Introduction

In this paper we study single-winner elections of a policy or a representative member. Examples illustrating this type of elections are: government presidency, state governorships, city mayor, primaries, chairs of academic departments, etc. We analyze this issue following the citizen-candidate approach due to Besley and Coate (1997) and Osborne and Slivinsky (1996).

The motivation for a citizen-candidate approach is that, generally, only a subset of the members of the group wants to become representative members. The incentives for becoming a candidate are either direct from getting in power and so choosing the policy, or indirect by affecting which candidate may win. In this way, when choosing a representative member, all the individuals face an entry decision. We shall assume therefore, that all the members decide whether to become candidate or not. It implies that in the citizen-candidate approach the candidates are endogenously determined.

In single-winner elections, when only two candidates are at stake, majority principle is the benchmark of democratic decision processes since it elects the candidate supported by a majority. When three or more candidates are at stake, Plurality rule in which the winner candidate is the one obtaining most votes, is one of the most commonly used processes. As pointed out by Borda and Condorcet in 1780s debate, with three or more candidates, Plurality rule may elect a poor candidate, namely one that would lose in pairwise majority comparisons to every other candidate. This is why both authors proposed abandoning altogether Plurality rule. Thus, as proposed by Condorcet (1785) a desirable property for a voting rule is that it elects a candidate who can not be defeated by majority voting, in pairwise elections, by any other candidate, namely a Condorcet winner.

Several studies concern with *exogenous* alternative elections. footnote Many criteria have been proposed for evaluating voting system in this context. One of the most commonly cited criterion is Condorcet Consistency. This criterion holds for a voting system if it selects the Condorcet winner whenever one exists. Some authors like Gehrlein (1992), Lepelley and Merlin (1998) among others, also evaluate voting systems in terms of their Condorcet efficiency - the percentage of elections in which the Condorcet winner is selected.

Few recent papers however, study *endogenous* alternative elections. In this endogenous context Dutta, Jackson and Le Breton (2001) show that all the voting procedures, but the dictatorial, are influenced by the candidates' incentives to exit an election. It means that the outcome of every non-dictatorial voting procedure is modified when the voting alternatives or candidates have the opportunity of withdrawing.

With this paper we aim at showing how the outcome of one of the most controversial and commonly used voting rules, Plurality rule, is modified by the entry-exit strategy of the candidates. More precisely, our objective is to evaluate, in an endogenous context, Plurality rule in terms of Condorcet consistency. In order to guarantee that a Condorcet winner exists we shall restrict our attention to single-peaked preferences. We shall define the Condorcet winner among the self-declared candidates, so that it may not necessarily coincide with the citizen located in the median of the peak distribution.

We find that the strategic entry-exit decision of the candidates eliminates some of the undesirable properties of Plurality rule, namely to elect a poor candidate in three-candidate elections, since as we show, the Condorcet winner is always elected. It is in four or more candidate elections however, where we find that Plurality rule may elect a poor candidate. Thus, our result suggests that up to three candidates, there is no conflict between Plurality rule and Condorcet winner. This result widely contrast the well-known conflict between Plurality rule and Condorcet winner in the standard exogenous three-alternative election.

Our study is made for two different kinds of voting behavior: sincere voting and strategic voting, since we consider that depending on the context the two voting behaviors are equally relevant. Although the outcome of Plurality rule does not coincide when the agent either vote sincerely or strategically, we find that our main results concerning Condorcet consistency are valid for the two kinds of voting behaviors.

We finally also study for the two types of voting behavior, what are the consequences of considering that some coalitions of candidates have the option of withdrawing. We then find that if we allow for coalitions that consist of all but two candidates, Plurality rule always elects the Condorcet winner.

The paper is organized as follows: Section 2 provides an example that motivate our approach. Section 3 is divided into two: Section 3.1 presents a citizen-candidate model based on Besley and Coate (1997) and Osborne and Slivinsky (1996), and in Section 3.2 we adapt the criterion of Condorcet consistency to our context. Section 4 and 5 provides, respectively, the main results for the case of sincere voting and strategic voting. Section 6 explores the consequences of coalition formation among candidates. And finally Section 7 offers a brief conclusion.

Examples

We develop a simple example showing that in three-candidate elections, Plurality rule, in which the winner candidate is the one obtaining most votes, does not always elect the Condorcet winner. We then contrast this result with the case in which candidates have the option of withdrawing.

Example *Consider n voters that rank three candidates A, B and C in the following way*:

Percentage	***Ranking***
n_1	$A \succ B \succ C$
n_2	$B \succ A \succ C$
n_3	$B \succ C \succ A$
n_4	$C \succ B \succ A$

Note that the proposed orderings satisfy the single-peaked preference criterion. We consider ideological candidates, which requires that the preferences of candidate A be of type 1, *i.e.,* $A \succ B \succ C,$ *candidate B either of type* 2 *or* 3, *and finally candidate C of type* 4.
For the sake of simplicity we consider that ties among candidates in pairwise comparisons or in Plurality rule are not possible. footnote Thus, the Condorcet winner is unique.
Any candidate can be the Condorcet winner:
If either candidates A or C are Condorcet winner, Plurality rule also elects these candidates.
If candidate B is the Condorcet winner however, Plurality rule may not always elect this candidate as we next show.
Two conditions guarantee then that B is the Condorcet winner:

$$n_2 + n_3 + n_4 > n_1$$
$$n_1 + n_2 + n_3 > n_4. \qquad \#$$

And for B to be elected by Plurality rule it is required that:

$$n_2 + n_3 > n_1$$
$$n_2 + n_3 > n_4. \qquad \#$$

Several voting percentages can be assigned such that conditions in (*ref: 1*) *hold but the conditions in* (*ref: 2*) *fail to hold. For instance, let* $n_1 = \frac{5}{11},\ n_2 = n_3 = n_4 = \frac{2}{11}.$
We can show that if candidates can withdraw, it is then B the elected candidate. The argument is the following, if A is elected by Plurality rule then, candidate C has incentives to withdraw since then it is B who wins. In the same way, if C is the elected candidate, candidate A has incentives to withdraw since then candidate B wins.

This example illustrates that Plurality rule fails to elect the Condorcet winner when candidates are fixed, and that when candidates can withdraw, Plurality rule may always elect the Condorcet winner. If voters vote strategically, we next show that the same conclusion holds.

Example *Consider the same ranking of Example* 1.
In order to vote strategically, we shall consider coalitions of voters. Consider the following groups of voters: $n_1,\ n_2,\ n_3$ *and* n_4. *Strategic voting requires that i*) *none of these groups vote for its unanimous least preferred candidate and ii*) *given the voting strategy of the other groups, each group of voters votes a best response*.
Thus, from i) *we have that* n_1, n_2 *can vote for either A or B, and* $n_3,\ n_4$ *for either B or C, and from ii*) *no group of voters can improve changing his vote*.
If B is a Condorcet winner, the conditions in (*ref: 1*) *holds. We then consider the following voting strategy*: n_1 *and* n_2 *vote for A*; n_3 *and* n_4 *vote for C. This voting strategy satisfies i*), *and it satisfies ii*) *if the following conditions in* (*ref: 2'*) *and in* (*ref: 2''*) *hold*

$$\text{either} \quad n_2 < n_1 \quad \text{or/and} \quad n_2 < n_3 + n_4 \qquad \#$$

$$\text{either} \quad n_3 < n_4 \quad \text{or/and} \quad n_3 < n_1 + n_2 \qquad \#$$

Several voting percentages can be assigned such that conditions in (*ref: 1*), (*ref: 2''*) *and* (*ref: 2'*) *hold, for instance* $n_1 = \frac{5}{11},\ n_2 = n_3 = n_4 = \frac{2}{11}$. *In this case, the proposed way of voting is strategic but candidate B is not elected by Plurality rule. If candidates can withdraw, the same argument of Example* 1 *applies, furthermore we then have that in two-candidate elections strategic voting coincides with sincere voting*.

In view of Example 1 and 2, the following sections analyze Plurality rule in a model where candidates are endogenously determined, i.e. where candidates face an entry-exit decision.

The Model

We firstly describe in Section 3.1 the citizen-candidate model which is inspired in the ones of Osborne and Slivinsky (1996) and Besley and Coate (1997). And secondly, in Section 3.2 the main definitions of Condorcet winner and Condorcet consistency are adapted to be applied in this context.

The Citizen-Candidate Model

There is an institution composed of a continuum of citizens in the set $M = [0,1]$, denoted by $i \in [0,1]$. The set of policies is assumed to be isomorphic to an interval $[\pi_0, \pi_1]$ where $\pi_0, \pi_1 \geqslant 0$.

Each citizen has an ideal policy π. Thus, π_i is the ideal policy of citizen *i*. It is assumed that π is distributed by means of a strictly positive density function *f*, in the interval $[\pi_0, \pi_1]$.

Each citizen has to decide whether he wants to become a candidate or not. Each citizen entry strategy is given by $e_i \in \{0,1\}$, where 0 indicates that he does not become candidate, and 1 indicates that he does. Thus, *e* is the description of an **entry strategy** for each of the citizens. If a citizen becomes candidate, he shall support his ideal policy, which is common knowledge.

Citizen *i* preferences, given the policy $\pi \in [\pi_0, \pi_1]$, are represented footnote by the function u_i where $u_i(\pi) = -|\pi - \pi_i|$.

The disutiliy assigned to the case in which no one becomes candidate, is such that $u_i(\emptyset) < 0$.

The electoral mechanism follows two stages:

Firstly, each citizen decides whether he wants to become candidate or not, by means of announcing e_i. We denote by $C \subset M$ the **set of candidates**. For each entry strategy *e*, we have that $C = \{i \in M : e_i = 1\}$, and so, the **political alternatives** are described by the set Π_C, where $\Pi_C = \{\pi_i \in [\pi_0, \pi_1] : i \in C\}$.

Secondly, each citizen casts his ballot for one of the self-declared candidates. We denote by α the **voting function**, where for each entry strategy *e*, $\alpha(e)$ describes a voting strategy for all the citizens. Note that different voting behaviors shall induce different voting functions. By Plurality rule, the **winning candidate** is the one obtaining most votes. If some candidates tie, we adapt the

convention of electing the candidate whose political alternative is the closest to π_0, i.e., the smallest one. footnote And just in the case that there are several candidates supporting this policy, we assume that a random device chooses one of them.
Thus, each citizen cast his ballot for one of the candidates in C, taking into account the candidates' political alternatives. The **winning political alternative** is the policy supported by the winner candidate. We denote by P the function that indicates, for each entry strategy e and voting function α, the winning political alternative, formally $P(e,\alpha(e)) = \pi$ (when the context be clear, we shall write $P(e)$).

In summary, the proposed electoral mechanism follows two stages
Stage 1: Citizens make an entry decision
Stage 2: Plurality rule chooses a candidate.

Regarding Stage 1, we shall assume that the entry decision is strategic, no matter what is the voting behavior. The equilibrium concept that we next consider at the entry stage is Nash equilibrium. We denote by e_{-i} the entry strategies for all the agents except for i.

Definition *Given a voting function α, an **entry equilibrium** is given by an entry strategy e, such that $u_i(P(e,\alpha(e))) \geqslant u_i(P((e_i',e_{-i}),\alpha(e_i',e_{-i})))$ for all $i \in M$ and for all $e_i' \in \{0,1\}$.*

Regarding Stage 2, Osborne and Slivinsky (1996) consider sincere voting, while Besley and Coate (1997) assume strategic voting. Austen-Smith (1996) argues that individuals voting behavior should be deduced and not presumed. In our approach, instead of deducing this behavior, we shall study our model for two different kinds of voting behavior: sincere voting and strategic voting.

For each type of voting behavior, we define a political equilibrium in the following way:

Definition *A profile (e,α) is a **political equilibrium** if*:
a) given α, e is an entry equilibrium and
b) the voting function α is defined according to the citizens voting behavior.

For each density function f, the set of political equilibria is denoted by $E(f)$. In the sequel we shall study the political equilibria with $\#C$ candidates where $\#C$ is the cardinality of C that indicates the number of citizens which become candidate.

Condorcet Consistency

The Condorcet criterion, which requires the election of the Condorcet winner when such an alternative exists, is the most commonly used principle to evaluate voting rules. Formally, a political alternative $\pi \in [\pi_0,\pi_1]$ is a Condorcet winner if it can not be defeated by any other political alternative $\pi' \in [\pi_0,\pi_1]$ in pairwise comparisons. *footnote* Thus, following Moulin (1988), a Condorcet consistent voting rule is one that elects the Condorcet winner. In our model, since f is a strictly positive density function, and preferences are single-peaked, a Condorcet winner exists, is unique and is given by the median voter alternative that we denote by π_m. Hence, a *Condorcet consistent* voting rule is one that elects π_m.

Since each of the citizens face an entry decision, we can not guarantee that the median voter becomes candidate. Therefore, we shall propose another consistency criterion such that it takes into account not all the policy proposals but those of the announced candidates. Thus, we define the Condorcet winner in Π_C.

Definition *For each density function f, and for each set of candidates C, the **Condorcet winner in** Π_C is given by the set Π_W where*

$$\Pi_W = \left\{\pi \in \Pi_C : \pi = \underset{\pi}{arg\,min}\ |\pi_m - \pi|\right\}.$$

The set Π_W contains the policies $\pi \in \Pi_C$ that can not be defeated by any other political alternative $\pi' \in \Pi_C$ in pairwise comparisons. footnote In particular, if $\pi_m \in \Pi_C$, we then have that $\Pi_W = \{\pi_m\}$.

We have that either $\#\Pi_W = 1$ or $\#\Pi_W = 2$. In the first case, we use the notation $\Pi_W = \{\pi_n\}$. In the second case, Π_W shall necessarily consist of two political alternatives which are equally distant (or symmetric) to π_m, we then denote the smallest alternative by π_n, so that its symmetric one is given by $2\pi_m - \pi_n$, and thus in this case $\Pi_W = \{\pi_n, 2\pi_m - \pi_n\}$.

In the same way, we shall also refer to the Condorcet loser in Π_C.

Definition *For each density function f, and for each set of candidates C, the* ***Condorcet loser in*** Π_C *is given by* Π_L *where*

$$\Pi_L = \left\{\pi \in \Pi_C : \pi = \underset{\pi}{arg\,max}\ |\pi_m - \pi|\right\}.$$

We next define the Condorcet consistency criterion:

Definition *Let c be the number of candidates. We say that a voting rule is* ***Condorcet consistent in c-candidate elections****, if*
i) for every f and each number of candidates $\#C \leqslant c$, $E(f) \neq \emptyset$
ii) $P(e,\alpha(e)) \in \Pi_W$ *for every* $(e,\alpha) \in E(f)$ *where* $\#C \leqslant c$.

By *i*) we require that for each number of candidates smaller or equal to *c* (there exists a density function) for which a political equilibrium with that number of candidates exists. By *ii*) we require that up to *c* candidates, the voting rule should always elect a Condorcet winner in Π_C.

Focussing our attention on the citizen-candidate model, the next sections study the number of candidates for which Plurality rule is Condorcet consistent.

Sincere Voting Behavior

Given some political alternatives Π_C, under sincere voting each citizen casts his vote for the candidate that supports his most preferred policy in Π_C. And just in the case that several candidates support the same political alternative, we assume that these candidates shall finally obtain equal votes.

Definition *Let* $\alpha_i \in C$ *be citizen i vote, and let* $\pi_{\alpha_i} \in \Pi_C$ *be the political alternative of candidate* α_i*, we then say that a voting function* α *is* ***sincere*** *if for all e,* $u_i(\pi_{\alpha_i}) \geqslant u_i(\pi)$ *for all* $i \in M$ *and for all* $\pi \in \Pi_C$.

Osborne and Slivinski (1996), provide a characterization of the equilibria from one until three candidates when citizens vote sincerely. These authors also assume that candidates face a small entry cost and that some benefits are derived from becoming a winning candidate. We however assume that no additional cost or benefits are derived from becoming candidate. Thus, the kind of equilibria that we find, includes some of the equilibria described by Osborne and Slivinski (1996) and some additional ones.

In our model, the incentives of the candidates to enter the race are either becoming winning candidates or modifying in their favor the winning political alternative. In this last case, note that a candidate can modify the wining policy by means of taking some votes away from a less preferred candidate. And the only incentive of a candidate to exit the race is to modify in his favor the winning political alternative.

Note that assuming that no cost is derived from becoming candidate implies that in equilibrium we can find some “dummy” candidates, i.e., some candidates that do neither modify the finally elected candidate, nor they are winning candidates.

We next provide a description of the obtained political equilibria from one until three candidates.

Proposition *Under sincere voting, every political equilibrium* (e,α) *with three or less candidates, is such that*:
a) $\#C = 1$ *where* $\Pi_C = \{\pi_m\}$ *and* $P(e) = \pi_m$
b) $\#C = 2$ *and*

b.1) $\Pi_C = \{\pi_n, \pi_i\}$ *where* $\pi_n < \pi_m < \pi_i$ *(or* $\pi_i < \pi_m < \pi_n$*) and* $P(e) = \pi_n$

b.2) $\Pi_C = \{\pi_n, 2\pi_m\text{-}\pi_n\}$ *and* $P(e) = \pi_n$

c) $\#C = 3$ *and*

c.1) $\Pi_C = \{\pi_n, \pi_i, \pi_j\}$ *where* $\pi_n < \pi_m < \pi_i \leqslant \pi_j$ *(or* $\pi_i \leqslant \pi_j < \pi_m < \pi_n$*),* $\pi_i < \pi_n < \pi_j$ *and* $P(e) = \pi_n$

c.2) $\Pi_C = \{\pi_n, 2\pi_m\text{-}\pi_n, \pi_i\}$ *where* $\pi_n < 2\pi_m\text{-}\pi_n \leqslant \pi_i$ *and* $P(e) = \pi_n$.

Proof (See Appendix).

In this Proposition we firstly show that up to three candidates no other forms of entry strategies can arise in equilibrium. We secondly propose a distribution function for which each of the proposed entry strategies are supported as political equilibria.

Up to three candidates, we find that every non-winning candidate is a "dummy" candidate since he can not modify the finally elected candidate by means of withdrawing. We find however that most of these "dummy" candidates are necessary for an entry equilibrium since the political alternative that they support guarantees that no other citizen can become a winning candidate by means of entering the race.

In the following theorem we evaluate Plurality rule in terms of Condorcet consistency.

Theorem *If citizens vote sincerely, Plurality rule is Condorcet consistent in three-candidate elections. In four or more candidate elections Plurality rule can choose the Condorcet loser in* Π_C.

Proof (See Appendix).

Thus, up to three candidates, the winning political alternative is always a Condorcet winner in Π_C. The proof of this statement is made by contradiction so that when a winning candidate is not a Condorcet winner, we show that there is always another candidates who strictly improves withdrawing. Note that the intuition of the proof is equivalent to the argument that we follow in Example 1.

For the case of four or more candidates we provide an example of a political equilibrium where the winning candidate is the one that is defeated by every other candidate in pairwise comparisons. Thus, the winning candidate is the most extreme candidate that neither can be defeated by the exit decision of the candidates, nor by the entry decision of any citizen.

Strategic Voting Behavior

When citizens vote strategically, they correctly anticipate the winning political alternative for all the possible voting strategies. We then have that some groups of voters aim at modifying the finally elected candidate.

In our model, since there is a continuum of citizens, we have that no agent can by himself change with his vote the finally elected policy. We therefore consider coalitions of citizens, each of these coalitions is denoted by S.

For each coalition S, let α_S be a voting function for the citizens in S, and let α_{-S} be a voting function for the rest of citizens. We then say that α_S is **weakly dominated** for coalition S, if for some candidate entry strategy e there exists another voting function α'_S such that

$$u_i(P(e, (\alpha'_S(e), \alpha_{-S}(e)))) \geqslant u_i(P(e, (\alpha_S(e), \alpha_{-S}(e)))) \text{ for all } i \in S$$
$$\text{and for all } \alpha_{-S}, \text{with inequality strict for some } \alpha_{-S}.$$

As proposed by Besley and Coate (1997) we shall rule out the use of weakly dominated footnote voting functions, it basically implies that no coalition votes for its unanimous least preferred candidate, when such a candidate exists.

Given an entry strategy e, and given $\alpha_{-S}(e)$, we say that $\alpha_S(e)$ is a **best response** for coalition

S if there is no $\alpha'_S(e)$ such that

$$u_i(P(e,(\alpha'_S(e),\alpha_{-S}(e))) \geqslant u_i(P(e,(\alpha_S(e),\alpha_{-S}(e))) \text{ for all } i \in S,$$
$$\text{with inequality strict for some } i \in S.$$

Note that a sincere voting function may not fulfill this condition since some entry strategy and a family of coalitions can be always defined such that a coalition may strictly improve casting his vote for another candidate. Thus, from the above considerations we define strategic voting.

Definition *We say that a voting function α is **strategic**, if for every coalition S*
i) α_S is not a weakly dominated voting function.
ii) $\alpha_S(e)$ is a best response for all e.

With regard to the coalitions, we shall consider coalitions of citizens with alike preferences, i.e., citizens which ideal policy belongs to the same political interval. A benchmark family of coalitions is the sincere coalition that we next define.

Definition *Given a candidate set C, a **partition** of the citizens $(S_i^C)_{i\in C}$ is said to be **sincere** if $j \in S_i^C$ implies that $u_j(\pi_i) \geqslant u_j(\pi_k)$ for all $\pi_k \in \Pi_C$.*

Thus, each sincere partition generates a family of sincere coalitions. In particular, the **sincere coalition** S_i^C is given by the group of citizens that supports candidate i when voting sincerely. footnote

A feature of the sincere partitions is that it depends on the candidates' political alternatives, so that it can only be formed once the entry strategy has been announced. It is then, when the members of these coalitions can reach some agreements regarding the voting strategy. Therefore, the timing implicitly considered is as follows: firstly the candidates are self-declared, and secondly the coalitions are formed.

We next provide a description, from one until three candidates, of the obtained equilibria under strategic voting.

Proposition *Under strategic voting with sincere coalitions, every political equilibrium (e,α) with three or less candidates is such that:*
***a**) $\#C = 1, \Pi_C = \{\pi_m\}$ and $P(e) = \pi_m$*
***b**) $\#C = 2$ and*
***b.1**) $\Pi_C = \{\pi_n,\pi_i\}$ where $\pi_n \leqslant \pi_m < \pi_i$ (or $\pi_i < \pi_m \leqslant \pi_n$), $\pi_n = \pi_m = \pi_i$*

$$\text{and } P(e) = \pi_n$$

***b.2**) $\Pi_C = \{\pi_n, 2\pi_m\text{-}\pi_n\}$ and $P(e,\alpha(e)) = \pi_n$*
***c**) $\#C = 3$, and*
***c.1**) $\Pi_C = \{\pi_n,\pi_i,\pi_j\}$ where $\pi_n \leqslant \pi_m < \pi_i \leqslant \pi_j$ (or $\pi_i \leqslant \pi_j < \pi_m \leqslant \pi_n$),*
$\pi_i < \pi_n < \pi_j$, $\pi_i = \pi_n \leqslant \pi_m < \pi_j$ (or $\pi_i < \pi_m \leqslant \pi_n = \pi_j$), $\pi_n = \pi_m = \pi_i = \pi_j$

$$\text{and } P(e) = \pi_n$$

***c.2**) $\Pi_C = \{\pi_n, 2\pi_m\text{-}\pi_n, \pi_i\}$ where $\pi_n < 2\pi_m\text{-}\pi_n \leqslant \pi_i$ (or $\pi_i \leqslant \pi_n < 2\pi_m\text{-}\pi_n$)*

$$\text{and } P(e,\alpha(e)) = \pi_n.$$

Proof (See Appendix).

In this Proposition we firstly show that no other entry strategies can be supported as political equilibria. We secondly show that the proposed forms of equilibria exist for every distribution function of π.

Comparing Propositions 1 and 2 we find that up to three candidates the descriptions of equilibria obtained under sincere voting are also among the descriptions of equilibria under strategic voting with sincere coalitions. We do not know, however if every entry equilibria under sincere voting can be also supported as an entry equilibria under strategic voting with sincere coalitions. This item is explored in the following proposition.

Proposition *An entry equilibrium e where citizens vote sincerely is also an entry equilibrium where citizens vote strategically with sincere coalitions, if and only if* $\#C \leqslant 3$.

Proof (See Appendix).

In the above Proposition we firstly show that voting sincerely is not weakly dominated and that it is a best response if and only if there are no more than two candidates. Up to three candidates, we secondly show that for every entry equilibrium under sincere voting, a strategic voting function can be proposed such that this entry strategy be also an entry equilibrium under strategic voting.

For the case of four candidates we finally provide an example of an entry equilibrium under sincere voting that can not be supported as an entry equilibrium under strategic voting with sincere coalitions. In the proposed example, no citizen has incentives to enter the race when voting sincerely since none of them can neither win the elections nor modify the winning policy in his favor. When voting strategically however, we show that there is a citizen that can enter the race and form a sincere coalition that switches the winning policy in his favor by means of casting his vote for another candidate. Thus, this example illustrates that under strategic voting the citizens have additional incentives to enter the race.

Concerning Condorcet consistency, in the following theorem we explore Plurality rule under strategic voting with sincere coalitions.

Theorem *If citizens vote strategically with sincere coalitions, Plurality rule is Condorcet consistent in three-candidate elections. Furthermore, in four or more candidate elections Plurality rule can choose the Condorcet loser in* Π_C.

Proof (See Appendix).

Following a similar argument to that of Theorem 1 we show that up to three candidate elections, the Condorcet winner in Π_C is always elected. For the case of four or more candidates we also provide an example of a political equilibrium footnote where the winning candidate is the one that is defeated by every other candidate in pairwise comparisons. Thus, we have that the same results obtained under sincere voting also hold when we instead consider strategic voting with sincere coalitions.

So far, we have just considered an only way of forming coalitions which consist of joining together all the citizens whose most preferred candidate coincides. We find however that some different arguments can be followed to justify coalition formations. Thus, given an entry strategy we shall consider partitions of citizens into coalitions such that no group of citizens strictly improves forming another coalition. Formally, we define this partitions as follows.

Definition *Let N and N' be natural numbers. Given (e,α) a* ***partition*** *of the citizens $(S_n)_{n=1}^{N}$ is said to be* ***strong*** *if there is no other partition $(S_n^{'})_{n=1}^{N'}$ and a coalition of this partition $S_n^{'}$ such that*
$u_i(P(e,(\alpha'_{S'_n}(e),\alpha_{-S'_n}(e))) \geqslant u_i(P(e,\alpha(e)))$ *for some strategic voting function $\alpha'_{S'_n}$ and for all $i \in S'_n$ with inequality strict for some i.*

Thus, for each candidate set, a partition is strong if no other partition can be formed where the members of a new coalition improve by means of changing their voting strategy. If a coalition is formed, we consider that the rest of citizens, i.e. those in $-S'_n$, follow the same voting strategy α, no matter whether the size of their coalition has changed.

Every strong partition generates a family of strong coalitions. Note that the strong coalitions as opposed to the sincere coalitions introduce an additional requirement to the political equilibrium that consist on checking the deviations of coalitions of citizens.

In the following theorem we explore Plurality rule under strategic voting with strong coalitions.

Theorem *If citizens vote strategically with strong coalitions, Plurality rule is Condorcet Consistent in every candidate election. Furthermore the winning political alternative is always given by* π_m.

Proof Two statements should be proven:

Claim 1: For every f and every number of candidates, $E(f) \neq \emptyset$.

Given f, we next propose several entry strategies, a partition of the population and a voting function for which we have that $E(f) \neq \emptyset$..

Consider that $\Pi_C = \{\pi_m\}$, or $\Pi_C = \{\pi_m, \pi_i\}$ where $\pi_i < \pi_m$, or in general $\Pi_C = \{\pi_m, \pi_i, \pi_j, \pi_k, ...\}$ where all the additional candidates π_j, π_k... are such that $\pi_j, \pi_k < \pi_m$. For every entry strategy, we consider the partition (S_1) where all the citizens are in the unique coalition S_1. And for every e, we consider that α is such that all the citizens vote for π_m. If $\Pi_C = \{\pi_m\}$ it follows directly that (S_1) is a strong partition. Otherwise, given e there is a majority of citizens who strictly prefer π_m to other policy π_i, so that if the rest of citizens form a coalition S_1' we have that $P(e, (\alpha'_{S_1'}(e), \alpha_{-S_1}(e)) = \pi_m$ for every possible $\alpha'_{S_1'}$. It implies that no agent in S_1' can strictly improve by means of forming this coalition and so, (S_1) is a strong partition.

Given α, we have proposed an entry equilibrium: If a citizen with preferred policy $\pi \neq \pi_m$ enters the race, following the above argument we have that no coalition of citizens can modify the winning political alternative, so that this candidate can not defeat π_m. If a citizen withdraws, by the voting policy α, policy π_m remains as the winning policy.

Finally, let us show that α is strategic. It requires that *i*) α_{S_1} should not be a weakly dominated voting function and *ii*) $\alpha_{S_1}(e)$ should be a best response for all e. Since there is only a coalition S_1, it follows directly that *i*) holds. Condition *ii*) also holds since for every e, a voting strategy $\alpha'_{S_1}(e)$ which strictly improves at least an agent requires that the winning political alternatives be different from π_m, it however implies that some citizens in S_1 will be then worse off.

We then conclude that given the partition (S_1) we have that $(e, \alpha) \in E(f)$ where e contains every possible number of candidates.

Claim 2: For every $(e, \alpha) \in E(f)$ we have that $P(e, \alpha(e)) = \pi_m$.

Suppose **firstly** to the contrary that e is such that $\pi_m \notin \Pi_C$. Since $(e, \alpha) \in E(f)$, we have that e is an entry equilibrium. It then follows that if the median citizen enters the race

$$u_m(P(e'_m, e_{-m}), (\alpha(e'_m, e_{-m}))) \leqslant u_m(P(e, \alpha(e)) \text{ where } e'_m = 1.$$

It implies that there exists a strong partition $(S_n)_{n=1}^N$ which votes strategically and such that $P(e', \alpha(e')) \neq \pi_m$ where $e' = (e'_m, e_{-m})$. By definition of the median citizen, more than half of the citizens shall prefers π_m to any other political alternative. If these citizens form a coalition S_n' such that given e' they vote altogether for the median citizen, we have that π_m becomes the winning policy. It implies that

$$u_i(P(e', (\alpha'_{S_n'}(e'_m, e_{-m}), \alpha_{-S_n'}(e'))) > u_i(P(e', \alpha(e')) \text{ for all } i \in S_n'$$

which contradicts that there is a strong partition $(S_n)_{n=1}^N$ for which $P(e', \alpha(e')) \neq \pi_m$. Thus, in every entry equilibrium $\pi_m \in \Pi_C$.

Suppose **secondly** to the contrary that the political equilibrium (e, α) is such that $P(e, \alpha(e)) \neq \pi_m$ (where we know that $\pi_m \in \Pi_C$). Since $(e, \alpha) \in E(f)$, there must be an strong partition $(S_n)_{n=1}^N$ and a voting function α for which $P(e, \alpha(e)) \neq \pi_m$. Consider then a coalition that consists of all the agents which strictly prefer policy π_m to the winning policy $P(e, \alpha(e))$. We denote this coalition by S_n' and it clearly contains more than half of the agents, so that voting altogether for π_m we have that $P(e, (\alpha'_{S_n'}(e), \alpha_{-S_n'}(e)) = \pi_m$. it however contradicts that there exists a strong partition $(S_n)_{n=1}^N$ for which $P(e, \alpha(e)) \neq \pi_m$. We conclude therefore that in every political equilibrium where citizens vote strategically with strong coalitions we have that $P(e, \alpha(e)) = \pi_m$.

In this Theorem we firstly show that for every number of candidates equilibrium exists. We also show that the median citizen always have incentives to enter the race since the, every strong partition and voting function should be such that π_m becomes the winning policy. Note that otherwise, a coalition containing the median citizen can always defeat any other winning political alternative.

Conclusions

Besley and Coate (1997) show that there is no equilibrium with three candidates. This negative result is due to considering that an additional cost is derived from becoming candidate, since this cost implies that the candidates who does not affect the final policy may always have incentives to withdraw. As we next shown, when no cost or benefit is derived from becoming candidate, strategic voting provides several political equilibria with three candidates.

Social choice theorists have widely focus on voting systems with exogenous alternatives or candidates. The many voting methods have been evaluated by many criteria. Our research compares the properties that a voting rule satisfies when applied to a standard voting model than when applied to the citizen candidate approach.

Our proposal basically consist on analyzing voting methods when a previous entry decision stage is taken into account. This proposal has potential applications to those institutions which shall elect a representative member: political parties, labor unions, associations, etc., as well as to democratic policy choices such as education or health care.

We have analyzed Plurality rule, and in order to evaluate this voting rule in a citizen-candidate model, we have adapted the Condorcet consistency criterion and the Condorcet Loser criterion to a two-stages setting. Thus, in Table 1 we provide a comparison of our results and those of Plurality rule in the standard approach (or one-stage approach):

Voting Behavior	*Standard Approach*	*Citizen-Candidate*	Condorcet Consistency	Condorcet Loser
Sincere voting	$\#C \leqslant 2$	$\#C \leqslant 3$	Yes	
	$\#C > 2$	$\#C > 3$		Yes
Strategic voting	$\#C \leqslant 2$	$\#C \leqslant 3$	Yes	
	$\#C > 2$	$\#C > 3$		Yes

Table 1: A comparison of the properties satisfied by Plurality rule

Thus, taking into account that many elections take place among three candidates, from a comparison between the two approaches, we can assert that the properties of Plurality rule drastically change when we consider a citizen-candidate approach. While in the standard approach a Condorcet loser can be elected in three-candidate elections, when an entry-decision stage is introduced, the Condorcet winner is always elected.

Hence, our main finding is that the strategic entry decision of the candidates eliminates one of the most undesirable properties of Plurality rule, namely to elect a poor candidate in three-candidate elections. We have shown that by means of withdrawing each candidates has the option of transferring his votes to other candidates. Thus, when in three-candidate elections the Condorcet winner in Π_C is not elected, there is always another candidate with incentives to withdraw. This result holds whether citizens vote sincerely or strategically with sincere coalitions.

When voting strategically we have considered that around each candidate a coalition of citizens is set. This type of coalitions, that we call sincere coalitions, provides an additional incentive to each citizen to enter the race since such coalition can vote for another candidate modifying in this way the winning policy. It is precisely this additional incentive which provokes that in four or more candidate-elections not every entry equilibrium under sincere voting be an entry equilibrium under strategic voting with sincere coalitions.

We have finally shown that when candidates can form coalitions (consisting on at most #C-2 candidates) to decide their entry strategy, Plurality rule always elects the Condorcet winner in Π_C.

One of the strategies of these coalitions of candidates is to withdraw all together. Thus, with this strategy these coalitions have the option of transferring all their votes to the Condorcet winner in Π_C who then obtains a majority of the votes that defeats every other candidate. Note that this type of coalitions requires a great coordination among competing candidates which is not always feasible.

As a result, our approach establishes some links between the classical theory which evaluates voting methods and the most recent models of policy choice in representative democracy. Thus, we have evaluated an endogenous model of candidate or policy choice from a basic axiomatic point of view.

Extensions which incorporate other ways of forming coalitions, as well as other voting methods or axioms may provide some greater insights into the study of democratic candidate-choices or policy-choices. In particular we let for future research the study of other Scoring rules different from Plurality rule in terms of Condorcet consistency.

Appendix

Proof of Proposition 1:

We **firstly** show by the following claims, that no other forms of political equilibria are possible.

Claim 1: There is no political equilibrium where #C=1 and $\Pi_C \neq \{\pi_m\}$.

Suppose to the contrary that such an entry equilibrium e exists where w.l.g. $\pi_n < \pi_m$. Let $j \in M$ be such that $\pi_j = \pi_n + \varepsilon$, where $\varepsilon > 0$ and close enough to 0. This citizen has incentives to become candidate since by sincere voting $P(e_j', e_{-j}) = \pi_j$ where $e_j' = 1$. It contradicts that e is an entry equilibrium.

Claim 2: There is no political equilibrium where #C=2 and $\Pi_C = \{\pi_n, \pi_i\}$ such that $\pi_n \leqslant \pi_m$ and $\pi_i \in [\pi_0, \pi_n]$ (or symmetrically where $\pi_n \geqslant \pi_m$ and $\pi_i \in [\pi_n, \pi_1]$).

It can be shown following the same argument of Claim 1.

Claim 3: There is no political equilibrium where #C=2, $\Pi_C = \{\pi_n, 2\pi_m - \pi_n\}$ and $P(e) = 2\pi_m - \pi_n$.

Since π_n and $2\pi_m - \pi_n$ are equidistant to π_m, by sincere voting both candidates tie, and then by convention $P(e) = \pi_n$.

Claim 4: There is no political equilibrium where #C=3 and $\Pi_C = \{\pi_n, \pi_i, \pi_j\}$, $\pi_i \leqslant \pi_j \leqslant \pi_n \leqslant \pi_m$ (or symmetrically $\pi_m \leqslant \pi_n \leqslant \pi_i \leqslant \pi_j$).

It can be shown following the same argument of Claim 1.

Claim 5: There is no political equilibrium where #C=3 and $\Pi_C = \{\pi_n, \pi_i, \pi_j\}$, $\pi_i = \pi_n < \pi_j$ (or symmetrically where $\pi_i < \pi_n = \pi_j$).

If $P(e) = \pi_j$, candidate i has incentive to withdraw since then by sincere voting $P(e_i', e_{-i}) = \pi_n$ where $e_i' = 1$, which contradicts that e is an entry equilibrium. If $P(e) = \pi_n$, it implies that candidates i and n obtain together at least a percentage $\frac{2}{3}$ of the votes. Furthermore, this entry equilibrium requires that if a citizen $k \in M$ with preferred policy $\pi_k \in [0, \pi_n]$ enters the race, he can not obtain a percentage of votes greater or equal to $\frac{1}{3}$. Thus, this entry equilibrium requires that the percentage of votes contained in $[0, \pi_n]$ be at most $\frac{1}{3}$. Consequently, in $\left[\pi_n, \frac{\pi_n + \pi_j}{2}\right]$ there must be at least another percentage $\frac{1}{3}$ of the votes. We then have that for every strictly positive density function, a citizen $\ell \in M$ with policy $\pi_\ell = \pi_n + \varepsilon$, ($\varepsilon > 0$ and close to 0), can get a percentage of votes as close to $\frac{1}{3}$ as we want (as ε approaches 0). Furthermore, if such citizen enters as candidate, each of the other candidates obtains a percentage of votes strictly smaller than $\frac{1}{3}$, so that $P(e_\ell', e_{-\ell}) = \pi_\ell$ where $e_\ell' = 1$. It contradicts that e is an entry equilibrium.

Claim 6: There is no political equilibrium where #C=3 and $\Pi_C = \{\pi_n, 2\pi_m\text{-}\pi_n, \pi_i\}$ $\pi_i \leqslant \pi_n < 2\pi_m\text{-}\pi_n$.

Since π_n and $2\pi_m\text{-}\pi_n$ are equidistant to π_m, and $\pi_i \leqslant \pi_n$, by sincere voting $P(e) = 2\pi_m\text{-}\pi_n$. However, candidate i has incentives to withdraw, since then $P(e_i', e_{-i}) = \pi_n$ where $e_i' = 0$.

Claim 7: There is no political equilibrium where #C=3 and $\Pi_C = \{\pi_n, 2\pi_m\text{-}\pi_n, \pi_i\}$, $\pi_n < 2\pi_m\text{-}\pi_n \leqslant \pi_i$ and $P(e) = 2\pi_m\text{-}\pi_n$.

Since π_n and $2\pi_m$-π_n are equidistant to π_m and $2\pi_m$-$\pi_n \leqslant \pi_i$ by sincere voting $P(e) = \pi_n$.
We **secondly** show that the proposed forms of political equilibria exist.
Let *f* be a uniform distribution function. To simplify notation we normalize the policy space so that $\pi_0 = 0$ and $\pi_1 = 1$. We propose the following equilibria configurations:
a) $\Pi_C = \left\{\frac{1}{2}\right\}$, since $\pi_m = \frac{1}{2}$, no citizen can defeat the Condorcet winner entering the race. Furthermore, he has no incentives to withdraw since $u_m(\emptyset) < u_m(\pi_m)$.
b.1) $\Pi_C = \{\pi_n, \pi_i\}$ where $\pi_n = \frac{1}{4} + \varepsilon$ ($\varepsilon > 0$ and close enough to 0), $\pi_i = \frac{3}{4}$ where $P(e) = \pi_n$. We show that no citizen has incentives to enter: If $j \in M$ is such that $\pi_j \in \left[0, \frac{1}{4} + \varepsilon\right)$, he gets at most a percentage $\frac{1}{4} + \gamma$ (with $0 < \gamma < \varepsilon$) of the votes, so that it is then candidate *i* who wins. If $j \in M$ is such that $\pi_j \in \left(\frac{3}{4}, 1\right]$ he gets at most a percentage $\frac{1}{4} - \gamma$ (with $\gamma > 0$) of the votes, so that the winning policy is not modified. Finally, for $\pi_j \in \left[\frac{1}{4} + \varepsilon, \frac{1}{2} - \frac{\varepsilon}{2}\right)$ it is then candidate *i* who wins, and for $\pi_j \in \left[\frac{1}{2} - \frac{\varepsilon}{2}, \frac{3}{4}\right]$ the winning policy is not modified. Since the citizens who strictly prefer policy π_i to π_n belong to the interval $\left(\frac{1}{2} + \frac{\varepsilon}{2}, 1\right]$, no citizen has incentives to enter the race. Furthermore, if candidate *i* withdraws, the winning policy is not modified.
b.2) $\Pi_C = \{\pi_n, 2\pi_m\text{-}\pi_n\}$ where $\pi_n = \frac{1}{4}$, $2\pi_m$-$\pi_n = \frac{3}{4}$ and $P(e) = \pi_n$. We show that no citizen has incentives to enter: If $j \in M$ is such that $\pi_j \in \left[0, \frac{1}{4}\right)$, he gets at most a percentage $\frac{1}{4} - \varepsilon$ (with $\varepsilon > 0$ and close enough to 0) of the votes, so that it is then candidate $2\pi_m$-π_n who wins. If $j \in M$ is such that $\pi_j \in \left(\frac{3}{4}, 1\right]$ he also gets at most a percentage $\frac{1}{4} - \varepsilon$ of the votes, so that the winning policy is not modified. Finally, for $\pi_j \in \left(\frac{1}{4}, \frac{1}{2}\right]$ it is then candidate $2\pi_m$-π_n who wins, and for $\pi_j \in \left(\frac{1}{2}, \frac{3}{4}\right]$ the winning policy is not modified. Since the citizens who strictly prefer policy $2\pi_m$-π_n belong to the interval $\left(\frac{1}{2}, 1\right]$ no citizen has incentives to enter the race. Furthermore, if candidate $2\pi_m$-π_n withdraws, the winning policy is not modified.
c.1) Two types of political equilibria are considered:
c.1.1) $\Pi_C = \{\pi_n, \pi_i, \pi_j\}$ where $\pi_n < \pi_m < \pi_i \leqslant \pi_j$ (or symmetrically $\pi_i \leqslant \pi_j < \pi_m < \pi_n$). We define the parameter $d = \frac{1}{4} + \frac{\varepsilon}{2}$ ($\varepsilon > 0$ and close enough to 0) so that $\pi_n = d$, $\pi_i = 3d$, $\pi_j \in [\pi_i, 1]$ and $P(e) = \pi_n$. We then have that π_n obtains a percentage $2d$ of the votes and candidates *i* and *j* obtain respectively a smaller percentage. Let us show that no candidate has incentives to enter: We have that for all $k \in M$ such that $\pi_k \in [0, 2d]$, the policy π_n is strictly preferred to π_i and π_j, furthermore, if any of them enters the race, he gets at most a percentage *d* of the votes, that in no case can defeat candidate *n*. It implies that they do not have incentives to become candidate. For all $k \in M$ such that $\pi_k \in (2d, 1]$ we have that policy π_i or π_j is strictly preferred to π_n. However, if one of these citizens *k* enters the race, he can not modify the winning policy since candidate *n* always obtains at least a percentage $\frac{3}{2}d$ of the votes, which is greater than the percentage obtained by each of the other candidates. Thus, no candidate $k \in M$ can strictly improve becoming a candidate. Finally, if either candidates *i* or *j* withdraw, candidate *n* can not be defeated in a pairwise comparison, so that the winning policy is not modified.
c.1.2) $\Pi_C = \{\pi_n, \pi_i, \pi_j\}$ where $\pi_i < \pi_n < \pi_j$. We define the parameters $d = \frac{1}{3} + \varepsilon$, $d' = \frac{1}{6} - \varepsilon$ ($\varepsilon > 0$ and close enough to 0) so that $\pi_i = d'$, $\pi_n = \frac{1}{2}$, $\pi_j = 1 - d'$, from which, each candidate *i* and *j* obtains $d' + \frac{d}{2}$ percentage of votes and candidate *n* obtains a percentage *d* of the votes. It implies that $P(e) = \pi_n$. Let us show that no candidate has incentives to enter: If $k \in M$ with $\pi_k \in \left[0, \frac{1}{2}\right)$ enters the race, he obtains at most $\frac{d}{2}$ so that either candidate *j* wins or the winning policy remains the same. If $k \in M$ with $\pi_k \in \left(\frac{1}{2}, 1\right]$ enters the race, he obtains at most $\frac{d}{2}$ so that either candidate *i* wins or the winning policy remains the same. And note that the citizen whose policy coincides with π_n has no incentive to become candidate. Thus, in no case a candidate can strictly improve. Finally, if either candidate *i* or *j* withdraws, the winning policy is not modified.
c.2) $\Pi_C = \{\pi_n, 2\pi_m\text{-}\pi_n, \pi_i\}$ where $\pi_n < 2\pi_m$-$\pi_n \leqslant \pi_i$. Let $\pi_n = \frac{1}{4}$, $2\pi_m$-$\pi_n = \frac{3}{4}$, $\pi_i \in \left[\frac{3}{4}, 1\right]$ so that $P(e) = \frac{1}{4}$. We show that no citizen has incentives to enter: If $j \in M$ is such that $\pi_j \in \left[0, \frac{1}{2}\right)$, he gets at most a percentage $\frac{1}{4}$ of the votes, which is smaller than the percentage obtained by candidate *n*, so that it is then either the candidate supporting policy π_n or policy $2\pi_m$-π_n who wins. If $j \in M$ is such that $\pi_j \in \left[\frac{1}{2}, 1\right]$ he also gets at most a percentage $\frac{1}{4}$ of the

votes which is always smaller than the percentage of votes obtained by *n*, so that the winning policy is not modified. Furthermore, if either candidate $2\pi_m$-π_n or π_i withdraws, the winning policy is not modified. [End Proof]

Proof of Theorem 1:

Let us **first** show that Plurality rule is Condorcet consistent in three-candidate elections: Suppose to the contrary that there is a political equilibrium (e,α) such that $P(e) \notin \Pi_W$. Two cases must be considered:

Case 1: $\Pi_W = \{\pi_n\}$. Suppose w.l.g. that $\pi_n \leqslant \pi_m$. Since $P(e) \neq \pi_n$, there must be at least another candidate contesting the election and winning. Let $i \in M$ be this candidate and let π_i be his political alternative. Since $P(e) \notin \Pi_W$, it follows $\pi_i \notin [\pi_n, 2\pi_m - \pi_n]$ and thus, either $\pi_i \in [\pi_0, \pi_n)$ or $\pi_i \in (2\pi_m - \pi_n, \pi_1]$. Since citizens vote sincerely, in no case candidate *i* can be the winner candidate, which contradicts that $P(e) \neq \pi_n$. Therefore, there must be an additional candidate who is also contesting the election. Let $j \in M$ be this candidate and let π_j be his political alternative. Since $P(e) \neq \pi_n$, it excludes the case of $\pi_i = \pi_j = \pi_n$, and since $\Pi_W = \{\pi_n\}$, it follows that $\pi_i, \pi_j \notin (\pi_n, 2\pi_m - \pi_n]$. If either $\pi_i, \pi_j \in [\pi_0, \pi_n)$ or $\pi_i,\ \pi_j \in (2\pi_m - \pi_n, \pi_1]$, by sincere voting we have that $P(e) = \pi_n$ which is a contradiction. If $\pi_i \in [\pi_0, \pi_n)$, $\pi_j = \pi_n$ and $P(e) = \pi_i$, candidate *j* has incentive to withdraw since then $P(e_j', e_{-j}) = \pi_n$ where $e_j' = 0$ and he strictly improves, contradicting that *e* is an entry equilibrium. Thus, π_i, π_j should have the following location: $\pi_i \in [\pi_0, \pi_n]$ and $\pi_j \in (2\pi_m - \pi_n, \pi_1]$. If $P(e) = \pi_i \neq \pi_n$, candidate *j* has incentives to withdraw, since then $P(e_j', e_{-j}) = \pi_n$ where $e_j' = 0$ and he strictly improves, contradicting that *e* is an entry equilibrium. And if $P(e) = \pi_j$, it is then candidate *i* who has incentives to withdraw since then $P(e_i', e_{-i}) = \pi_n$ and he strictly improves, contradicting that *e* is an entry equilibrium. Therefore, if $\Pi_W = \{\pi_n\}$ and (e,α) is a political equilibrium, we have that $P(e) = \pi_n$.

Case 2: $\Pi_W = \{\pi_n, 2\pi_m - \pi_n\}$.. If (e,α) is a political equilibrium where $P(e) \notin \Pi_W$, it implies that there must be at least another candidate $i \in N$ contesting the election, such that $P(e) = \pi_i$. Since $P(e) \notin \Pi_W$, it follows that $\pi_i \notin [\pi_n, 2\pi_m - \pi_n]$.. Thus, either $\pi_i \in [\pi_0, \pi_n)$, or $\pi_i \in (2\pi_m - \pi_n, \pi_1]$. If $\pi_i \in [\pi_0, \pi_n)$, by sincere voting we have $P(e) = 2\pi_m - \pi_n$, which contradicts that $P(e) = \pi_i$. If $\pi_i \in (2\pi_m - \pi_n, \pi_1]$, by sincere voting we have $P(e) = \pi_n$, which contradicts that $P(e) = \pi_i$. Therefore, if $\Pi_W = \{\pi_n, 2\pi_m - \pi_n\}$ and (e,α) is a political equilibrium, we have that $P(e) \in \Pi_W$..

Let us **second** show that Plurality rule in four or more candidate elections chooses the Condorcet loser in Π_C.

It is shown by means of the following example: Let *f* be a uniform distribution function. To simplify notation we normalize the policy space so that $\pi_0 = 0$ and $\pi_1 = 1$. Let $\Pi_C = \{\pi_n, \pi_i, \pi_j, ..., \pi_z\}$ where there are at least four candidates. We consider that $\pi_i = ¼ - \varepsilon$ ($\varepsilon > 0$ and close enough to 0), $\pi_n = ½ + 2\varepsilon$, and $\pi_j = .. = \pi_z = ¾$, so that $\Pi_L = \{\pi_i\}$. Then, by sincere voting candidate *i* obtains the percentage $\frac{3}{8} + \frac{\varepsilon}{2}$ of the votes, candidate *n* obtains $\frac{1}{4} + \frac{\varepsilon}{2}$ of the votes, and finally, candidates *j* to *z* obtain each $\frac{3}{8x} - \frac{\varepsilon}{x}$ of the votes where $x \geqslant 2$. Therefore, we have that $P(e) = \pi_i$. Let us show that *e* is an entry equilibrium, it requires that *a*) no citizen strictly improves becoming candidate and *b*) no candidate strictly improves withdrawing.

a) Let ℓ be a new entrant: If $\pi_\ell \in [0, \pi_i)$, we have that candidate ℓ obtains a percentage of votes strictly smaller than $\frac{1}{4} - \varepsilon$, so that by plurality rule $P(e_\ell', e_{-\ell}) = \pi_n$ where $e_\ell' = 1$. It implies that candidate ℓ is worse off entering the race.

If $\pi_\ell \in (\pi_i, \pi_n)$ we have that candidate ℓ obtains a percentage $\frac{1}{8} + \frac{3}{2}\varepsilon$ of the votes. If $\pi_\ell \in \left[\pi_i, \frac{1}{4} + \frac{\varepsilon}{2}\right)$, it is candidate *n* who wins the election. And if $\pi_\ell \in \left[\frac{1}{4} + \frac{\varepsilon}{2}, \pi_n\right]$, it is candidate *i* who wins the election. Since the preferred policy of the citizens who strictly prefer candidate *n* over *i* is contained in $\left(\frac{3}{8} + \frac{\varepsilon}{2}, \pi_1\right]$, it follows that no citizen $\pi_\ell \in [\pi_i, \pi_n]$, has incentives to enter the race.

Finally, if $\pi_\ell \in (\pi_n, 1]$, we have that the winning political alternative is not modified, so that no

citizen ℓ has incentives to become candidate.
b) If candidate *j* withdraws, by sincere voting the rest of candidates supporting the same policy obtains each the percentage $\frac{3}{8(x-1)} - \frac{\varepsilon}{(x-1)}$ of the votes which for every $x \geqslant 2,$ is strictly smaller than the percentage obtained by candidate *i*, so that $P(e_j^{\prime}, e_{-j}) = \pi_i,$ where $e_j^{\prime} = 0$. It implies that candidate *j* can not improve. The same argument applies if the rest of candidates supporting the same policy withdraws. If candidate *n* withdraws, candidate *i* obtains a percentage $\frac{1}{2} - \frac{\varepsilon}{2}$ of the votes so that $P(e_n^{\prime}, e_{-n}) = \pi_i$ where $e_n^{\prime} = 0$, and candidate *n* does not improve. And finally, candidate *i* has no incentives to withdraw. footnote [End Proof]

Proof of Proposition 2:

We **firstly** show that no other forms of equilibria are possible.
Claim 1: There is no political equilibrium where $\#C = 1$ and $\Pi_C \neq \{\pi_m\}$.
Suppose to the contrary that such an entry equilibrium *e* exists where w.l.g. $\pi_n < \pi_m$. Let $j \in N$ be such that $\pi_j \in (\pi_n, \pi_m]$. This citizen has incentives to become candidate, since the sincere coalition S_j^C is majoritarian, so that when voting a best response, it requires that π_j be the winning policy. This contradicts that *e* is an entry equilibrium.
Claim 2: There is no political equilibrium where $\#C = 2$ and $\Pi_C = \{\pi_n, \pi_i\}$ such that $\pi_n < \pi_m$ and $\pi_i \in [\pi_0, \pi_n]$ (or symmetrically where $\pi_n > \pi_m$ and $\pi_i \in [\pi_n, \pi_1]$).
It can be shown following the same argument of Claim 1.
Claim 3: There is no political equilibrium where $\#C = 2$ and $\Pi_C = \{\pi_n, 2\pi_m\text{-}\pi_n\}$ and $P(e, \alpha(e)) = 2\pi_m\text{-}\pi_n$.
Since π_n and $2\pi_m - \pi_n$ are equidistant to $\pi_m,$ by strategic voting with sincere coalitions the unique non weakly dominated voting function implies that both candidates tie, and then, by convention $P(e, \alpha(e)) = \pi_n$.
Claim 4: There is no political equilibrium where $\#C = 3$ and $\Pi_C = \{\pi_n, \pi_i, \pi_j\}$ such that $\pi_i \leqslant \pi_j \leqslant \pi_n < \pi_m$ (or symmetrically $\pi_m < \pi_n \leqslant \pi_i \leqslant \pi_j$).
It can be shown following the same argument of Claim 1.
Claim 5: There is no political equilibrium where $\#C = 3$ and $\Pi_C = \{\pi_n, 2\pi_m\text{-}\pi_n, \pi_i\}$ where $\pi_n < 2\pi_m\text{-}\pi_n \leqslant \pi_i$ and $P(e, \alpha(e)) \neq \pi_n$.
By strategic voting with sincere coalitions, the members of S_n^C have a best response which is voting for the candidate supporting policy $\pi_n,$ since then, even in the case of tie, $P(e, (\alpha_S^{\prime}(e), \alpha_{-S}(e))) = \pi_n$ for all $\alpha_{-S}(e)$.
Claim 6: There is no political equilibrium where $\#C = 3$ and $\Pi_C = \{\pi_n, 2\pi_m\text{-}\pi_n, \pi_i\}$ where $\pi_i \leqslant \pi_n < 2\pi_m\text{-}\pi_n$ and $P(e, \alpha(e)) \neq \pi_n$.
Suppose to the contrary that such political equilibrium (e, α) exist and that $P(e, \alpha(e)) \neq \pi_n$. Suppose first that $P(e, \alpha(e)) = \pi_i \neq \pi_n$. The candidate supporting policy $2\pi_m\text{-}\pi_n$ has incentives to withdraw, since then, strategic voting implies that candidate π_n wins the election. It however contradicts that *e* is an entry equilibrium. Suppose second that $P(e, \alpha(e)) = 2\pi_m - \pi_n$. It is then candidate *i* who has incentives to withdraw since by strategic voting candidate *n* wins the election. It however contradicts that *e* is an entry equilibrium.

We **secondly** show that the proposed forms of equilibria exist. We propose the following equilibria configurations:
a) Consider that $\Pi_C = \{\pi_m\}$, and let $\alpha(e)$ be such that all citizens vote for $m,$ and for any other candidate set, $\alpha(e^{\prime})$ is such that the vote of each coalition is not weakly dominated and it is a best response. Given $\alpha,$ we have proposed an entry equilibrium: no citizen can defeat candidate *m* entering the race, since the sincere coalition of the median candidate is always majoritarian. And since $u_m(\emptyset) < u_m(\pi_m)$ candidate *m* has no incentives to withdraw. Furthermore, α is strategic.
b.1) We distinguish two cases:
b.1.1) Let $d > 0$ be such that the percentage of votes in $\left[\pi_m - d, \pi_m + \frac{3}{2}d\right]$ be strictly smaller than $\frac{1}{3}$. And consider that $\Pi_C = \{\pi_n, \pi_i\},$ $\pi_n \leqslant \pi_m < \pi_i$ (or symmetrically $\pi_i < \pi_m \leqslant \pi_n$) where $\pi_n \in [\pi_m - d, \pi_m]$ and $\pi_i = \pi_m + 2d$. Let (S_n^C, S_i^C) be a sincere partition. For each entry strategy,

we next construct the voting function. Consider that coalition S_n^C votes for *n* and coalition S_i^C votes for *i*, so that π_n is the winning policy. Let us show that no citizen $j \in N$ has incentives to contest the elections. Candidate $j \in N$ induces a sincere partition $(S_n^{C'}, S_i^{C'}, S_j^{C'})$ where $C' = \{n, i, j\}$.
If $\pi_j \in [\pi_0, \pi_n)$, consider that all the coalitions vote for *n*. Since the members of no coalition are voting for their least preferred candidate, this is not a weakly dominated strategy. Furthermore, since $S_j^{C'}$ and $S_i^{C'}$ are not majoritarian coalitions, they are voting a best response since they can not modify the winning policy changing their votes.
If $\pi_j \in [\pi_n + d, \pi_1]$, consider that $S_i^{C'}$ and $S_j^{C'}$ vote for *i*, and coalition $S_n^{C'}$ votes for *n*. This is not weakly dominated, and since $S_i^{C'}$ and $S_j^{C'}$ are smaller coalitions than $S_n^{C'}$, the proposed way of voting is a best response. We then have that, even in the case of tie, the winning policy π_n is not modified.
If $\pi_j \in (\pi_n, \pi_n + d)$, we consider then that the coalitions $S_n^{C'}$ and $S_j^{C'}$ vote for *n* and coalition $S_i^{C'}$ votes for *i*. Since coalition $S_i^{C'}$ is not majoritarian, the winning policy is π_n. Note that the members of the coalitions $S_n^{C'}$ and $S_i^{C'}$ are voting their most preferred candidate. With regard to coalition $S_j^{C'}$, since the votes in the interval $\left[\pi_m - d, \pi_m + \frac{3}{2}d\right]$ are strictly smaller than $\frac{1}{3}$, if $\pi_j \in (\pi_n, \pi_m + d)$, coalition $S_j^{C'}$ is smaller than either $S_n^{C'}$ or $S_i^{C'}$. Thus, voting for *n* is not weakly dominated for coalition $S_j^{C'}$, and it is also a best response since in the case that voting for *i* changes the winning policy to π_i, it does not strictly improve all the members in the sincere coalition $S_j^{C'}$. If $\pi_j \in [\pi_m + d, \pi_m + 2d]$, coalition $S_n^{C'}$ is majoritarian, and so, $S_j^{C'}$ is smaller than $S_n^{C'}$. Thus, voting for *n* is not weakly dominated for coalition $S_j^{C'}$, and it is also a best response since $S_j^{C'}$ can not modify the winning policy changing his vote.
Finally, it is clear that nor *n* neither *i* has incentives to withdraw.
If for any other entry strategy, we consider that the vote of each coalition is not weakly dominated and it is a best response, we have that, the proposed entry strategy and voting function is a political equilibrium.
b.1.2) Consider that $\Pi_C = \{\pi_n, \pi_i\}$ where $\pi_n = \pi_m = \pi_i$. We next construct the voting function. Consider that $\alpha(e)$ is such that all citizens vote for *n*. This way of voting is clearly not weakly dominated and it is a best response for every sincere partition (S_n^C, S_i^C). For $C' = \{n, i, j\}$ where $j \in N$, we consider that the sincere coalition $S_j^{C'}$ votes for *j*, and $S_n^{C'}, S_i^{C'}$ vote for *n*. This voting behavior is clearly not weakly dominated and it is a best response where the winning policy is not modified. For any other entry strategy, we consider that the vote of each coalition is not weakly dominated and it is a best response. Thus, α is strategic. Furthermore, given α, we have proposed an entry equilibrium: since no citizen has incentives neither to enter the race, nor to withdraw. Hence, the proposed entry strategy and voting function is a political equilibrium.
b.2) Let $d > 0$, such that the percentage of votes in $\left[\pi_m - d, \pi_m + \frac{3}{2}d\right]$ be strictly smaller than $\frac{1}{3}$.. Consider that $\Pi_C = \{\pi_n, 2\pi_m\text{-}\pi_n\}$ where $\pi_n = \pi_m - d$ and $2\pi_m\text{-}\pi_n = \pi_m + d$. Then, the same voting function of Case (b.1.1) (where candidate *i* is substituted by the candidate with preferred policy $2\pi_m - \pi_n$) can be applied to show existence of this equilibrium.
c.1) We distinguish four cases.
c.1.1) Consider that $\Pi_C = \{\pi_n, \pi_i, \pi_j\}$, $\pi_n \leqslant \pi_m < \pi_i \leqslant \pi_j$ (the case in which $\pi_i \leqslant \pi_j < \pi_m \leqslant \pi_n$ is similar) where $\pi_n \in [\pi_m - d, \pi_m]$, $\pi_i = \pi_m + 2d$, $\pi_j \in [\pi_m + 2d, \pi_1]$. It induces the sincere partition (S_n^C, S_i^C, S_j^C). A similar argument than that of Case (b.1.1) applies, where we consider additionally, that coalition S_j^C votes in the same way than coalition S_i^C.
c.1.2) Let $d > 0$, such that the percentage of votes in $\left[\pi_m - d, \pi_m + \frac{3}{2}d\right]$ be strictly smaller than $\frac{1}{3}$.. Consider that $\Pi_C = \{\pi_n, \pi_i, \pi_j\}$, $\pi_i < \pi_n < \pi_j$ where $\pi_i = \pi_m - d$, $\pi_n \in (\pi_m - d, \pi_m + d)$, $\pi_j = \pi_m + 2d$ and let (S_n^C, S_i^C, S_j^C) be a sincere partition. For each entry strategy, we next construct the voting function. Consider that $\alpha(e)$ is such that all coalitions vote for *n*, so that π_n is the winning policy. Since the members of no coalition are voting their least preferred candidate, this is not a weakly dominated strategy. Furthermore, since the coalition S_i^C and $S_j^C$are not by itself majoritarian, they can not modify the winning policy changing their vote, so that they are voting a best response. Let us show that no citizen $k \in N$ has incentives to contest the elections. Candidate

$k \in N$ induces a sincere partition $(S_n^{C'}, S_i^{C'}, S_j^{C'}, S_k^{C'})$ where $C' = \{n, i, j, k\}$.
If either $\pi_k \in [\pi_0, \pi_i]$ or $\pi_k \in [\pi_m + d, \pi_1]$, coalition $S_k^{C'}$ can not be majoritarian. We consider then that all the coalitions vote for *n*, so that the winning policy is not modified. Voting for *n* is strategic since the members of no coalition vote for their *least* preferred candidate, and since no coalition $S_i^{C'}, S_j^{C'}, S_k^{C'}$ is majoritarian, they are voting a best response.
If $\pi_k \in (\pi_i, \pi_m + d)$, since in the interval $\left[\pi_m - d, \pi_m + \frac{3}{2}d\right]$ there is less than $\frac{1}{3}$ of the electoral, it follows that $S_k^{C'}$ can not be majoritarian. Consider then, that all the coalitions vote for *n*, so that the winning policy is not modified. Voting for *n* is strategic since no coalition votes for his least preferred candidate, and since coalitions $S_i^{C'}, S_j^{C'} S_k^{C'}$ are not majoritarian, they are voting a best response.
Finally, no candidate has incentives to withdraw, since π_n is the winning policy.
And for any other entry strategy, we consider that the vote of each coalition is not weakly dominated and it is a best response. Thus, the proposed entry strategy and voting function is a political equilibrium.
c.1.3) Let $d > 0$, be such that the percentage of votes in $\left[\pi_m - d, \pi_m + \frac{3}{2}d\right]$ be strictly smaller than $\frac{1}{3}$. Consider that $\Pi_C = \{\pi_n, \pi_i, \pi_j\}$, $\pi_i = \pi_n \leqslant \pi_m < \pi_j$ (the case in which $\pi_i < \pi_m \leqslant \pi_n = \pi_j$ is similar) where $\pi_n \in [\pi_m - d, \pi_m]$, $\pi_j = \pi_m + 2d$. It induces the sincere partition (S_n^C, S_i^C, S_j^C). A similar argument than that of Case (b.1.1) applies, where we consider additionally, that new coalition S_i^C votes in the same way than coalition S_n^C.
c.1.4) Consider that $\Pi_C = \{\pi_n, \pi_i, \pi_j\}, \pi_n = \pi_m = \pi_i = \pi_j$. We then have that the same argument of Case (b.1.2), where coalition S_j^C votes in the same way than coalition S_i^C applies.
c.2) Let $d > 0$, such that the percentage of votes in $\left[\pi_m - d, \pi_m + \frac{3}{2}d\right]$ be strictly smaller than $\frac{1}{3}$. Consider that $\Pi_C = \{\pi_n, 2\pi_m\text{-}\pi_n, \pi_i\}$, $\pi_n < 2\pi_m\text{-}\pi_n \leqslant \pi_i$ (the case in which $\pi_i \leqslant \pi_n < 2\pi_m - \pi_n$ is similar) where $\pi_n = \pi_m - d$, $2\pi_m - \pi_n = \pi_m + d$ and $\pi_i \in [\pi_m + d, \pi_1]$. Then, a similar argument than that of Case (b.1.1) applies, where we consider additionally that the coalitions which most preferred policies are $2\pi_m\text{-}\pi_n$ and π_i respectively, vote in the same way than that proposed for S_i^C in Case (b.1.1). [End Proof]

Proof of Proposition 3:

Two previous lemmas are needed to show this result:

Lemma *For every sincere coalition S, voting sincerely is not a weakly dominated voting function.*

Proof If S_i is a majoritarian coalition, voting for *i* provides the best result for all the citizens in S_i. Otherwise, let α_{-S_i} be such that almost half of the citizens vote for $j \in N$, and the rest for *i*. Then, that the members of S_i vote sincerely provides the best result for all the member in S_i.

Lemma *For every sincere coalition S, voting sincerely is a best response if and only if* $\#C \leqslant 2$.

Proof We **first** show that when $\#C \leqslant 2$, and the sincere coalition *S* votes sincerely, $\alpha_S(e)$ is a best response.
If $\#C = 1$, then, there is no other way of voting, but sincerely.
If $\#C = 2$, $\Pi_C = \{\pi_i, \pi_j\}$, $\pi_i \leqslant \pi_j$ we have that either one of the sincere coalitions is majoritarian or both have equal size. Let w.l.g. S_i be majoritarian or equal to S_j, so that, even in the case of ties, voting sincerely elects their most preferred candidate, which is the best result for the citizens in S_i. We also have that $u_k(P(e, (\alpha_{S_j}(e), \alpha_{S_i}(e))) = u_k\left(P(e, (\alpha'_{S_j}(e), \alpha_{S_i}(e))\right)$ for all $k \in S_j$ and for every possible $\alpha'_{S_j}(e)$, in particular, for $\alpha'_{S_j}(e)$ where the members of coalition S_j vote sincerely.
We **secondly** show that there are some entry strategies where $\#C > 2$ such that voting sincerely is not a best response.
Let Π_C be such that there are at least three candidates which preferred policies satisfy that $\pi_i < \pi_j < \pi_k$. We consider that the sincere coalition S_i is the greatest coalition, and $S_j \cup S_k$ is greater than S_i. We then have that sincere voting implies that π_i is the winning policy. We find,

however, that if S_k votes for $j,$ it is then π_j the winning policy, so that all the members of coalition S_k strictly improve.

We now complete the proof of Proposition 3:

We **first** show that the entry equilibria e where $\#C \leqslant 3$ and where citizens vote sincerely are also entry equilibria under strategic voting with sincere coalitions. Three cases are considered.

Case 1: Let e be such that $\#C = 1$. If citizen $i \in N$ enters the race, suppose that citizens vote sincerely, which by Lemma 1 and Lemma 2, it is not weakly dominated and it is a best response. Thus, a strategic voting function can be proposed such that e is an entry equilibrium.

Case 2: Let e be such that $\#C = 2$ where $\Pi_C = \{\pi_i, \pi_j\},\ \pi_i \leqslant \pi_j$ and where citizens vote sincerely. If a candidate withdraws, we consider that they vote sincerely. Thus, by Lemmas 1 and 2, it is not weakly dominated and it is a best response. Let e' be such that $k \in N$ enters the race. It generates the sincere coalitions $(S_i^{C'}, S_j^{C'}, S_k^{C'})$ where $C' = \{i,j,k\}$. Since e is an entry equilibrium under sincere voting, $S_k^{C'}$ is smaller than either $S_i^{C'}$ or $S_j^{C'}$. Consider that $\alpha(e') = \alpha(e),$ so that the winning policy is not modified. Since $S_i^{C'}$ and $S_j^{C'}$ vote sincerely, by Lemma 1 it is not weakly dominated. It is clearly a best response for the coalition supporting the winning policy, and it is also for the other coalition since it can not modify the winning policy changing its vote. Finally, the members of $S_k^{C'}$ vote for their second best preferred candidate, and since $S_k^{C'}$ is not majoritarian, it is not weakly dominated. Furthermore, if either $\pi_k \in [\pi_0, \pi_i]$ or $\pi_k \in [\pi_j, \pi_1]$ the members of $S_k^{C'}$ can not strictly improve changing their vote. And if $\pi_k \in (\pi_i, \pi_j)$, some of the citizens in $S_k^{C'}$ strictly prefer π_i to $\pi_j,$ so that changing their vote may not strictly improve all of them. Therefore, the proposed way of voting is also a best response for coalition $S_k^{C'}$. Thus, a strategic voting function can be proposed such that e be an entry equilibrium.

Case 3: Let e be such that $\#C = 3$ where $\Pi_C = \{\pi_i, \pi_j, \pi_k\},\ \pi_i \leqslant \pi_j \leqslant \pi_k$ and where citizens vote sincerely. By Lemma 1, it is not weakly dominated. Let us show that none of the sincere coalitions (S_i^C, S_j^C, S_k^C) can strictly improve changing their vote. These coalitions can only improve if the candidate they support is not the winning candidate. If either S_i^C or S_k^C votes for candidate j, it is, respectively, equivalent to the case in which citizens vote sincerely and candidate i or k withdraws. However, since e is an entry equilibrium under sincere voting, neither the members of $S_i^C,$ nor those of S_k^C can then strictly improve when voting for j. Coalition S_j^C, can only modify the winning policy voting for either π_i or π_k. However, some of the members of S_j^C strictly prefer π_i to $\pi_k,$ so that switching their vote can not strictly improve all of them. Thus, voting sincerely is a best response for all the coalitions. If a candidate withdraws, citizens vote sincerely which by Lemma 1 and 2, it is not weakly dominated and it is a best response. Finally, let e' be such that $\ell \in N$ enters the race, it generates the sincere coalitions $(S_i^{C'}, S_j^{C'}, S_k^{C'}, S_\ell^{C'})$ where $C' = \{i,j,k,\ell\}$. Since e is an entry equilibrium under sincere voting, $S_\ell^{C'}$ is smaller than at most one of the other coalitions. Consider that $\alpha(e') = \alpha(e),$ so that $S_i^{C'}, S_j^{C'}$ and $S_k^{C'}$ vote sincerely which, by Lemma 1, it is not weakly dominated. Thus, the winning policy is not modified and it is supported by one of these coalitions: $S_i^{C'}, S_j^{C'}$ or $S_k^{C'}$. Consider w.l.g. that π_i is the winning policy. It implies that $S_j^{C'}$ and $S_k^{C'}$ are smaller coalitions than $S_i^{C'}$, so that if either $S_j^{C'}$ or $S_k^{C'}$ switch its vote for $\ell,$ the winning policy is not modified. Thus, the same argument that guarantees that $\alpha(e)$ is a best response applies to show that $\alpha(e')$ is a best response for $S_i^{C'}, S_j^{C'}$ and $S_k^{C'}$. The members of $S_\ell^{C'}$ vote for their second best-preferred candidate. Since $S_\ell^{C'}$ is not the greatest coalition, this way of voting is not weakly dominated. Let us show that it is also a best response: If $\pi_\ell \in [\pi_0, \pi_i]$ $(\pi_\ell \in [\pi_k, \pi_1])$, the members of $S_\ell^{C'}$ can not improve when π_i or π_j (respectively π_k or π_j) is the winning policy. Consider that the winning policy is π_k (π_i). Since $S_\ell^{C'} \subset S_i^C$ $(S_\ell^{C'} \subset S_k^C),$ and that e is an entry equilibrium implies that the members of $S_i^C \cup S_j^C$ $(S_k^C \cup S_j^C)$ can not modify the winning policy π_k (π_i) when all them vote for $j,$ it follows that the members of $S_\ell^{C'}$ can not strictly improve if instead of voting for π_i (π_k), they vote for π_j. If $\pi_\ell \in (\pi_i, \pi_j)$ $(\pi_\ell \in (\pi_j, \pi_k))$ and the winning policy is either π_i or π_j (respect to π_k or π_j), some of the members of $S_\ell^{C'}$ can not improve changing their vote since they may strictly prefer the winning policy. Even when the winning policy is $\pi_k,$ the members of $S_\ell^{C'}$ can not improve since changing their vote altogether for either π_i or π_j (respect to

π_k or π_j), does not modify the final policy since $S_\ell^{C'} \subset S_i^C \cup S_j^C$ (respect to $S_\ell^{C'} \subset S_k^C \cup S_j^C$) and the same above argument applies. Finally, if $\pi_\ell = \pi_j$, since $S_\ell^{C'} \subset S_j^C$, the same argument showing that $\alpha(e)$ is a best response for S_j^C applies to show that $\alpha(e')$ describes a best response for $S_\ell^{C'}$. Thus, a strategic voting function can be proposed such that e be an entry equilibrium.

We **second** show that there is an entry equilibria e under sincere voting such that $\#C = 4$, which is not an entry equilibrium under strategic voting with sincere coalitions. The entry strategy that we propose is given by $\Pi_C = \{\pi_i, \pi_j, \pi_k, \pi_\ell\}$ where $\pi_i < \pi_j < \pi_k < \pi_\ell$ and $|\pi_j - \pi_i| > |\pi_j - \pi_\ell|$. It generate the sincere coalitions $(S_i^C, S_j^C, S_k^C, S_\ell^C)$, and we propose a strictly positive density function which satisfies the following properties:
P.1: S_i^C contains $\frac{2}{5}$ of the votes, and S_j^C, S_k^C, S_ℓ^C contain each $\frac{1}{5}$ of the votes
P.2: the intervals $[\pi_0, \pi_i]$, $\left[\pi_i, \pi_i + \frac{(\pi_j - \pi_i)}{2}\right]$ contain respectively $\frac{1}{5}$ of the votes uniformly distributed
P.3: the intervals $\left[\pi_j + \frac{(\pi_k - \pi_j)}{4}, \pi_j + \frac{(\pi_k - \pi_j)}{2}\right]$, $\left[\pi_j + \frac{(\pi_k - \pi_j)}{2}, \pi_j + \frac{3(\pi_k - \pi_j)}{4}\right]$ contain each $\frac{1}{5} - \varepsilon$ of the votes uniformly distributed, where $\varepsilon > 0$ and sufficiently close to 0.
From P.1 it follows that when voting sincerely, $P(e, \alpha(e)) = \pi_i$.

We next show that when voting sincerely, e is an entry equilibrium:
a) No citizen $r \in N$ has incentives to enter the race: In each of the intervals $[\pi_j, \pi_k], [\pi_k, \pi_\ell], [\pi_\ell, \pi_1]$ there is at most $\frac{2}{5}$ of the votes, so that no citizen entering in these intervals can defeat candidate i. If $\pi_r \in [\pi_0, \pi_i)$, by P.2 he gets less than $\frac{1}{5}$ of the votes, which by P.1 it means that he can not defeat candidate i. Finally, if $\pi_r \in [\pi_i, \pi_j)$ by P.1, P.2 and P.3 the maximum percentage of votes is obtained when $\pi_r = \pi_i + \gamma$ where $\gamma > 0$ is close enough to 0. Candidate r obtains there $\frac{1}{5} - \frac{\gamma}{2}\frac{1}{5} + \frac{\gamma}{2}\varepsilon$. Candidate i then obtains $\frac{1}{5} + \frac{\gamma}{2}\frac{1}{5}$ of the votes. Since $\varepsilon \leqslant \frac{1}{5}$, candidate r can not defeat candidate i.
b) No citizen has incentives to withdraw: If j, k or ℓ withdraw, the other candidates get at most $\frac{2}{5}$ of the votes, and candidate i can not be defeated.

Let us show that when voting strategically with sincere coalitions, e is not an entry equilibrium. Consider that a citizen $r \in N$ with preferred policy $\pi_r = \pi_j + \frac{(\pi_k - \pi_j)}{2}$ becomes candidate. It generates the sincere coalitions $\left(S_i^{C'}, S_j^{C'}, S_r^{C'}, S_k^{C'}, S_\ell^{C'}\right)$, and by P.3, he obtains, under sincere voting, $\frac{2}{5} - 2\varepsilon$ of the votes. Since $|\pi_j - \pi_i| > |\pi_j - \pi_\ell|$, voting for candidate i is weakly dominated for the sincere coalitions $S_r^{C'}, S_k^{C'}$ and $S_\ell^{C'}$. Furthermore, by P.1 and P.3 coalitions $S_k^{C'}$ and $S_\ell^{C'}$ contain together $\frac{1}{5} + \varepsilon$ of the votes, and so, we can ensure that one of the candidates: j, r, k or ℓ, obtains at least a percentage $\frac{1}{4}(\frac{1}{5} + \varepsilon)$ of the votes. It implies that a best response for coalition $S_r^{C'}$ requires voting for a candidate j, r, k or ℓ, who receives from coalitions $S_k^{C'}$ and $S_\ell^{C'}$ at least $\frac{1}{4}(\frac{1}{5} + \varepsilon)$ of votes. Thus, with the vote of coalition $S_r^{C'}$ one of the candidates: j, r, k or ℓ obtains at least: $\frac{9}{20} - \frac{7\varepsilon}{4}$ which for ε sufficiently small it defeats candidate i who obtains at most the votes of coalitions $S_i^{C'}$ and $S_j^{C'}$, that by P.3 are $\frac{2}{5} + \varepsilon$. It implies that given e, every strategic voting function is such that a citizen strictly improves entering the race. [End Proof]

Proof of Theorem 2:

Let us **first** show that Plurality rule is Condorcet consistent in three-candidate elections. Suppose to the contrary that there is a political equilibrium (e, α) such that $P(e) \notin \Pi_W$. Two cases must be considered.

Case 1. $\Pi_W = \{\pi_n\}$. Suppose that, w.l.g., $\pi_n \leqslant \pi_m$. Since $P(e) \neq \pi_n$, there must be at least another candidate contesting the election and winning. Let $i \in M$ be this candidate and let π_i be his political alternative. Since $P(e) \notin \Pi_W$, it follows that $\pi_i \notin [\pi_n, 2\pi_m - \pi_n]$ and thus, either $\pi_i \in [\pi_0, \pi_n)$ or $\pi_i \in (2\pi_m - \pi_n, \pi_1]$. In both cases, the sincere coalition S_n^C is majoritarian, and voting candidate i is weakly dominated, which contradicts that $P(e) \neq \pi_n$. Therefore, there must be an additional candidate who is also contesting the election. Let $j \in M$ be this candidate and let π_j

be his political alternative. Since $P(e) \neq \pi_n$, it excludes the case of $\pi_i = \pi_j = \pi_n$, and since $\Pi_W = \{\pi_n\}$, $\pi_i, \pi_j \notin (\pi_n, 2\pi_m - \pi_n)$. If either $\pi_i, \pi_j \in [\pi_0, \pi_n)$ or $\pi_i, \pi_j \in (2\pi_m - \pi_n, \pi_1]$, the sincere coalition S_n^C is also majoritarian, so that voting for either candidate *i* or *j* is weakly dominated, which contradicts that $P(e) \neq \pi_n$. If $\pi_i \in [\pi_0, \pi_n)$ and $\pi_j = \pi_n$, we have that just in the case that $P(e) = \pi_i$, candidate *j* has incentives to withdraw, since then $P(e_j', e_{-j}) = \pi_n$ where $e_j' = 0$ and he strictly improves, contradicting that *e* is an entry equilibrium. Finally, consider that $\pi_i \in [\pi_0, \pi_n]$ and $\pi_j \in (2\pi_m - \pi_n, \pi_1]$. If $P(e) = \pi_i \neq \pi_n$, candidate *j* has incentives to withdraw since then $P(e_j', e_{-j}) = \pi_n$ where $e_j' = 0$ and he strictly improves, contradicting that *e* is an entry equilibrium. And if $P(e) = \pi_j$, it is then candidate *i* who has incentives to withdraw since then $P(e_i', e_{-i}) = \pi_n$ where $e_i' = 0$ and he strictly improves, contradicting that *e* is an entry equilibrium. Therefore, if $\Pi_W = \{\pi_n\}$ and (e, α) is a political equilibrium, we have that $P(e) = \pi_n$.

Case 2. $\Pi_W = \{\pi_n, 2\pi_m - \pi_n\}$.. If (e, α) is a political equilibrium where $P(e) \notin \Pi_W$, it implies that there must be at least another candidate *i*, contesting the elections, such that $P(e) = \pi_i$, and that $\pi_i \notin [\pi_n, 2\pi_m - \pi_n]$. Thus, either $\pi_i \in [\pi_0, \pi_n)$ or $\pi_i \in (2\pi_m - \pi_n, \pi_1]$. If $\pi_i \in [\pi_0, \pi_n)$, the candidate with policy $2\pi_m - \pi_n$ has incentives to withdraw since then the winning policy becomes π_n. It contradicts that *e* is an entry equilibrium. If $\pi_i \in (2\pi_m - \pi_n, \pi_1]$ it is then candidate *n* who strictly improves withdrawing since then, the winning policy is $2\pi_m - \pi_n$. It however contradicts that *e* is an entry equilibrium. Therefore, if $\Pi_W = \{\pi_n, 2\pi_m - \pi_n\}$ and (e, α) is a political equilibrium, we have that $P(e) \in \Pi_W$.

Let us **second** show that in four or more candidate elections the Condorcet loser in Π_C can be elected.

The proof will be made by construction: for a given entry strategy *e*, we construct a voting function α such that it is a political equilibrium where $P(e, \alpha(e)) \in \Pi_L$.

Consider the following entry strategy *e*:

Let $\Pi_C = \{\pi_n, \pi_i, \pi_j, \pi_k\}$, where $\pi_i < \pi_j < \pi_n < \pi_k$. We define $\pi_i = \pi_j - \varepsilon$ (where $\varepsilon > 0$ and close enough to 0), $\pi_j = \pi_m - 3d$, $\pi_n = \pi_m + d$ and $\pi_k = \pi_m + 4d$. The candidate set generates a sincere partition $\left(S_i^C, S_j^C, S_n^C, S_k^C\right)$, and we consider a density function *f* such that for some $d > 0$, the percentage of citizens in the interval $[\pi_m, \pi_m + \frac{3}{4}d]$ be lower than γ (where $\gamma > 0$ and close enough to 0), in the interval $[\pi_m - \frac{3}{2}d, \pi_m]$ be γ' (where $\gamma' > \gamma$ and close enough to γ), the percentage of citizens in S_k^C, be strictly greater than that in S_n^C, and the percentage of citizens in each of the intervals $[\pi_0, \frac{\pi_i + \pi_j}{2}]$ and $[\frac{\pi_i + \pi_j}{2}, \pi_m - 2d]$ be greater than 2γ.

For the proposed entry strategy *e*, we consider that $\alpha(e)$ is as follows:

The sincere coalitions S_n^C and S_k^C vote for *k* and S_i^C and S_j^C vote for *i* and *j* respectively. Since the median citizen belongs to coalition S_n^C, $P(e, \alpha(e)) = \pi_k$. The proposed way of voting is strategic since: i) it is not a weakly dominated voting strategy: the members of each coalition vote for their first or second preferred candidate, and ii) it is a best response for each coalition: since S_n^C is a smaller coalition than S_k^C, the members of S_n^C can not switch the winning political alternative to π_n, and by voting for *j* or *i* not all members of the coalition S_n^C are better off. Furthermore, since candidate *k* receives more than $\frac{1}{2}$ of the votes, the members of S_i^C or S_j^C can not modify the winning policy by means of changing their voting strategy.

We next show that a complete description of a voting function can be proposed such that *e* be an entry equilibrium:

a) No citizen ℓ improves entering the elections:

If $\pi_\ell \in [\pi_0, \pi_j]$, this entry strategy $(e_\ell', e_{-\ell})$ where $e_\ell' = 1$, generates a narrower sincere partition in the interval $[\pi_0, \pi_j]$ than the entry strategy *e*. Then, if $\alpha(e_\ell', e_{-\ell}) = \alpha(e)$, by the above arguments i) and ii) voting is strategic. And since the winning policy is π_k, candidate ℓ is not better off becoming candidate.

If $\pi_\ell \in (\pi_j, \pi_m - d)$, consider that $\alpha(e_\ell', e_{-\ell}) = \alpha(e)$ where $e_\ell' = 1$. Voting in this way is not a weakly dominated strategy since no sincere coalition votes for his unanimous least preferred candidate. Furthermore, since the median citizen $m \in S_n^{C'}$, we have that $S_n^{C'}$ and $S_k^{C'}$ contain together more than half of the votes. It implies that each of the sincere coalitions $S_i^{C'}$, $S_j^{C'}$ and $S_\ell^{C'}$

can not modify the winning policy changing their vote. The members of $S_n^{C'}$ prefer policy n to the policy of the rest of candidates. And since $S_n^{C'} \subset S_n^C,\ S_k^{C'} = S_k^C$ and S_n^C is smaller than $S_k^C,$ the members of $S_n^{C'}$ by changing their vote, either do not modify the winning policy π_k or they make π_i or π_j be the winning policy which does not improve all the members of $S_n^{C'}$. So that all the sincere coalitions vote a best response. And since the winning policy is $\pi_k,$ candidate ℓ is not better off becoming candidate.
If $\pi_\ell \in [\pi_m - d, \pi_m]$, consider that the sincere coalitions $S_n^{C'}$ and $S_k^{C'}$ vote for k and $S_i^{C'}$, $S_j^{C'}$ and $S_\ell^{C'}$ vote for i, j and ℓ respectively. Since candidate k gets a percentage of votes greater than $\frac{1}{2} - \gamma$, and candidates i, j or ℓ get at least a percentage of votes of 2γ, 2γ and γ' respectively, and since $\gamma' > \gamma$, the maximum percentage of votes that either i, j or ℓ get is lower than $\frac{1}{2} - 3\gamma$, so that π_k is the winning policy. Since the members of each coalition do not vote for their least preferred candidate, it is not a weakly dominated voting strategy. And it is a best response for each coalition: since $S_n^{C'}$ is a smaller coalition than $S_k^{C'}$, the members of $S_n^{C'}$ can not switch the winning political alternative to π_n, and by voting for i, j or ℓ not all members of the coalition $S_n^{C'}$ are better off. Furthermore, since candidate k gets a percentage of votes greater to $\frac{1}{2} - \gamma$, by changing their voting strategy $S_i^{C'}$, $S_j^{C'}$ or $S_\ell^{C'}$ can not modify the winning policy because the maximum percentage of votes that either i, j or ℓ may obtain is lower than $\frac{1}{2} - \gamma$.
If $\pi_\ell \in (\pi_m, \pi_m + \frac{d}{2}]$, consider that the sincere coalitions $S_n^{C'}$, $S_k^{C'}$ and $S_\ell^{C'}$ vote for k and $S_i^{C'}$ and $S_j^{C'}$ vote for i and j respectively. Voting in this way is not a weakly dominated strategy since no sincere coalition votes for his unanimous least preferred candidate. Furthermore, since the median citizen $m \in S_\ell^{C'}$, we have that $S_n^{C'}$, $S_k^{C'}$ and $S_\ell^{C'}$ contain together more than half of the votes and π_k is the winning policy. It implies that each of the sincere coalitions $S_i^{C'}$, $S_j^{C'}$ can not modify the winning policy changing their vote. Since the percentage of citizens in $S_\ell^{C'}$ is lower than $\gamma + \gamma'$, the members of $S_\ell^{C'}$ can not switch the winning political alternative to π_n or π_ℓ, and by voting for j or i not all members of the coalition $S_\ell^{C'}$ are better off. The members of $S_n^{C'}$ prefer policy n to the policy of the rest of candidates. And since $S_n^{C'} \subset S_n^C,\ S_k^{C'} = S_k^C$ and S_n^C is smaller than $S_k^C,$ the members of $S_n^{C'}$ by changing their vote, either do not modify the winning policy π_k or they make π_i or π_j be the winning policy that is worse than π_k for all the members in $S_n^{C'}$. So that all the sincere coalitions vote a best response. And since the winning policy is $\pi_k,$ candidate ℓ is not better off becoming candidate.
If $\pi_\ell \in (\pi_m + \frac{d}{2}, \pi_1]$. It implies that $|\pi_k - \pi_\ell| < |\pi_j - \pi_\ell|$, so that candidate ℓ prefers policy π_k to π_j. And let $\alpha(e'_\ell, e_{-\ell})$ where $e'_\ell = 1$ be such that all the coalitions vote for j. It is not a weakly dominated since no coalition votes for his unanimous least preferred candidate. And since no sincere coalition can modify the winning policy changing his vote, agent ℓ has no incentives to become candidate.
b) No candidate improves withdrawing:
Candidate n has no incentives to withdraw since $|\pi_k - \pi_n| < |\pi_j - \pi_n|$ so that he prefers π_k to π_j or π_i.
If candidate i withdraws, then $C' = \{n, j, k\}$, which generates the sincere partition $(S_j^{C'}, S_n^{C'}, S_k^{C'})$ where $S_j^{C'} = S_j^C \cup S_i^C,\ S_n^{C'} = S_n^C,\ S_k^{C'} = S_k^C$. Consider that $S_n^{C'}$ and $S_k^{C'}$ vote for k and $S_j^{C'}$ vote for j, we have that the winning policy is also π_k. Then, the same above arguments i) and ii) guarantee that this way of voting is strategic.
If candidate j withdraws, then $C' = \{n, i, k\}$, which generates the sincere partition $(S_i^{C'}, S_n^{C'}, S_k^{C'})$ where $S_k^{C'} = S_k^C$. Consider that the members of $S_n^{C'}$ and $S_k^{C'}$ vote for k and those of $S_i^{C'}$ vote for i, so that the winning policy π_k is not modified. Since $\pi_i = \pi_j - \varepsilon,$ for ε sufficiently close to 0, we have that coalition $S_n^{C'}$ can be as close as we want to $S_n^C,$ so that $S_n^{C'}$ is strictly smaller than $S_k^{C'}$. It implies that $S_n^{C'}$ can not be majoritarian, so that arguments i) and ii) guarantee that this way of voting is strategic.
Finally, candidate k does not have incentives to withdraw, since he supports the winning policy.
Finally, for the rest of entry strategies, we consider that the voting function is such that it is not weakly dominated and it describes a best response for each sincere coalition. Thus, for every $f,$ we

have proposed an entry strategy e and a voting function α which are a political equilibrium footnote and where $P(e,\alpha(e)) \in \Pi_L$.

The generalization of this result to five or more candidates follows directly by means of introducing some additional candidates which most preferred policy coincides with one of the initial candidates and the voting function remains the same. [End Proof]

Proof of Theorem ref: th3 :

Consider w.l.g. that $\pi_n \leqslant \pi_m$. Suppose to the contrary that there is a strong political equilibrium (e,α) such that $P(e) \notin \Pi_W$. Following the same argument of Case 1 in Theorem 1 and 2, it implies that there are at least two candidates i and j which policy positions are $\pi_i \in [\pi_0,\pi_n]$ and $\pi_j \in (2\pi_m$-$\pi_n,\pi_1]$. We consider that some additional candidates can also support other policies in $[\pi_0,\pi_1]$. Since $P(e) \notin \Pi_W$, two cases must be considered:

Case 1: A candidate in the interval $[\pi_0,\pi_n)$ wins the elections. Consider a coalition G that consist of those candidates with policy positions in the interval $(2\pi_m$-$\pi_n,\pi_1]$ together with all the candidates but one which policy position coincides with π_n (just in case that several candidates support π_n). This coalition G is such that $\#G \geqslant 1$ and $\#G \leqslant \#C-2$ since it neither contains the winning candidate nor a candidate supporting π_n. If citizens vote sincerely this coalition has incentives to withdraw since then, $P(e'_{-G},e_G) = \pi_n$ where $e'_{-G} = (0,...,0)$ so that all the candidates in S strictly improve. If citizens vote strategically with sincere coalitions, we have that the sincere coalition $S_n^{C'}$ becomes a majoritarian coalition so that its best response should be such that $P(e'_{-G},e_G) = \pi_n$. We find, however that it contradicts that e be a strong entry equilibrium.

Case 2: A candidate in the interval $(2\pi_m$-$\pi_n,\pi_1]$ wins the election. Consider a coalition G that consist of those candidates with policy position in the interval $[\pi_0,\pi_n)$ together with all the candidates but one which policy position coincides with π_n (just in case that several candidates support π_n). This coalition G is such that $\#G \geqslant 1$ and $\#G \leqslant \#C-2,$ since it neither contains the winning candidate nor a candidate supporting π_n. If citizens vote sincerely this coalition has incentives to withdraw since then, $P(e'_{-G},e_G) = \pi_n$ where $e'_{-G} = (0,...,0)$ so that all the candidates in S strictly improve. If citizens vote strategically with sincere coalitions, we have that the sincere coalition $S_n^{C'}$ becomes a majoritarian coalition so that its best response should be such that $P(e'_{-G},e_G) = \pi_n$. It however contradicts that e be a strong entry equilibrium. [End Proof]

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