the equilibrium p_0 of X. It can be shown (Hirsch, 1976) that it does not depend on the precise form of γ, and that it is a topological invariant of the vector field. Actually, the sum I_X of the indices of all the equilibria of X is a so–called homotopy invariant. That is, if there is an open set V containing all equilibria of X, a parametrised family of vector fields Y_λ with $Y_0 = X$, $Y_1 = \tilde{X}$, and such that for all $\lambda \in [0, 1]$, Y_λ does not have an equilibrium on the boundary ∂V of V, then $I_X = I_{\tilde{X}}$.

Generically – in a typical system – the curves $\xi = \{X_1 = 0\}$ and $\eta = \{X_2 = 0\}$ intersect transversally at an equilibrium p_0. That is, the normal vectors to the respective curves, $\operatorname{grad} X_1$ and $\operatorname{grad} X_2$, span the plane $\mathbb{R}^2$. The index is then equal to the sign of det $DX(p_0)$. See figure 5.

It is easily seen that the signs of the indices of P_j and P_{j+1} alternate for all $j = 1, \cdots, n-1$. Since the index of a source is $+1$, that of a saddle -1, and the only equilibria occurring in the kind of systems under consideration are saddles and sources (by proposition 7), we have the following proposition.

Proposition 1 *Generically, no two equilibria P_j of the family (51) have the same x–coordinates; they can be assumed to be ordered by ascending values of this coordinate. Of two consecutive equilibria, one is a saddle and the other a source.*

3.5 Heteroclinic connections

This subsection investigates the simplest possible configuration for which Skiba points can occur. Since generalising the results obtained here to more complex situations is straightforward, we will limit ourselves to the simple case. At the end of the section, some remarks are made about more complex cases.

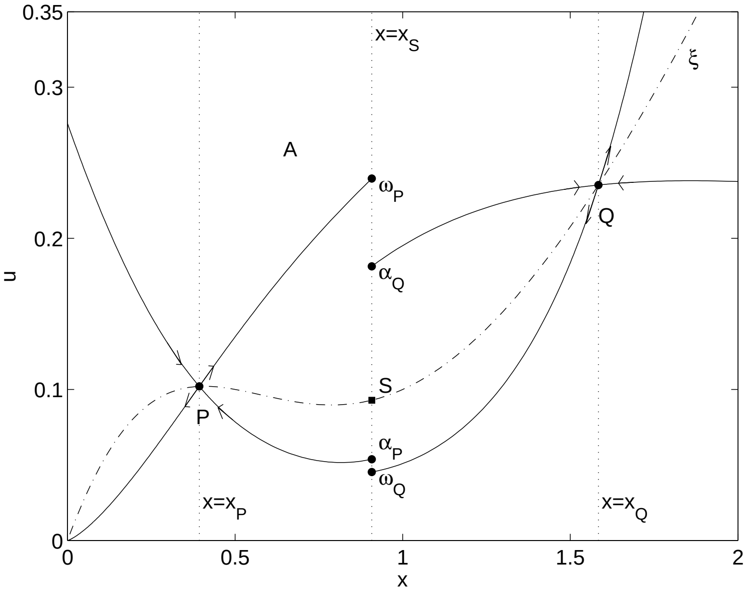


Figure 6: Illustrative sketch of the phase space of the shallow lake system. *Sketch of the phase space of equation (16) for parameter values $(b, c, \rho) = (0.6, 0.5, 0.03)$. Note that if $\alpha_P = \omega_Q$, then there is a heteroclinic connection between Q and P. Likewise if $\alpha_Q = \omega_P$.*

The simplest case. We consider the case that there are three equilibria $P = (x_P, u_P)$, $Q = (x_Q, u_Q)$ and $S = (x_S, u_S)$ of (51). Note that $u_P = v(x_P)$ *etc*, with v as in 3.4.2. Moreover, it is assumed that P and Q are saddles, that S is a source, and that $x_P < x_S < x_Q$ (see figure 6). Consider the region A in phase space, bounded by the lines ℓ_P and ℓ_S, given by $x = x_P$ and $x = x_S$, respectively, and the part of the curve ξ that connects P and S.

Note that phase curves enter A along ℓ_P and ξ, and that they can exit this region only through ℓ_S, excepting the point S itself. Moreover, note that the right–hand part of the local unstable manifold of P enters this region. By extending this local unstable manifold it is found that either the manifold goes to infinity, or it intersects the line ℓ_S at a certain point (x_S, ω_P). To exclude the first case, the following technical assumption is made:

Assumption 1 *Any trajectory starting at a point between the lines ℓ_P and ℓ_Q either stays bounded for all times, or it intersects one of those lines at a finite time.*

This assumption is, for instance, satisfied for the shallow lake system, as is shown in appendix A.4.

Extending the left–hand part of the local stable manifold of Q by integrating the vector field backwards in time, it can be shown that it intersects ℓ_S at some other point (x_S, α_Q).

Define $d_\pm$ as follows:

$$d_+ = |\omega_P - u_S| - |\alpha_Q - u_S|, \quad d_- = |\omega_Q - u_S| - |\alpha_P - u_S|. \tag{57}$$

There are a number of cases:

1. $d_+ \leq 0$ and $d_- \leq 0$. This is impossible, since the region bounded by the respective stable and unstable manifolds and the line ℓ_S would be mapped into itself by the flow of the vector field.
2. *Stable manifold of P optimal.* If $d_+ > 0$ and $d_- < 0$, then the stable manifold of P can be extended from (x_S, α_P) until it intersects ℓ_Q. It is shown below that this stable manifold then gives the optimal solution everywhere. Compare figure 4, (vi).
3. *Stable manifold of Q optimal.* Analogously, if $d_+ < 0$ and $d_- > 0$, the stable manifold of Q can be extended from (x_S, α_Q) until it intersects ℓ_P, and it gives the optimal solution everywhere (figure 4, (ii)).
4. *Heteroclinic connection.* This is the case where either $d_+ = 0$ or $d_- = 0$. The unstable (stable) manifold of P and the stable (unstable) manifold of Q coincide and form a heteroclinic connection (figure 4, (iii) and (v)).
5. *Skiba points.* Finally, in the case that $d_+ > 0$ and $d_- > 0$, the stable manifolds of both P and Q end up spiralling towards S as $t \to -\infty$. Only in this situation are Skiba points found (figure 4, (iv)).

In the next subsection, the statement of case 5 is proved. Note that case 4 is the 'hairline case' between Skiba and non–Skiba configurations.

The general case. In the general case, there are n equilibria, P_1, $P_2, \cdots, P_n$, with $P_j = (x_j, u_j)$. Assume that none of them is degenerate, and that P_1 (and hence P_3, P_5 *etc.*) is a saddle. To determine whether there are any Skiba points, take any a such that $x_1 < a < x_n$ (the choice $a = x_S$ above was for convenience only), and determine the intersections α_{2j+1} and ω_{2j+1} of the stable and unstable manifolds of P_{2j+1}, respectively, with the line $\ell_a = \{(x, u) \,:\, x = a\}$.

Then d_- and d_+ are replaced by the quantities

$$d_+ = \max_{j<k} |\omega_{2j+1} - v(a)| - \min_{j \geq k} |\alpha_{2j+1} - v(a)| \tag{58}$$

and

$$d_- = \max_{j \geq k} |\omega_{2j+1} - v(a)| - \min_{j<k} |\alpha_{2j+1} - v(a)| \tag{59}$$

3.6 Costs

This subsection gives the proofs for the assertions made in the previous subsection.

Global optimality of one stable manifold. In case 3 of the enumeration in the previous subsection, we have that $d_+ < 0$, and the stable manifold of Q can be extended until it intersects ℓ_P (see figure 7). Since the manifold does not intersect ξ, for all $x \in X$ it can be parametrised as the graph of a smooth function $w_Q(x)$. Proposition 4 then implies that

$$C(x_P, u_P) > C(x_P, w_Q(x_P)). \tag{60}$$

Moreover, if $d_+ < 0$, the stable manifold of P cannot extend to the line ℓ_Q, since then the region bounded by the two stable manifolds and the lines ℓ_P and ℓ_Q would be mapped inside itself by the flow, which is impossible (as argued above). Hence there is some state $\tilde{x}_P$ such that the stable manifold of P does not extend past the line $x = \tilde{x}_P$; by the implicit function theorem, the u–coordinate $\tilde{u}_P$ of the point M_P of maximal extension of the stable manifold of P has to be such that

$$M_P = (\tilde{x}_P, \tilde{u}_Q) \in \xi. \tag{61}$$

Again by proposition 4, we have that

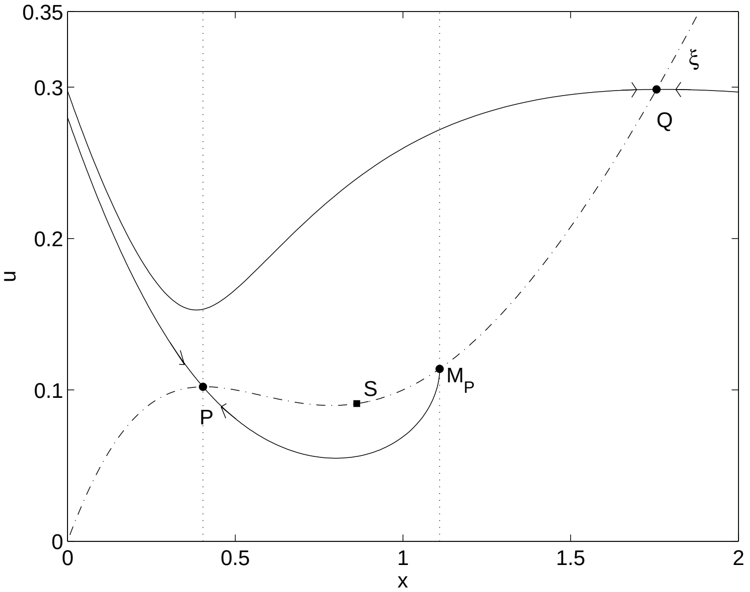


Figure 7: The stable manifold of Q is optimal everywhere. *Sketch of the stable manifolds W^s_P and W^s_Q of P and Q in the case that W^s_Q extends over the complete state space. As argued in the text, W^s_P has to intersect ξ somewhere between S and Q.*

$$C(M_P) > C(\tilde{x}, w_Q(\tilde{x})). \tag{62}$$

Now, by monotonicity of $\Delta(x)$, it follows that the stable manifold of Q is preferable to the stable manifold of P for all states x.

In the same way, it is seen that if $d_- < 0$, the stable manifold of P is preferable to that of Q for all x.

Skiba case. It remains to investigate the case that $d_+ > 0$ and $d_- > 0$, see figure 8. In this case, neither stable manifold can be extended towards the other equilibrium. There are two points $M_P = (\tilde{x}_P, \tilde{u}_P)$, $M_Q = (\tilde{x}_Q, \tilde{u}_Q)$, both on ξ, such that the stable manifolds W_P^s and W_Q^s of P and Q can be extended to M_P and M_Q, respectively, but

$$W_Q^s \cap \{(x,u)\ :\ x < \tilde{x}_Q\} = \emptyset, \qquad W_P^s \cap \{(x,u)\ :\ x > \tilde{x}_P\} = \emptyset \tag{63}$$

Invoking proposition 4 once more, it follows that

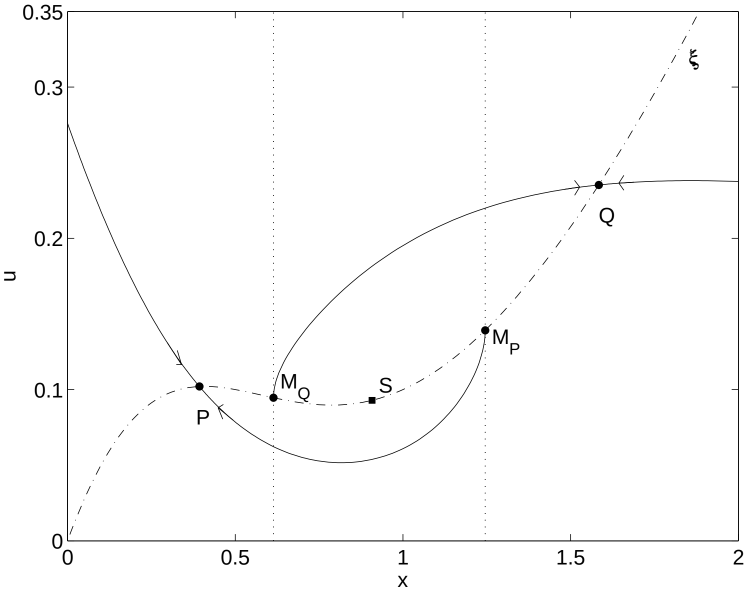


Figure 8: Neither stable manifold is optimal everywhere. *Since both the stable manifold* W_P^s *of* P *and* W_Q^s *of* Q *intersect* ξ *at some point between* S *and* Q*, and* S *and* P*, respectively, it follows that close to* x_P, W_P^s *is optimal, while close to* x_Q, W_Q^s *is optimal. Hence, there is a point* $x_* \in X$ *such that both manifolds are optimal solutions.*

$$C(M_P) > C(\tilde{x}_P, w_Q(\tilde{x}_P)), \tag{64}$$

and

$$C(\tilde{x}_Q, w_P(\tilde{x}_Q)) < C(M_Q), \tag{65}$$

respectively. Here w_P denotes the function whose graph equals the stable manifold of P from P to M_P.

These two inequalities, together with the monotonicity of $\Delta(x)$, ensure that there is a unique Skiba point x_* in the interval $(\tilde{x}_P, \tilde{x}_Q)$. The preceding discussion is summarised in the following proposition.

Proposition 2 *In terms of the notation of the previous two subsections, a necessary and sufficient condition for the occurrence of Skiba points in the interval* (x_P, x_Q) *is that*

$$d_+ > 0 \qquad \textit{and} \qquad d_- > 0. \tag{66}$$

Since $d_{\pm} = 0$ is equivalence to the occurrence of a heteroclinic bifurcation, an immediate consequence is:

Corollary 1 *In parameter space, the parameter region for which Skiba points do occur is bounded by curves of heteroclinic bifurcations.*

4 A local criterion for the occurrence of Skiba points

In this section, a criterion is presented which permits to conclude that Skiba points arise in a given system. This criterion is the occurrence of a certain type of *cusp bifurcation* for the system (13) at $\rho = 0$. Then, general bifurcation theory permits us to conclude the existence of curves of heteroclinic bifurcations close to the cusp bifurcation. Perturbation theoretic arguments then ensure that these bifurcations occur for small but non–zero ρ.

The Hamiltonian situation. Recall that for $\rho = 0$, the system (13) is a Hamiltonian system of differential equations. Hence Hamiltonian bifurcation theory can be brought to bear on it.

From Thom (1972), p. 62, the general fact is taken that if a cusp bifurcation occurs in a Hamiltonian system, then there are *normal form coordinates* such that the Hamiltonian takes the form:

$$H_\mu(p,q) = \eta \frac{1}{2} p^2 + \frac{1}{4} q^4 + \mu_1 \frac{1}{2} q^2 + \mu_2 q + \cdots , \tag{67}$$

where $\eta = \pm 1$, and where the dots denote higher order terms in p and q that can be neglected initially. We are interested in the case $\eta = -1$: the corresponding bifurcation diagram is shown in figure 9. Note that in the bifurcation diagram, a curve of heteroclinic bifurcations appears for $\mu_2 = 0$, corresponding to a situation in phase space as illustrated by figure 10. These bifurcations are non–degenerate, and they will hence persist under small perturbations. Note that for $\mu_1 < 0$ and $\mu_2 = 0$, there are two *simultaneous* heteroclinic connections between the two saddles. The simultaneity is typical for Hamiltonian systems.

The effect of a non–Hamiltonian perturbation. Here the effects of a small generic non–Hamiltonian perturbation of the Hamiltonian cusp normal form system are discussed. Let the perturbation depend on an additional parameter ε, such that for $\varepsilon = 0$, the original Hamiltonian system is obtained. Note that the cusp bifurcation point as well as the saddle–node bifurcation curves persist under the perturbation: that is, they will be at a distance of order $\mathsf{O}(\varepsilon)$ of their unperturbed counterparts. It can be shown that the heteroclinic bifurcation curve splits into two separate curves, one for each heteroclinic connection. These heteroclinic bifurcation curves do not extend to the cusp point any more, as they did in the Hamiltonian case. Moreover, they can intersect only at an isolated point in the bifurcation diagram.

These general remarks can now be applied to a system of the form:

$$\begin{aligned} \dot{x} &= H_p \\ \dot{p} &= \rho p - H_x, \end{aligned} \tag{68}$$

where $H = H(x, p, \lambda)$ depends on some multi–dimensional parameter $\lambda \in \mathbb{R}^q$, $q \geq 2$. Assume that for $\rho = 0$ the system has a non–degenerate cusp bifurcation point λ_0. Note that this occurs in generic (‘most’) two

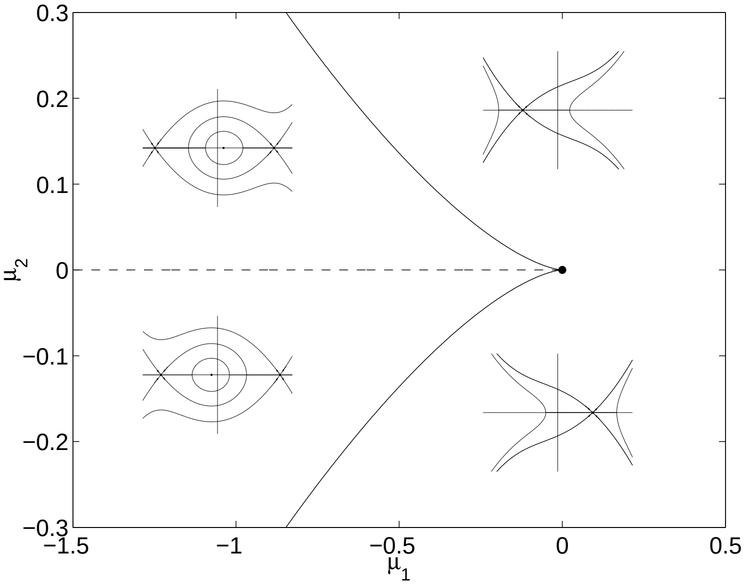


Figure 9: The Hamiltonian cusp bifurcation. *Sketch of the bifurcation diagram of the Hamiltonian cusp bifurcation with normal form* $H_\mu(p,q) = -\frac{1}{2}p^2 + \frac{1}{4}q^4 + \frac{\mu_1}{2}q^2 + \mu_2 q$. *Drawn lines are saddle–node bifurcations, the broken line denotes a heteroclinic bifurcation curve.*

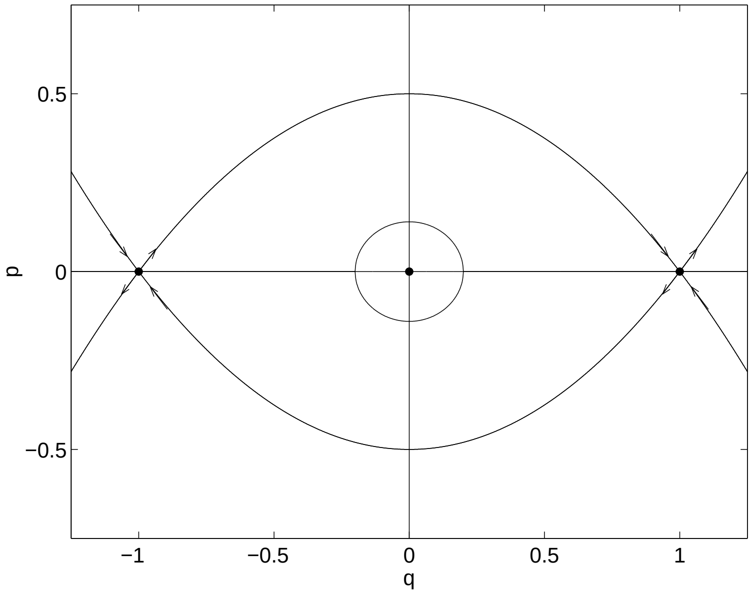


Figure 10: Hamiltonian heteroclinic cycle. *Sketch of the phase space of the Hamiltonian system for* $H_{-1,0}(p,q) = -\frac{1}{2}p^2 + \frac{1}{4}q^4 - \frac{1}{2}q^2$. *Two heteroclinic connections form a heteroclinic cycle.*

(or more) parameter families of Hamiltonians. Recall that for $\rho > 0$, the divergence of the vector field is equal to ρ, and the system is not Hamiltonian any more. By the above discussion, the cusp point as well as the saddle–node curves persist, and the heteroclinic bifurcation curve splits into two curves. Generically, the heteroclinic bifurcation curves cannot end in the cusp bifurcation point, since in a neighbourhood of that point the three equilibria will be on a repelling one–dimensional centre manifold, which excludes the possibility of heteroclinic connections. Note that in our special case, the two heteroclinic curves cannot cross in the bifurcation diagram: that would correspond to two heteroclinic connections existing simultaneously, and the existence of an invariant region bounded by these connections, which was ruled out by the divergence being non–zero.

If quantities d_- and d_+ are defined as in subsection 3.6, it follows that there is an open set of parameters, bounded by the heteroclinic bifurcation points, such that $d_- > 0$ and $d_+ > 0$. For these parameters, the corresponding optimal control problem has a Skiba point in state space.

Hence the following proposition holds:

Proposition 3 *Let the parametrised family of current value Hamiltonians $H_\lambda(p, x)$, $\lambda \in \mathbb{R}^q$, $q \geq 2$ have a cusp bifurcation point λ_0, such that in suitable coordinates*

$$H_{\lambda_0} = -\frac{1}{2}p^2 + \frac{1}{4}q^4 + \cdots . \tag{69}$$

Then there is some $\rho_0 > 0$ such that for all $\rho \in (0, \rho_0)$, there is an open subset of parameters close to λ_0 for which the associated optimal control problem has a Skiba point.

Note. This use of this method is not primarily to be able to prove, for a given system, the existence of Skiba points analytically, though for suitable systems that could be done as well. Rather, if for the case $\rho = 0$ the system (68) a local numerical bifurcation study reports the existence of cusp bifurcation points in the system, then the proposition implies the existence of Skiba points for small but positive ρ.

A Transversality condition

This appendix clears up two technical points. First, in subsections A.1 to A.3, it is shown that the only admissible solution curves for the shallow lake system are the stable manifolds of the two saddles. In particular, it is shown that a solution trajectory of the system of differential equations (16) starting at a point (x_0, u_0) with $x_0, u_0 > 0$ either

1. ends on one of the two saddle points, or
2. gives rise to a control $u(t)$ that goes to infinity in finite time, or
3. does not satisfy the transversality condition at infinity:

$$\lim_{t\to\infty} p(t)\mathrm{e}^{-\rho t} \neq 0. \tag{70}$$

Since integral curves of category 2 are not admissible solutions to the problem, and curves of category 3 do not satisfy the transversality condition, only curves satisfying 1 remain.

Finally, in subsection A.4, assumption 1 in subsection 3.5 is shown to be fulfilled for the shallow lake system.

A.1 Geometrical considerations

The region $R = \{(x, u)\ :\ x > 0,\ u > 0\}$ will be denoted the *open first quadrant* of $\mathbb{R}^2$. To show the above statement, we take an arbitrary initial point (x_0, u_0) in R, and investigate the possible limit behaviour of the solution curve γ starting at this point.

First, remark that for $x = 0$,

$$\begin{cases} \dot{x} = u & \geq \ 0, \\ \dot{u} = -(\rho + b)u & \leq \ 0. \end{cases} \tag{71}$$

Hence the solution curve cannot cross the positive u–axis, and it can only approach this axis at $u = 0$.

Likewise, note that for $u = 0$,

$$\begin{cases} \dot{x} = -bx + \dfrac{x^2}{x^2 + 1} & < \ 0, \\ \dot{u} = 0. \end{cases} \tag{72}$$

It follows that the solution curve cannot cross the positive x–axis either, and, provided that $b > \frac{1}{2}$, it again can only approach it at $x = 0$. Note that $(x, u) = (0, 0)$ would be an equilibrium of the system, if the system were extended to an open neighbourhood of R.

Invoking yet again the Poincaré–Bendixon theorem, it follows that solution curves starting in the open first quadrant either approach a saddle equilibrium inside R (since they cannot approach a source), or approach the point $(0, 0)$ on the boundary of R, or are such that $|\gamma(t)|$ tends to infinity as $t \to \infty$. The first case corresponds to the first case in the above list. The other two are studied in the following subsections.

A.2 Solutions approaching the origin

Assume that γ is such that

$$\lim_{t\to\infty} \gamma(t) = 0. \tag{73}$$

Linearising the differential equations (16) around $(x, u) = (0, 0)$, the following system is obtained:

$$\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} -b & 1 \\ 0 & -(\rho + b) \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \mathsf{O}(2), \tag{74}$$

where $\mathsf{O}(2)$ denotes second and higher order terms in x and u. The matrix on the right–hand side has eigenspaces spanned by $(1, 0)$ and $(1, b)$ respectively. Since the eigenvalue of the latter is the smallest, all solutions not on the invariant manifold connected to the eigenspace $(1, 0)$ will approach the origin as

$$\gamma(t) = (x(t), u(t)) = \mathrm{e}^{-(\rho+b)t}(1, b) + \mathsf{o}(\mathrm{e}^{-(\rho+b)t}). \tag{75}$$

From the form of the equations (16) it follows that the invariant manifold connected to $(1, 0)$ is actually the x–axis. Hence all solution curves starting inside the open first quadrant that tend to 0 actually satisfy (75). This implies for $p(t) = 1/u(t)$:

$$\lim_{t\to\infty} p(t)\mathrm{e}^{-\rho t} = \lim_{t\to\infty} \frac{\mathrm{e}^{-\rho t}}{\mathrm{e}^{-(\rho+b)t}b + \mathsf{o}(\mathrm{e}^{-(\rho+b)t})} = \lim_{t\to\infty} \frac{\mathrm{e}^{bt}}{b + \mathsf{o}(1)} = \infty. \tag{76}$$

Hence the transversality condition at infinity is not satisfied for these solution curves.

A.3 Solutions going to infinity I

Note that it is not possible for solutions going to infinity that $u(t)$ remains smaller than some constant M for all t. For if this were the case, for every t_* such that $x(t_*) = \frac{M+2}{b}$ it would follow that

$$\dot{x} = u - bx + \frac{x^2}{x^2+1} < M - b\frac{M+2}{b} + 1 < -1, \tag{77}$$

and $x(t)$ could not increase beyond $x(t_*)$.

From the results of the previous two subsections, it follows that for solutions tending to infinity, $x(t)$ is bounded away from 0, say $x(t) \geq \delta > 0$ for all t. Hence

$$\dot{u} = -\left(\rho + b - \frac{2x}{(x^2+1)^2}\right)u + 2cxu^2 > -(\rho+b)u + 2c\delta u^2 > c\delta u^2, \tag{78}$$

where the last inequality holds if $u > \frac{\rho+b}{c\delta}$. However, setting $v(t_*) = u_*$, the equation

$$\dot{v} = c\delta v^2, \tag{79}$$

has as solution

$$v(t) = \frac{u_*}{1 - u_* c\delta(t - t_*)}, \tag{80}$$

which goes to infinity in finite time. Since $u(t) \geq v(t)$ for $t \geq t_*$, it follows that $u(t)$ goes to infinity in finite time as well.

A.4 Solutions going to infinity II

It is also not possible for a solution to go to infinity such that $x(t) < M$ for all $t > 0$. If this were the case, then

$$\dot{x} = u - bx + \frac{x^2}{x^2+1} \geq u - m. \tag{81}$$

However, since $(x(t), u(t)) \to \infty$, there exists $T_0 > 0$ such that $u(t) > m + 1$ for all $t > T_0$. Hence, for $T = T_0 + M$, $x(T) \geq M$, contradicting the assumption $x(t) < M$ for all t.

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