Ex-post Implementation with Interdependent Valuations

Moritz Meyer-ter-Vehn and Benny Moldovanu*

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Abstract

We consider a social choice setting with multidimensional signals and interdependent valuations. Such frameworks have been recently and increasingly used in order to study multi-object auctions. We obtain concise characterizations of ex-post implementable (not necessarily efficient) social choice functions in terms of affine functions that associate a weight to each agent and to each alternative. These characterizations can greatly reduce the complexity of the search for a constrained efficient (i.e., second best) mechanism in the generic cases where efficient outcomes cannot be implemented.

1 Introduction

We study a social choice problem where society has to choose among several possible alternatives. Each agent obtains a private signal about each possible alternative, and an agent's valuation for a given alternative depends both on her own, and on the other agents' information. An agent's utility is given by the sum of her valuation for the chosen alternative and a monetary transfer. Thus, we consider a model with quasi-linear utility functions, multi-dimensional signals and *interdependent valuations*.

A well-known special sub-case of the model described above is the *private-values* model where an agent's valuation for an alternative depends only on her own information. In this context, a prominent role is played by the

*We wish to thank Philippe Jehiel and Eric Maskin for helpful comments. This research has been partly sponsored by the German Science Foundation. Meyer-ter-Vehn and Moldovanu: University of Mannheim, Department of Economics, L7, 3-5, 68131 Mannheim, Germany. mold@pool.uni-mannheim.de

Clarke-Groves-Vickrey (CGV) mechanisms (see Vickrey, 1961, Clarke, 1971 and Groves, 1973). These are direct revelation mechanisms in which truthfully revealing the private signal is a dominant strategy for each agent, and where a value-maximizing (or efficient) alternative is chosen for any realization of signals. The key insight behind these mechanisms is that individual transfers are perfectly aligned to the marginal impact of the individual reports on society's welfare. This perfect alignment works because the valuations of all agents other than i, say, do not depend on i's information (i.e., we are in the private values case). An important result due to Green and Laffont (1977) shows that, when the set of admissible valuation functions is rich enough, the CGV mechanisms are in fact the only dominant-strategy, efficient mechanisms[1]. There are numerous applications of the CGV idea, which is central to mechanism design.

A growing literature (in particular in the subfield of auction theory) has sought to depart from the restrictive informational assumption of the private values paradigm, by allowing valuations to also depend on others' information. With interdependent valuations we cannot remain in the framework of dominant strategy implementation since non-trivial dominant strategy mechanisms usually do not exist. In particular, the marginal impact of agent i on society's welfare depends also on i's signal, and CGV mechanisms which use these impacts as transfers fail to create the right incentives for truthful revelation.

Milgrom (1981) and Milgrom and Weber (1982) considered one-object auction models where each bidder's valuation depends on all agents' one-dimensional signals. These authors focused on Bayes-Nash equilibria of standard auction formats in the case where the agents' valuations all have the same functional form (i.e., the model is symmetric), and where, by assumption, the agent with the highest signal has the highest value. Hence, in those models an efficient allocation is achieved by all auction formats were the agent with the highest signal gets the object.

Cremer and McLean (1985) focus their attention on the possibility of rent extraction when the agents' signals are correlated (full rent extraction implies, in particular, that a value-maximizing alternative must be chosen). They allow for interdependent and asymmetric valuations while considering

[1] It is also worth mentioning here that weakening the implementation requirement to Bayes-Nash equilibrium does not add much freedom: for example, if signals are independent, the expected transfers in any efficient mechanism where truth-telling is a Bayes-Nash equilibrium correspond to Clarke-Groves-Vickrey transfers. This is a consequence of the so called Revenue-Equivalence Theorem, which says that the expected transfers in any Bayes-Nash incentive compatible mechanism are already determined, up to a constant, by the social choice rule itself.

a finite set of linearly ordered signals for each agent. In this set-up they construct an efficient mechanism where truth telling is an *ex-post equilibrium*. The idea is to adjust the CGV transfer to agent i in a way that neutralizes the impact of i's signal on society's welfare (and hence on i's transfer). The ex-post equilibrium notion is stronger than Bayes-Nash equilibrium and weaker than dominant strategy equilibrium. For truth-telling in a direct revelation mechanism to be an ex-post equilibrium, correctly revealing the private information must be individually optimal ex-post, i.e. even after an agent has discovered the information available to others (and no matter what this information turns out to be). Roughly speaking, this is a condition of no-regret.

The Cremer-McLean construction hinges on a technical condition on valuations called *single-crossing* which restricts the set of permissible valuation functions in order to enable an alignment of private and social incentives (where the latter are represented by the sum of the individual valuations). A general formulation in a social choice framework is as follows: if a change in agent i's signal improves i's valuation of alternative k more than i's valuation of alternative l, then the same must be true for the relative improvement of social welfare. In the private values case this condition is trivially satisfied since the improvement in the social valuation (which depends on i's signal only via i's valuation) coincides with the individual improvement. This fact allows the efficient implementation via CGV mechanisms. As long as signals are one-dimensional, analogue conditions can be generically satisfied also in frameworks with interdependent valuations, and therefore ex-post incentive compatible and ex-post efficient mechanisms have been be constructed for a variety of models (see for example Ausubel (1997), Bergemann and Välimäki (2000), Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), Maskin (1991), and Perry and Reny (1999)). For example, in an one-object auction model with one-dimensional signals and interdependent valuations, single-crossing boils down to the requirement that i's signal has a stronger impact on i's valuation than on j's valuation, $j \neq i$.

The situation changes when signals are multidimensional. Jehiel and Moldovanu (2001) (JM) consider a social choice model with multidimensional, independently drawn signals (each agent obtains a private signal about each possible alternative), and with interdependent valuations that are linear functions of the signals. In this simple framework they show that efficiency is not compatible with Bayes-Nash incentive compatibility unless a strong, non-generic condition[2] holds on the valuation functions[3]. The JM condi-

[2] This condition is trivially satisfied in the private values case and in the pure common values case.

[3] In auction frameworks this difficulty arises either in multi-object auctions or in one-

tion requires that the marginal impact of an agent's various signals on the society's welfare is independent of the chosen alternative. In other words, efficient implementation is possible only for individual valuation functions where we can define the "weight" of each agent, independently of the chosen alternative.

The JM result immediately implies that, no matter how signals are distributed, efficiency is generally not compatible with ex-post incentive compatibility[4]. This impossibility result focuses the attention on other desirable and implementable social choice rules (e.g., such that may arise in the search for mechanisms that maximize welfare subject to the incentive compatibility constraints). Given that in the Bayes-Nash framework the incentive compatibility constraint takes the form of a complex partial differential equation, there is little hope for analytical advances within that framework. This in turn redirects our attention to the stronger and more robust (since independent of signals' distributions) notion of implementation in ex-post equilibrium: besides the associated intrinsic interest, a relatively simple characterization for all mechanisms in this smaller class would greatly simplify the search for second-best mechanisms in a variety of settings. Obtaining a powerful characterization is the main goal of the present paper, and we need to assume here that valuations are the sum of two components, where the first component describes the dependence on one's own signal, and the second component describes the dependence on others' signals. Although this kind of *semi-separability* is a restrictive assumption, the advantage is that it allows a very concise and powerful characterization of ex-post implementability. Without this assumption only a weak characterization is possible (see for example Theorem 1 in Chung and Ely[5], 2001).

For our first result we "endow" the designer with valuations over the alternatives that depend on the agents' signals (e.g., in efficient implementation it is as if the designer values each alternative by the sum of the agents' valuations). We characterize all such utilities where first-best implementation is possible: they must be monotone transformations of *affine* functions determined by *agent-specific* and *alternative-independent* weights with which an agent's information enters the designer's preferences, and by additional *alternative-specific* weights, which are independent of agents' information. Alternatively, this result can be interpreted as saying that, for arbitrary de-

object auctions with allocative externalities.

[4] Otherwise, since ex-post equilibria do not depend on the distribution of signals, we would obtain efficiency also for the case of independently drawn signals contradicting the JM result.

[5] Given the weak characterization, these authors focus on applications where signals are one-dimensional.

signer's preferences, first-best ex-post implementation is possible only if the agents' valuations satisfy some very restrictive conditions (these generalize the single crossing property which was originally geared towards the particular case of efficiency). The main role in the analysis is played by a basic property called "*positive association of differences*" (PAD), which must be fulfilled by any ex-post implementable social choice rule. Roughly speaking, PAD says that if a social choice rule chooses alternative k when the signal is s, then k continues to be chosen for all other signal realizations which, for all agents, make alternative k relatively more preferable than all other alternatives.

A corollary of the above result is that, in JM's linear setting, the JM condition described above (which was shown to be necessary for efficient, Bayes-Nash incentive compatible implementation) is in fact *necessary and sufficient* for efficient, ex-post incentive compatible implementation. This also shows that efficient Bayes-Nash implementation wit independent signals is in fact equivalent to efficient ex-post implementation.

Given the restrictive results obtained above, we next drop the requirement that the social choice rule maximizes some function of the agents' signals, in order to characterize *all* deterministic, ex-post implementable social choice rules. Somewhat surprisingly, we show that every such choice rule must nevertheless maximize an affine function of the agents' signals. The proof relies on a subtle result obtained by Roberts (1979). Roberts worked within a private values framework and characterized, in terms of affine functions, all social choice rules that are dominant strategy implementable (without the efficiency requirement)[6]. We are able to use Roberts' result since its proof relies solely on the PAD property, which is satisfied for dominant strategy implementable choice rules in his case, and for ex-post implementable choice rules in ours.

The price for dropping the maximizer assumption is that our result (as Roberts') hinges on two main conditions: first, there must be at least three social alternatives; second, the space of valuation functions must be sufficiently rich in the sense that, for any fixed signals of others, some agent can, by varying her signal, force the choice of any alternative. This last requirement parallels the one in Green and Laffont's (1977) characterization of dominant strategy mechanisms as CGV mechanisms for the private values

[6]Although Roberts' result is very elegant, it has received, to our knowledge, relatively little attention. We suspect that this is due to the fact that, in the private values framework, there always exist simple and well known mechanisms (the GCV ones) which are both efficient and dominant strategy incentive compatible. Hence there was relatively little interest in mechanisms that were not efficient. This should be contrasted with the interdependent framework where efficient mechanisms usually do not exist.

case (see above). It is automatically satisfied if, for each agent and for each alternative (recall that an agent gets a signal for each possible alternative) the respective signal space is the entire real line. We show via examples how the characterization fails if at least one of the conditions is not satisfied, and we also obtain a weaker characterization for the case with only two alternatives.

The paper is organized as follows: In Section 2 we describe the social choice model. In Section 3 we define the notion of implementation in ex-post equilibria, and we show that all ex-post implementable social choice rules must satisfy PAD. In Section 4 we characterize all deterministic, ex-post implementable social choice rules that arise as maximizers of some designer's utility function (e.g. the efficient SCR). In Section 5 we drop the maximizing requirement and characterize all deterministic social choice rules that are ex-post implementable, subject to two important conditions pertaining to the number of the alternatives and to the signal spaces. We also discuss the role of these conditions. Section 6 concludes.

2 Model

There is a set N of agents, indexed $i, j = 1, ..., N$ and a set K of alternatives, indexed by $k, l = 1, ..., K$[7]. Each agent $j \in N$ has a signal $s^j = \left(s_k^j\right)_{k \in K}$ where s_k^j is drawn from an interval $S_k^j \subseteq \mathbb{R}$. Thus agent i's signal space is $S^j := \prod_{k \in K} S_k^j \subseteq \mathbb{R}^K$. s^j is private information of agent j. The space of signal combinations of all agents is the Cartesian product $S := \prod_{j \in N} S^j \subseteq \left(\mathbb{R}^K\right)^N$ with generic element s[8]. The space of signal combinations of agents other than i is denoted $S^{-i} := \prod_{j \neq i} S^j$, with generic element s^{-i}. Moreover we denote the set of signals on alternative k by $S_k := \prod_{j \in N} S_k^j \subseteq \mathbb{R}^N$ with generic element $s_k = \left(s_k^j\right)_{j \in N}$. As usual, we write $s = (s^i; s^{-i})$ when we want to emphasize agent i's role. Signal combinations are distributed according to some density function $f(s) > 0$ for all $s \in S$.

Agent i's utility when alternative k is chosen depends in an additively separable manner on her own signal, on other agents' signals in that alternative, and on a monetary transfer. If the true information is given by s her valuation for alternative k is given by:

$$v_k^i(s) = s_k^i + h_k^i\left(s_k^{-i}\right), \tag{1}$$

[7] This slight abuse of notation will not lead to confusion.

[8] Our results can be replicated for arbitrary convex sets $S \subseteq \left(\mathbb{R}^K\right)^N$.

where h_k^i is an arbitrary function[9]. When agent i receives a monetary transfer of t^i her utility is $v_k^i(s) + t^i$.

A *deterministic social choice rule* (SCR) chooses one of the K alternatives, contingent on the signal $s = (s_k^i)_{i \in N, k \in K} \in S$. In the sequel we always consider deterministic social choice rules (without mentioning it anymore). In addition, we only consider SCRs such that, for each alternative k, there exists at least one signal combination s where k is indeed chosen by the SCR.

A SCR ψ is called *efficient*, if for every signal s, it maximizes the sum of the agents' utilities net of monetary transfers:

$$\psi(s) \in \arg\max_{k \in K} \left\{ \sum_{i=1}^{N} v_k^i(s) \right\} \tag{2}$$

We study here *deterministic direct revelation mechanisms*[10] $\Gamma = (g, t) : S \to K \times \mathbb{R}^N$, which, for each signal s, specify an alternative $g(s) \in K$ and a payment $t^i(s) \in \mathbb{R}$ to each agent i.

3 Ex-Post Implementation and Positive Association of Differences

Non-trivial dominant strategy equilibria usually do not exist in the presence of interdependent valuations. For this case, the ex-post equilibrium notion can be seen as the right analogue to the dominant equilibrium notion in private values models.

Definition 1 *A SCR ψ is **ex-post implementable** if there exists a mechanism $\Gamma = (g, t)$ such that $g(s) = \psi(s)$ for all s, and such that truth-telling is an ex-post equilibrium in Γ:*

$$v_{g(s)}^i(s) + t^i(s) \geq v_{g(\widehat{s}^i; s^{-i})}^i(s) + t^i\left(\widehat{s}^i; s^{-i}\right)$$

for every agent i, every true signal $s = (s^i; s^{-i})$, and every possible reported signal $\widehat{s}^i \in S^i$.

A crucial role in our analysis will be played by a property of SCRs called "*positive association of differences*" (PAD), as formulated in Roberts (1979):

[9] The "naked" appearance of s_k^i in the above formula is a normalization: as usual, it is important to have an valuation function which is strictly monotonic in own signals, and if the separable valuation function has the form $v_k^i(s) = d_k^i(s_k^i) + h_k^i\left(s_k^{-i}\right)$ where d_k^i is an arbitrary strictly monotonic function, we redefine the signals accordingly.

[10] As usual, a revelation principle applies in our framework.

Definition 2 *A SCR ψ satisfies PAD, if and only if for signals $s, s' \in S$ such that*

$$s_k'^i - s_k^i > s_l'^i - s_l^i \text{ for all } i \in N \text{ and all } l \neq k \in K,$$

$\psi(s) = k$ implies $\psi(s') = k$.

In Words: Let the social choice be alternative k at signal s. Then the choice must also be k at signal s', if the change in signal from s to s' makes alternative k relatively more preferable than all other alternatives $l \neq k$ for all agents. The importance of PAD for our characterization results is conveyed by the following Lemma:

Lemma 3 *Every ex-post implementable SCR ψ satisfies PAD.*

Proof. See Appendix. ■

It is important to note here that the more ubiquitous single crossing properties mentioned in the Introduction (which directly impose conditions on the agents' valuation functions) precisely induce the necessary alignment between individual incentives and social incentives represented by the efficient SCR; i.e. they indirectly ensured that the efficient SCR satisfies PAD in the respective framework. In contrast, PAD is an abstract property of SCRs, and does not involve the underlying model of preferences (be it private values, interdependent values, etc.). The following simple result shows that the ex-post implementability of an SCR does not depend on the underlying model of preferences:

Lemma 4 *An SCR $\psi : S \to K$ is ex-post implementable in the interdependent values model if and only if it is ex-post implementable in the associated private values model where $\forall i, k,\ h_k^i \equiv 0$.*

Proof. See Appendix. ■

For further reference we also note here that most structural features of implementable SCRs are determined at signals where the SCR is "just on the brink" between two or more alternatives:

Definition 5 *Given a SCR $\psi : S \to K$ and a signal combination $s \in S$, ψ is said to be **indifferent** between alternatives k, l at s , if and only if $s \in \overline{\psi^{-1}(k)} \cap \overline{\psi^{-1}(l)}$.*[11]

[11] $\overline{A}$ denotes the topological closure of a subset $A \subseteq S$.

4 Principal-Agents Problems and Efficiency

In this section we assume that the designer is endowed with an utility function, i.e., there are functions $\{\phi_k(s_k)\}_k$ such that the designer values alternative k at $\phi_k(s_k)$. Hence, for any realization of agents' signals[12], the designer would like to choose an alternative $\psi(s) \in \arg\max_{k \in K} \{\phi_k(s_k)\}$. For example, efficient implementation is obtained in this framework by endowing the designer with valuations $\phi_k(s_k) = \sum_i v_k^i(s)$.

Is such first-best ex-post implementation possible for the designer? As indicated in the Introduction the answer is, in general, negative. To make this point in the simplest manner, let us briefly look at efficient implementation with linear valuations:

$$v_k^i(s) = \sum_{j \in N} a_{ki}^j s_k^j \text{ , with } a_{ki}^i = 1$$

Thus $a_{ki}^j \in \mathbb{R}$ is the coefficient measuring the influence of agent j's signal about alternative k on i's valuation of alternative k. Note that agent i's signal has an impact of $\sum_{j=1}^N a_{kj}^i$ on the designer's valuation.

Lemma 3 implies that a change in i's signal $s^i \to s'^i$ which improves i's valuation of alternative k more than her valuation for alternative l must also improve the designer's valuation of k more than her valuation for l. Letting $s := (s^i, s^{-i})$, $s' := (s'^i, s^{-i})$, this means:

$$\begin{aligned} \left(v_k^i(s') - v_k^i(s)\right) - \left(v_l^i(s') - v_l^i(s)\right) &\geq 0 \Rightarrow \\ \left(\sum_{j=1}^N v_k^j(s') - v_k^j(s)\right) - \left(\sum_{j=1}^N v_l^j(s') - v_l^j(s)\right) &\geq 0 \end{aligned} \quad (3)$$

For the linear specification of utilities, the above condition holds for all $s_k'^i - s_k^i, s_l'^i - s_l^i \in \mathbb{R}$ if and only if

$$\sum_{j=1}^N a_{kj}^i = \sum_{j=1}^N a_{lj}^i \geq 0. \quad (*)$$

In other words, the impact of i's signal on the designer's valuation must be positive and independent of the chosen alternative.

We show below how this insight can be generalized: thus, we return to the more general specification of valuations for agents and designer (see Section

[12]These functions may of course depend also on signals available to the principal, but, since, the principal does not act strategically, this information acts as a constant and we do not explicitly carry it here.

2, and the first paragraph of the present Section). We first need the following definition:

Definition 6 *A SCR $\psi : S \to K$ is said to be an* ***affine maximizer****, if and only if it is of the form:*

$$\psi(s) \in \arg\max_{k\in K} \left\{ \sum_{j=1}^{N} \alpha^j s_k^j + \lambda_k \right\}$$

for agent-specific weights $\alpha^j \geq 0$ (with $>$ for at least one j) and alternative-specific weights $\lambda_k \in \mathbb{R}$.

For a given affine maximizer, the weight α^j can be interpreted as the importance of agent j's information to the social choice, and the weight λ_k as the designer's preference for alternative k. When a general SCR is indifferent between two alternatives k, l at some signal combination s, there is a rate $r_{ij} = r_{ij}(s, k, l)$ of information substitution between agents i and j[13], such that the SCR stays indifferent if we change the signal combination by increasing agent i's signal on alternative k, say, and decreasing j's signal on l at this rate. The main feature of an affine maximizer is that this rate, which is given by α_i/α_j, depends neither on the signal s nor on the alternatives k, l. We now have:

Theorem 7 *1) Assume that a SCR $\psi : S \to K$ is ex-post implementable*[14] *and satisfies $\psi(s) \in \arg\max_{k\in K} \{\phi_k(s_k)\}$ for some differentiable*[15] *functions $\phi_k : S_k \to K$ with $\nabla\phi_k \neq 0$. Then ψ is an affine maximizer. 2) Conversely, any affine maximizer is ex-post implementable.*

Proof. See Appendix. ■

From this proposition it easily follows that the efficient SCR is implementable if and only if maximizing $\sum_j v_k^j(s)$ yields the same choice rule ψ as maximizing $\sum_{j\in N} \alpha^j s_k^j + \lambda_k$ for some α^j, λ_k. We can now connect our result on ex-post implementation with the main result of JM who consider Bayes-Nash incentive compatible and ex-post efficient implementation in a setting with linear utilities and independently drawn signals. A priori, there could be circumstances in which the efficient SCR is implementable in the sense of Bayes-Nash equilibrium, but not in the stronger sense of ex-post equilibrium. But JM showed that condition * (which implies the existence of

[13] The information substitution condition * reads $r_{ii}(s, k, l) \equiv 1$ in this terminology.

[14] The proof uses only the fact that ψ satisfies PAD.

[15] Differentiability is not crucial. It is used in order to obtain a concise proof.

agent-specific, alternative-independent weights as needed for an affine maximizer) is already necessary for efficient Bayes-Nash implementation[16]. Together with Theorem 7, this yields:

Corollary 8 *In the linear setting, the three following statements are equivalent:*

1. *The efficient SCR is ex-post implementable.*
2. *The efficient SCR is BNE implementable for the case of independent signals.*
3. *Condition * holds for all agents i and all alternatives k, l.*

5 Implementation of General Social Choice Rules

The previous Section showed that the designer's first-best outcome can be implemented only under restrictive conditions. Hence, in this Section we want to characterize the set of **all** ex-post implementable social choice rules (i.e., without requiring that the SCR to be implemented maximizes some utility function for the designer). The analysis is conducted in the general semi-separable framework described in Section 2. It turns out that the previous affine characterization still plays a major role. We have:

Theorem 9 *Assume that: 1) $S = \left(\mathbb{R}^K\right)^N$ and 2) $K > 2$. Then a SCR $\psi : S \to K$ is ex-post implementable if and only if it is an affine maximizer*[17].

Proof. See Appendix. ■

The "if" part of the above theorem follows by the converse part of Theorem 7, and does not depend on the two assumptions. The "only if" part is more subtle and crucially relies on the assumptions. To better understand what Theorem 9 says, consider the trivial case where only agent i holds private information s^i (while s^{-i} is fixed and known). Agent i has utility

[16] The main idea of their proof is to check when the conditional expected probability vector field (whose coordinates are the expected probabilities with which an agent with a given signal expects an efficient mechanism to choose the various alternatives) satisfies a complex integrability constraint imposed by Bayes-Nash incentive compatibility. JM focus on the equality part of equation *. The inequality part follows from the monotonicity of their vector field.

[17] This theorem also implies Theorem 7, when we adopt its assumptions.

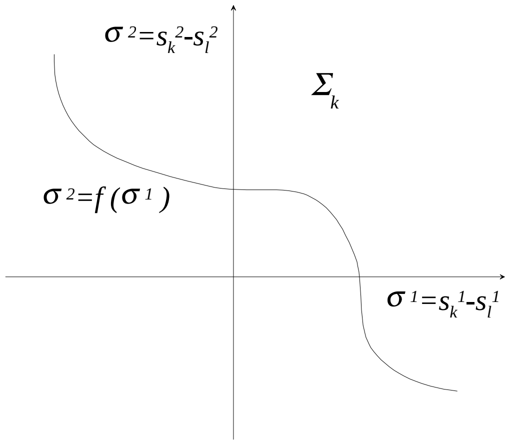

Figure 1:

$s^i_k + h^i_k\left(s^{-i}_k\right) + t^i$. Let ψ be implemented by the mechanism (g, t). The monetary transfer to agent i may depend on the signal s^i only via the chosen alternative $g(s)$, i.e. $t^i(s) = t^i_{g(s)}\left(s^{-i}\right)$. Setting $\lambda_k := h^i_k\left(s^{-i}_k\right) + t^i_k\left(s^{-i}\right)$, the mechanism (g, t) induces truthtelling only if g is an affine maximizer: $g(s) \in \arg\max_{k\in K}\{s_k + \lambda_k\}$. For $N > 1$ agents holding private information, the same thing must be true for any agent i and for any fixed signal s^{-i} of the other agents. Theorem 9 states that the only way to consistently solve the N one-agent problems is via an affine maximizer.

5.1 The Role of the Assumptions

The following result shows that the requirement $K > 2$ is necessary for Theorem 9:

Proposition 10 *Assume that $K = \{k, l\}$ and define reduced signals by $\sigma^i := s^i_k - s^i_l$. Then a SCR $\psi : S \to K$ is ex-post implementable if and only if there exists an open comprehensive*[18] *set $\Sigma_k \subseteq \mathbb{R}^N$ such that:*

$$\psi(s) = \begin{cases} k & \text{if } \left(\sigma^1, ..., \sigma^N\right) \in \Sigma_k \\ l & \text{if } \left(\sigma^1, ..., \sigma^N\right) \in \mathbb{R}^N \backslash \bar{\Sigma}_k \end{cases}$$

Proof. See Appendix. ■

The set Σ_k above can be described by a function describing its border. For any agent i define a function $f : \mathbb{R}^{N-1} \to \mathbb{R} \cup \{-\infty, \infty\}$ as follows:

[18] i.e. a set $\Sigma_k \subseteq \mathbb{R}^N$ with the property that $\Sigma_k + \mathbb{R}^N_+ \subset \Sigma_k$ (i.e. $\Sigma_k + \alpha \subset \Sigma_k$ for every $\alpha \gg 0$).

$f\left(\sigma^{-i}\right) := \inf\left\{\sigma^i : \left(\sigma^i, \sigma^{-i}\right) \in \Sigma_k\right\}$. f is easily seen to be weakly monotonic decreasing. Consider now figure 1 which illustrates the easily graphed case $N = 2$ (the intuition remains valid for arbitrary N). The figure shows that the rate of informational substitution $r_{12}(s, k, l)$ among agents 1 and 2 (which is given by $-\partial f / \partial \sigma^1$) needs not be constant, in contrast to the case of an affine maximizer.

Next we give an example showing that the assumption $S = \left(\mathbb{R}^K\right)^N$ is indispensable in Theorem 9.

Example 11 *Let there be two agents* $1, 2$, *three alternatives* $K = \{o, k, l\}$ *and let the signal space be* $S = \left([0,1]^3\right)^2$. *Then the social choice rule* $\psi : S \rightarrow K$ *defined below is ex-post implementable but it is not an affine maximizer:*

$$\begin{aligned} \psi(s) &:= \begin{cases} \arg\max_{\{o,k,l\}} \left\{-1.5 + s_o^1 + s_o^2, s_k^1 + s_k^2, s_l^1 + s_l^2\right\} & \text{if } s \in S_{gen} \\ l & \text{if } s \in S \backslash S_{gen} \end{cases} \\ S_{gen} &:= \left\{s \in S : s_l^1 < s_k^1 + 0.5, \text{ or } s_l^2 > s_k^2 - 0.5\right\} \end{aligned}$$

Proof. To show ψ is not an affine maximizer, assume contrarily that it was one with weights $\alpha^1, \alpha^2, \lambda_o, \lambda_k, \lambda_l$. The formula defining ψ for $s \in S_{gen}$ entails first: $\alpha^1 = \alpha^2$ and using this: $\lambda_k = \lambda_l = \lambda_o + 1.5$. An affine maximizer with these parameters would choose k at the signal s defined by $s_o^1 = 0.5, s_o^2 = 0.5$, $s_k^1 = 0.2, s_k^2 = 0.9$, $s_l^1 = 0.8, s_l^2 = 0.1$, but as $s \notin S_{gen}$ we have $\psi(s) = l$, a contradiction.

To see that ψ is ex-post implementable, the reader can check that the mechanism (g, t) defined by $g \equiv \psi$ and

$$t^1(s) := \begin{cases} s_o^2 & \text{if } g(s) = o \\ \min\left\{1.5 + s_k^2, 2 + s_l^2\right\} & \text{if } g(s) = k \\ 1.5 + s_l^2 & \text{if } g(s) = l \end{cases}$$

induces agent 1 to tell the truth. The transfers $t^2(s)$ can be analogously defined. ■

What goes wrong in Example 11? Consider s^2 with $s_k^2 > 0.5$. No signal s^1 can lead to $\psi(s) = o$. Thus, for s^2 in a subset of S^2 agent 1 can only "choose" among alternatives k and l. For this subset we only get the weak characterization of Proposition 10 instead of the strong one of Theorem 9.

If we severely restrict the class of allowable SCRs, in order to rule out the problems of example 11, we get an analogue to Theorem 9.

Definition 12 *An agent* i *is decisive for an SCR* ψ, *if for every signal of the other agents* s^{-i} *and for any alternative* k *there is a signal* s^i *in the interior of* S^i *such that* $\psi\left(s^i, s^{-i}\right) = k$.

For the example of an affine maximizer ψ, any agent j with $\alpha^j > 0$ and $S^j = \mathbb{R}^K$ is decisive for ψ. In particular under the assumptions of Theorem 9, given any ex-post implementable SCR ψ, there is some agent i who is decisive for ψ.

Proposition 13 *If a SCR ψ is ex-post implementable and if some agent i is decisive for ψ, then ψ is an affine maximizer.*

Proof. See Appendix. ■

6 Conclusion

Since efficient implementation in settings with interdependent valuations and multidimensional signals is usually not-possible, one has to look for implementable social choice rules that satisfy some other desirable criteria. Given some technical conditions, we have characterized all ex-post incentive compatible SCRs in terms of affine functions of the agents' signals. We have therefore greatly simplified the quest for other satisfactory mechanisms by reducing the complex design problem to one of determining a finite set of real numbers (representing agent-specific and alternative-specific weights) which yields the desired outcome.

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Appendix

Proof of Lemma. 3: Let ψ be ex-post implemented by a mechanism (g,t). Consider two signals $s=(s^i,s^{-i}), s'=(s'^i,s^{-i})\in S$ which differ only in agent i's signal and denote $g(s)=k$ and $g(s')=k'$. In order to induce truthful revelation of both s^i and s'^i, it is necessary to have:

$$\begin{array}{rcl} s^i_k+h^i_k(s^{-i}_k)+t^i_k\left(s^{-i}\right) & \geq & s^i_{k'}+h^i_{k'}(s^{-i}_{k'})+t^i_{k'}\left(s^{-i}\right) \\ s'^i_k+h^i_k(s^{-i}_k)+t^i_k\left(s^{-i}\right) & \leq & s'^i_{k'}+h^i_{k'}(s^{-i}_{k'})+t^i_{k'}\left(s^{-i}\right) \end{array}$$

Taking differences, we obtain:

$$s'^i_k-s^i_k\leq s'^i_{k'}-s^i_{k'}$$

Combining the above inequality with the hypothesis that $s'^i_k-s^i_k>s'^i_l-s^i_l$ for all $i\in N$ and all $l\neq k\in K$, yields $g(s')=k'=k$.

To complete the proof for arbitrary s, s', it suffices to regard the sequence of signals $s_{(0)} := s$, $s_{(i)} := \left(s'_i, s_{(i-1)}\right)$ for all agents $i \leq n$ (this gives $s_{(n)} = s'$). The argument above serves then as the induction step proving that with $\psi\left(s_{(0)}\right) = k$ we have $\psi\left(s_{(i)}\right) = k$ for all i, yielding $\psi\left(s'\right) = k$. ■

Proof of Lemma 4. **"if":** Let ψ be ex-post implemented by a mechanism (g, t) under private values. Thus $s^i_{g(s)} + t^i(s) = \max_{\hat{s}^i \in S^i}\left(s^i_{g(\widehat{s})} + t^i(\widehat{s})\right)$ for all i, s^i, s^{-i}, denoting $s = \left(s^i, s^{-i}\right), \widehat{s} = \left(\widehat{s}^i, s^{-i}\right)$. If utilities are of the general interdependent form $s^i_k + h^i_k\left(s^{-i}_k\right) + t^i$ we define the monetary payments t' that implement ψ by $t'^i(s) = t^i(s) - h^i_{g(s)}\left(s^{-i}_{g(s)}\right)$. One easily verifies $s^i_{g(s)} + h^i_{g(s)}\left(s^{-i}_{g(s)}\right) + t'^i(s) = \max_{\hat{s}^i \in S^i}\left(s^i_{g(\widehat{s})} + h^i_{g(\widehat{s})}\left(s^{-i}_{g(\widehat{s})}\right) + t'^i(\widehat{s})\right)$ for all i, s^i, s^{-i}. This shows that (g, t') implements ψ in ex-post equilibrium for the interdependent values case.

"only if": Analogously. ■

Proof of Theorem. **7**: **1)** Assume that ψ is ex-post implementable and satisfies the maximizing condition. By Lemma 3, ψ satisfies the PAD property.

For the sake of clarity of exposition we assume that for every signal $s_k \in S_k$ there are signals $s_{-k} \in S_{-k}$ such that ψ is indifferent between alternative k and some alternative l at signal s, i.e. $\phi_k(s_k) = \phi_l(s_l) = \max_{k'} \phi_{k'}(s_{k'})$.[19]

Suppose ψ to be indifferent between k and l when the signal is s. The proof of Lemma 3 yields $\nabla\phi_k(s_k) > 0$[20] which, using PAD again, shows $\phi_k(s_k + \delta) = \phi_l(s_l + \delta) > \phi_k(s_k)$ for small enough $\delta \in \mathbb{R}^N_+$.[21] This implies:

$$\phi_k(s_k) = \phi_k(s'_k) \Rightarrow \phi_k(s_k + \delta) = \phi_k(s'_k + \delta) \tag{4}$$

for all $s_k, s'_k, s_k + \delta, s'_k + \delta \in S_k$, which in turn implies $\nabla\phi_k(s_k) = \nabla\phi_k(s'_k)$. Given $x \in \mathbb{R}$, $(\phi_k)^{-1}(x)$ is a $n-1$ dimensional sub-manifold with unit normal

[19] In general define the set $I_k \subseteq S_k$ such that for each $s_k \in I_k$ there is $s_{-k} \in S_{-k}$ with the property that for some l we have $\phi_k(s_k) = \phi_l(s_l) = \max_{k'} \phi_{k'}(s_{k'})$. With $I := \{x \in \mathbb{R} : \exists k, l \in K, k \neq l$ such that $\max_k \phi_k(s_k) = \max_l \phi_l(s_l) \geq x$ and $\forall k \in K : \min_{s_k} \phi_k(s_k) < x\}$ we have $I_k := (\phi_k)^{-1}(I)$.

We then conduct the proof for $s_k \in I_k$ (instead of S_k) and $x \in I$ (instead of $\mathbb{R}$). For $s_k \in S_k \backslash I_k$ it must either be that $\phi_k(s_k) \geq \sup I$ or that $\phi_k(s_k) \leq \inf I$. In the first case PAD yields $f\left(\sum_{j \in N} \alpha^j s^j_k + \lambda_k\right) \geq \sup I$ and thus we can replace $\phi_k(s_k)$ by $f\left(\sum_{j \in N} \alpha^j s^j_k + \lambda_k\right)$, without changing $\psi(s) = \arg\max_{k \in K}\{\phi_k(s_k)\}$ for any s. The second case is analogous.

[20] i.e. $\partial_i \phi_k \geq 0$ for all $i \in N$, with strict inequality for at least one $i \in N$

[21] This proof as stated works only for $s_k \in \mathring{S}_k$, the interior of S. For $s_k \in S_k \backslash \mathring{S}_k$ the statement follows by continuity of ϕ_k.

vector $\alpha = (\alpha^i)_{i\in N} = \frac{\nabla\phi_k(s_k)}{\|\nabla\phi_k(s_k)\|} > 0$. The coordinates α^i depend neither on $s_k \in (\phi_k)^{-1}(x)$, nor on x (since, by equation 4, the sets $(\phi_k)^{-1}(x)$ for different x are just translates of each other) nor on k (by PAD).

A sub-manifold with constant unit normal vector is a hyperplane: $(\phi_k)^{-1}(x) = \{s_k \in S_k : s_k \cdot \alpha = const.\}$. A change of variables shows that, for all $s_k \in S_k$, $\phi_k(s_k) = f_k\left(\sum_{j\in N}\alpha^j s_k^j\right)$ for some function $f_k : \mathbb{R} \to \mathbb{R}$ with $f_k' > 0$. PAD yields that this function depends on k only by an additive constant in the argument $\phi_k(s_k) = f\left(\sum_{j\in N}\alpha^j s_k^j + \lambda_k\right)$ for some $\lambda_k \in \mathbb{R}$. As $f' > 0$ this concludes the proof.

2) For the converse part, let $\psi(s) \in \arg\max_{k\in K}\left\{\sum_{j=1}^N \alpha^j s_k^j + \lambda_k\right\}$. Define $g \equiv \psi$ and $t^i(s) := \frac{1}{\alpha^i}\left(\sum_{j\neq i}\alpha^j s_{g(s)}^j + \lambda_{g(s)}\right) - h_{g(s)}^i\left(s_{g(s)}^{-i}\right)$ for all agents i such that $\alpha^i > 0$. If $\alpha^i = 0$, agent i's signal is irrelevant for the decision, and we set $t^i \equiv 0$.

Given her true signal s^i, and given the other agents' truthfully reported signals s^{-i}, agent i faces the problem of what signal $\widehat{s}^i$ to announce. Denoting $\widehat{s} := (\widehat{s}^i; s^{-i})$, her utility is given by:

$$\begin{aligned} v_{g(\widehat{s})}^i(s) + t^i(\widehat{s}) &= s_{g(\widehat{s})}^i + h_{g(\widehat{s})}^i\left(s_{g(\widehat{s})}^{-i}\right) + \frac{1}{\alpha^i}\left(\sum_{j\neq i}\alpha^j s_{g(\widehat{s})}^j + \lambda_{g(\widehat{s})}\right) - h_{g(\widehat{s})}^i s_{g(\widehat{s})}^{-i} \\ &= \frac{1}{\alpha^i}\left(\sum_j \alpha^j s_{g(\widehat{s})}^j + \lambda_{g(\widehat{s})}\right). \end{aligned}$$

Agent i optimally chooses $\widehat{s}^i$ such that $g(\widehat{s})$ maximizes this expression. This is done by telling the truth: $\widehat{s}^i := s^i$ (here we used $\alpha^i \geq 0$). ■

Proof of Theorem 9. "if": Follows by the converse part of Proposition 7. Note that this argument goes through for arbitrary $K \in \mathbb{N}$ and for an arbitrary space of signal combinations $S \subseteq \left(\mathbb{R}^K\right)^N$.

"only if": This part uses an important result due to Roberts (1979). Roberts studied deterministic SCRs that are implementable in dominant strategies in a private values setting, and he showed that such SCRs must satisfy PAD. Using our notation, his proof relies on the following technical result[22]:

Theorem A (Roberts 1979): Assume that: 1) $S = \left(\mathbb{R}^K\right)^N$ and 2) $K > 2$. Then any SCR $\psi : S \to K$ which satisfies PAD is an affine maximizer.

Together with Lemma 3 , Theorem A implies the result. ■

[22] Roberts' proof uses a hyperplane-separation argument which yields the weights in the affine representation.

Proof of Proposition 10. **"if":** Let ψ be of the above form with a given set Σ_k. By Lemma 4, we can restrict attention to a model with private values. Given signals s^{-i}, and recalling that $\sigma^i := s^i_k - s^i_l$, the transfers that implement ψ are defined as follows: $t^i_k(s^{-i}) = -\inf\{\sigma^i : (\sigma^i, \sigma^{-i}) \in \Sigma_k\}$ if there are s^i, s'^i such that $\psi(s^i, s^{-i}) = k, \psi(s'^i, s^{-i}) = l$, and $t^i_k(s^{-i}) = 0$ otherwise; $t^i_l(s^{-i}) = 0$. Thus, given (s^i, s^{-i}), agent i prefers k to l if and only if:

$$\begin{aligned} s^i_k + t^i_k\left(s^{-i}\right) &> s^i_l + t^i_l\left(s^{-i}\right) \Leftrightarrow \\ s^i_k - s^i_l &> \inf\left\{s^i_k - s^i_l : \left(s^i, s^{-i}\right) \in S\right\} \end{aligned}$$

which is just the case when the mechanism will choose k. If the infimum does not exists, i's signal is irrelevant to $\psi(s^i, s^{-i})$ and the mechanism can ignore it.

"only if": Suppose that ψ is ex-post implementable. By Lemma 3 , ψ satisfies PAD. Let Σ_k be the interior of $\left\{\left(\sigma^1, ..., \sigma^N\right) : \psi(s) = k\right\} + \mathbb{R}^N_+$ (this is well defined thanks to PAD). We have to show that $\psi(s) = k$ if $\left(\sigma^1, ..., \sigma^N\right) \in \Sigma_k$ and $\psi(s) = l$ if $\left(\sigma^1, ..., \sigma^N\right) \notin \Sigma_k$. Indeed if $\left(\sigma^1, ..., \sigma^N\right) \in \Sigma_k$ there is an $s' \in S$, such that $\psi(s') = k$ and $s^i_k - s^i_l > s'^i_k - s'^i_l$ for all agents i. Then, by PAD, we have $\psi(s) = k$. The other assertion is proved by an analogous argument. ■

Proof of Proposition 13. The idea of the proof is to first take s^{-i} as fixed, and describe as a function of s^i the choice rules inducing truthtelling for agent i. Then we examine the dependence of the choice rule on s^{-i}.

We start with some technical prerequisites. Choose an (arbitrary) ordering $\succ$ of the alternatives, and assume $\psi^{-1}\{l \in K : l \succ k\}$ is closed for every $k \in K$.[23] This allows us to strengthen Lemma 3 as follows:

Lemma 14 *Let ψ be a SCR that is ex-post implementable and satisfies $\psi(s) = \max\left\{k \in K : s \in \overline{\psi^{-1}(k)}\right\}$ for the ordering $\succ$. Then for signals $s, s' \in S$ such that:*

$$s'^i_k - s^i_k \geq s'^i_l - s^i_l \quad \textit{for all } i \in N \textit{ and all } l \neq k \in K$$

$\psi(s) = k$ implies $\psi(s') = k$.

[23] The proposition for general ψ then follows by applying the proof below to ψ' defined by $\psi'(s) := \max\left\{k \in K : s \in \overline{\psi^{-1}(k)}\right\}$, where the max is with respect to $\succ$. Then $\psi'(s) \in \arg\max_{k\in K}\left\{\sum_{j=1}^N \alpha^j s^j_k + \lambda_k\right\}$ implies $\psi(s) \in \arg\max_{k\in K}\left\{\sum_{j=1}^N \alpha^j s^j_k + \lambda_k\right\}$.

Proof. Consider $s = (s^i, s^{-i})$ and $s' = (s'^i, s^{-i})$ as in the statement of the lemma. The case with general s' following by induction. In the proof of Lemma 3 we noticed that $s'^i_k - s^i_k > s'^i_l - s^i_l \Rightarrow \psi(s') \neq l$. This entails $s'^i_k - s^i_k \geq s'^i_l - s^i_l \Rightarrow \psi(s' + \varepsilon e^i_k) \neq l$ for any $\varepsilon > 0$ and therefore $\psi(s') \neq l$ if $k \succ l$. For $l \succ k$ on the other hand we have $\psi(s + \varepsilon e^i_l) = k$ for some $\varepsilon > 0$ yielding $\psi(s') \neq l$. We conclude that $\psi(s') = k$.[24] ■ ■

Proof of Proposition 13 resumed. Lemma 14 makes rigorous the idea that ψ only depends on a reduced signal σ, rather than all information of the signal s. Define:

$$\sigma^j_k := \pi^j_k(s) := s^j_k - s^j_o \text{ for all } k \neq o \text{ in } K \text{ and all } i \in N$$

where $o \in K$ denotes the largest element in K with respect to $\succ$. Formally, π is a function from $S \subset (\mathbb{R}^K)^N$ onto $\Sigma \subset \prod_{j \in N} \Sigma^j \subset (\mathbb{R}^{K-1})^N$. For agent $j \in N$ and alternative $k \neq o \in K$ we denote by e^j_k the standard basis vectors in Σ. For ease of further notation we define $\sigma^j_o := 0$ for all agents j. By Lemma 14 ψ is constant on $\pi^{-1}(\sigma)$ for given $\sigma \in \Sigma$.[25] Thus we obtain a choice rule $\phi : \Sigma \to K$ that represents ψ in the new variables, i.e. it satisfies $\phi(\pi(s)) = \psi(s)$ for all signals $s \in S$.

We now come to the heart of the proof. We want to define a function $f : \Sigma^{-i} \to \mathbb{R}^{k-1}$ (with coordinate functions f_k for alternatives $k \neq o$, defining $f_o \equiv 0$ such that $\sigma^i_k - f_k(\sigma^{-i}) = 0$ for $k = o$), such that for every signal $\sigma^i \in \Sigma^i$:

$$\phi\left(\sigma^i, \sigma^{-i}\right) \in \arg\max_{k \in K} \left\{\sigma^i_k - f_k\left(\sigma^{-i}\right)\right\} \tag{5}$$

For the case $K = \{o, k, l\}$ this is depicted in figure 2. The point $f(\sigma^{-i})$ completely describes the choice rule faced by agent i anticipating σ^{-i}.

Our goal is to show that f_k is affine of the form:

$$f_k\left(\sigma^{-i}\right) = -\lambda_k - \sum_{j \neq i} \alpha^j \sigma^j_k.$$

This will imply $\phi(\sigma) \in \arg\max_{k \in K} \left\{\sigma^i_k + \sum_{j \neq i} \alpha^j \sigma^j_k + \lambda_k\right\}$, which in turn implies $\psi(s) \in \arg\max_{k \in K} \left\{\sum_j \alpha^j s^j_k + \lambda_k\right\}$, finishing this proof.

By the assumptions that i is decisive in ψ and that ψ induces truthtelling of agent i, ϕ must be of the form in equation 5 with $f_k(\sigma^{-i}) = -(t^i_k(s^{-i}) - t^i_o(s^{-i}))$

[24] This proof as stated works only for $s, s' \in \mathring{S}$, the interior of S. For $s \in S \backslash \mathring{S}$ the statement follows by the fact that $\psi(s) = \max\left\{k \in K : s \in \overline{\psi^{-1}(k)}\right\}$.

[25] Note that this is in general not true for ψ satisfying PAD but not PAD'.

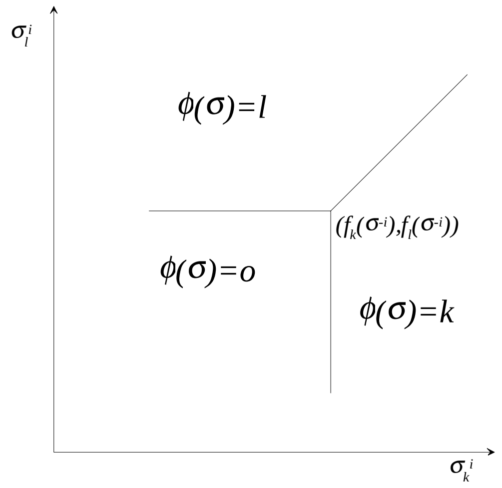

Figure 2: When agent i anticipates σ^{-i} (taken fixed in this figure), he "chooses" between o, k, l by announcing σ^i in the corresponding area. i is indifferent between the alternatives when his signal σ^i is just the indifference point $(f_k(\sigma^{-i}), f_l(\sigma^{-i}))$.

$-\left(h_k^i(s^{-i}) - h_o^i(s^{-i})\right)$, where $\pi(s) = \sigma$. This parallels the principal-agent case after Theorem 9.

Lemma 14 yields that f_k is constant with respect to changes in σ_l^j for all agents $j \neq i$ and alternatives $l \neq k$. Suppose not: Then there exist $\sigma^{-i}, \sigma'^{-i}$ differing only in $\sigma_l^j \neq \sigma_l'^j$ for some agent $j \neq i$ such that $f_k(\sigma^{-i}) > f_k(\sigma'^{-i})$. There is some σ^i, such that $\psi(\sigma^i, \sigma^{-i}) = o$ and $\psi(\sigma^i, \sigma'^{-i}) = k$ [26]. This is in contradiction to Lemma 14, as that Lemma requires $\psi(\sigma^i, \sigma'^{-i}) = k \Rightarrow \psi(\sigma^i, \sigma^{-i}) = k$ if $\sigma_l^j < \sigma_l'^j$ and $\psi(\sigma^i, \sigma^{-i}) = o \Rightarrow \psi(\sigma^i, \sigma'^{-i}) = o$ if $\sigma_l^j > \sigma_l'^j$. Intuitively a change in σ_l^i does not affect the relative preferences between k and o, and therefore must not "shift the border" between the sets of signals σ^i where ϕ chooses k and o in figure 2. This entitles us to abuse notation by writing $f_k(\sigma^{-i}) = f_k\left(\sigma_k^{-i}\right)$.

An analogous argument, also using Lemma 14, shows that:

$$f_k\left(\sigma_k'^{-i}\right) - f_k\left(\sigma_k^{-i}\right) = f_l\left(\sigma_l'^{-i}\right) - f_l\left(\sigma_l^{-i}\right) \tag{6}$$

for all alternatives $k, l \neq o$ and $\sigma^{-i}, \sigma'^{-i} \in \Sigma^{-i}$ with the property that $\sigma_k'^{-i} - \sigma_k^{-i} = \sigma_l'^{-i} - \sigma_l^{-i} \left(\in \mathbb{R}^{N-1}\right)$. Intuitively the change from σ^{-i} to σ'^{-i} does not affect the relative preferences between alternatives k and l, and therefore

[26] namely one with $f_k\left(\sigma^{-i}\right) > \sigma^i > f_k\left(\sigma'^{-i}\right)$.

does not "shift the border" between the sets of σ^i where ϕ chooses k and l in figure 2. Equation 6 entails:

$$f_k\left(\sigma_k'^{-i}\right) - f_k\left(\sigma_k^{-i}\right) = f_k\left(\sigma_k'^{-i} + v\right) - f_k\left(\sigma_k^{-i} + v\right) \tag{7}$$

for all $v \in \mathbb{R}^{N-1}$ such that $\sigma', \sigma, \sigma' + v, \sigma + v \in \Sigma_k^{-i}$. Equation 7 shows that $\alpha^j := -\frac{f_k\left(\sigma_k^{-i} + \varepsilon e_k^j\right) - f_k\left(\sigma_k^{-i}\right)}{\varepsilon}$ (≥ 0 by Lemma 14) depends neither on σ_k^{-i} nor on ε[27]. It does not depend on k either because of equation 6. Therefore we have $\frac{\partial f_k}{\partial \sigma_k^j} \equiv -\alpha^j$ which gives $f_k\left(\sigma^{-i}\right) = -\lambda_k - \sum_{j \neq i} \alpha^j \sigma_k^j$ for all $k \neq o$ and all $\sigma^{-i} \in \Sigma^{-i}$. ■

[27] The nondependence on ε follows first for entire and then for rational multiples of a fixed ε by equation 7, and then for arbitrary multiples by the monotonicity of f_k.