Restricting Growth Rates in Cointegrated VAR Models

Håvard Hungnes*
Division for Macroeconomics
Research Department
Statistics Norway
P.O.B. 8131 Dep
N-0033 OSLO
Norway

Abstract

In a cointegrated vector autoregressive model the intercept parameters can be decomposed into growth rate parameters and cointegration mean parameters. The growth rate parameters have important economic interpretations and may be equally important to identify and conduct hypothesis testing on as the cointegration vectors and the matrix of adjustment parameters. Here we develop a linear switching algorithm for estimating (possibly) restricted growth rates as a part of the cointegration analysis. We also show that the standard deviations for the growth rate parameters can be computed from their information matrix alone. An example with Danish money demand illustrates the method.

Keywords: Johansen procedure, cointegrated VAR, growth rates, cointegration means, linear switching algorithm, money demand.

JEL classification: C32, C51, C52, E41.

Acknowledgments: Thanks to Gunnar Bårdsen, Søren Johansen, Eilev S. Jansen, Bjørn Naug, Terje Skjerpen and Anders Rygh Swensen for valuable comments on various versions of the paper. An earlier version of the paper was presented at the *Bridging Economics and Econometrics* conference in Florence (June 6th - 9th, 2001). The Ox-program with the switching algorithm derived here is available from the author upon request, and will soon be made downloadable from my homepage.

---

*E-mail: hhu@ssb.no Homepage: http://www.ssb.no/forskning/ansatte/kompkat/?login=hhu

1 Introduction

Cointegrated vector autoregressive (VAR) models are a powerful tool in analysing time series. Granger's representation theorem (see Engle and Granger, 1987) shows that cointegrated time series can be represented in an equilibrium correction vector autoregressive model. Furthermore, Johansen (1988) shows that canonical correlation technique combined with reduced rank regression technique can be used to estimate such models. These techniques are implemented in standard time series packages such as PcGive (see Doornik and Hendry, 2001) and Cats in Rats (see Hansen and Juselius, 1994). The cointegration vectors can be identified and overidentified restrictions can be tested against economic theory. A lot of work has been accomplished in estimating long-run cointegration relationships in economics.

However, other parameters in a cointegrated VAR model also have economic interpretations. By rewriting the equilibrium correction form of the VAR model (VEqCM), we can identify the underlying growth of the variables as well as the long-run means of the cointegration relationships.

Within the VEqCM the intercepts can either be restricted to lie in the cointegration space, or not. If the intercepts are not restricted to lie in the cointegration space ('unrestricted'), they allow the system to have both growth and cointegration means. If, however, the intercepts are restricted, there is no growth in the system, (see Johansen and Juselius, 1990).

The growth rates tell us how much to expect (unconditionally) the variables in the system to grow from one period to the next. If the system is used for forecasting, the vector of growth rates will be one of the most important ones in providing good forecasts. In fact, as the forecast horizon approaches infinity, the forecasts will rely on this vector only, see Clements and Hendry (1999, pp. 49-51).

There are also variables we do not believe will grow over time. If the interest rate or the inflation rate is assumed to be $I(1)$, we may not want to allow them to grow. Especially not if we want to use the system for forecasting. However, restricting the intercepts to lie in the cointegration space may be too restrictive, as the system may include variables we believe do grow over time. We then want to restrict some of the variables to have no growth and let other variables in the system grow. We develop an estimation procedure in which we allow restrictions in the system on some or all of the growth rates.

The cointegration means may also have economic interpretations. In a system with (the logs of) consumption and income, the intercept in the cointegration vector can be interpreted as the equilibrium savings ratio if the income elasticity is unity. A system with nominal interest rate and inflation (both assumed to be $I(1)$), where the cointegration mean can be interpreted as the equilibrium real interest rate, is another example.

Sometimes we may want to restrict the cointegration mean. Assume we are testing the law of one price, and are analysing a system with an unrestricted intercept to allow the prices to grow over time.[1] We may find that $p - e - p^*$ is the cointegration

---

[1] The law of one price states that one product shall have the same price in two different regions. Let P be the price of the product in one of the regions and P^* the price in the other region.

relationship, (where p and p^* are the domestic and foreign price respectively, and e the exchange rate, all variables measured in logs,) and want to test the strict version of the law of one price. This implies testing if the cointegration mean is equal to zero. To achieve this, we have to decompose the intercepts in the system in growth rates and a cointegration mean, and test if the mean is equal to zero. This can also be achieved by the estimation procedure presented here.

The paper is organized as follows: In section 2 we show how the growth rates and cointegration means can be estimated. In section 3 a linear switching algorithm is presented. The switching algorithm we derive here is an extension of the linear switching algorithm in Boswijk (1995). In section 4 the method is illustration on money demand in Denmark. Section 5 concludes.

Throughout the paper we define the orthogonal complement of the full column rank matrix A as $A_\perp$ such that $A'_\perp A = 0$ and $(A, A_\perp)$ has full rank. Further, let A^+ be the Moore-Penrose inverse of A, see e.g. Theil (1983, pp. 51-52) or Magnus and Neudecker (1988, pp. 32-34).[2] If A has full column rank the transponed of the Moore-Penrose inverse is given by $\overline{A} = A\left(A'A\right)^{-1}$.

2 Growth rates and cointegration means

In this section we look at some properties of the cointegrated VAR model. In particular, we focus on how the growth rates and cointegration means can be estimated.

In (1) X_t is an n-dimensional vector of non-stationary $I(1)$ variables, η is a vector of intercepts, α and β are matrixes of dimension $n \times r$ (where r is the number of cointegration vectors) and $\beta' X_t$ is $I(0)$. Furthermore, Γ_i is an $n \times n$ matrix of coefficients and Δ is the difference operator. D_t is a vector of centred seasonal dummies and impulse dummies. The residual ε is assumed to be white noise Gaussian ($\varepsilon_t \sim N(0, \Omega)$).

$$\Delta X_t = \eta + \alpha\beta' X_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta X_{t-i} + sD_t + \varepsilon_t, \quad t = 1, 2, ..., T \tag{1}$$

The system grows at the unconditional rates $E[\Delta X] = \gamma$ with long run (cointegration) means $E[\beta' X] = \mu$ apart from terms involving seasonal dummies (and other impulse dummies). Then we can rewrite the relationship as

$$\Delta X_t - \gamma = \alpha\left(\beta' X_{t-1} - \mu\right) + \sum_{i=1}^{p-1} \Gamma_i \left(\Delta X_{t-i} - \gamma\right) + sD_t + \varepsilon_t. \tag{2}$$

Notice that $\beta'\gamma = 0$. To prove this, we premultiply the left hand side of (2) with β'.

$$\beta'\left(\Delta X_t - \gamma\right) = \left(\beta' X_t - \beta' X_{t-1}\right) - \beta'\gamma \tag{3}$$

Furthermore, let E be the exchange rate (if the two regions lie in two different countries). Then the law of one price states that $P = A \cdot E \cdot P^*$, where A is a constant capturing differences in the price level due to transportation costs etc. The strict version of the law of one price states that $A = 1$, i.e. there are no differences in the prices of the product in the two regions.

[2] The Moore-Penrose matrix A^+ satifies the following requirements; $AA^+A = A$, $A^+AA^+ = A^+$, and both A^+A and AA^+ are symmetric.

We see that since $E\left[\Delta X\right] = \gamma$ the unconditional expectation of the left hand side of (3) equals zero. And since $E\left[\beta' X\right] = \mu$ the unconditional expectation of the right hand side of (3) will equal zero only if $\beta'\gamma = 0$.

By comparing equation (1) and (2) we see that

$$\eta = \Gamma\gamma - \alpha\mu, \tag{4}$$

where $\Gamma = \left(I - \sum_{i=1}^{p-1} \Gamma_i\right)$. For given estimates of the coefficients in (1) we find[3]

$$\gamma = C\eta \tag{5}$$

and

$$\mu = \overline{\alpha}' \left(\Gamma C - I_n\right) \eta, \tag{6}$$

where $C = \beta_\perp \left(\alpha'_\perp \Gamma \beta_\perp\right)^{-1} \alpha'_\perp$. These properties are known from Granger's representation theorem, see Engle and Granger (1987) and Johansen (1991).

3 The linear switching algorithm

When no restrictions are imposed on the growth rates γ and the cointegration means μ, the vector of intercepts δ in (1) have n variation free elements, and these could be estimated with ordinary least square. However, when we restrict γ and/or on μ, another estimation procedure must be used. Since we are imposing restrictions on the growth parameters and cointegration means as a part of the cointegration analysis, our estimation procedure must also allow for restrictions on α and β. We extend the linear switching algorithm in Boswijk (1995) to also involve restrictions on the growth rates. We define $\beta^* = \left(\beta', -\mu\right)'$ and $X_t^* = \left(X_t', 1\right)'$, so restrictions on the cointegration means can be imposed on β^*.

The algorithm in Boswijk (1995) allows for linear restrictions on α and β. The restrictions on the cointegration vectors can be written as $R'_\beta vec\beta^* = c_\beta$ or

$$vec\beta^* = H_\beta \phi + h_\beta, \tag{7}$$

where $H_\beta = \left(R_\beta\right)_\perp$ and $h_\beta = \left(R'_\beta\right)^+ \cdot c_\beta$. Since we are stacking the cointegration vectors into one vector, we can allow for restrictions between the cointegration vectors as well as within them.

Similarly, restrictions on the adjustment parameters can be written $R'_\alpha vec\alpha' = 0$ or

$$vec\alpha' = H_\alpha \varphi, \tag{8}$$

where $H_\alpha = \left(R_\alpha\right)_\perp$. Here the intercepts are excluded, since we normally only test exclusion restrictions on α.[4]

---

[3] To find (5), premultiply (4) with $\alpha'_\perp$ and use $I = \overline{\beta}\beta' + \beta_\perp \overline{\beta}'_\perp$; $\alpha'_\perp \eta = \alpha'_\perp \Gamma \beta_\perp \overline{\beta}'_\perp \gamma + \alpha'_\perp \Gamma \overline{\beta} \beta' \gamma = \left(\alpha'_\perp \Gamma \beta_\perp\right) \overline{\beta}'_\perp \gamma$ since $\beta'\gamma = 0$. The $n-r$ matrix $\left(\alpha'_\perp \Gamma \beta_\perp\right)$ must have full rank, or some of the variables in the system are $I(2)$. Therefore; $\overline{\beta}'_\perp \gamma = \left(\alpha'_\perp \Gamma \beta_\perp\right)^{-1} \alpha'_\perp \eta$ or, since $\gamma = \overline{\beta}\beta'\gamma + \beta_\perp \overline{\beta}'_\perp \gamma = \beta_\perp \overline{\beta}'_\perp \gamma$, premultiplying with $\beta_\perp$ gives (5). To find (6), rearrange (4) to $\alpha\mu = \Gamma\gamma - \eta$, premultiply with $\overline{\alpha}'$ and apply (5).

[4] It is straightforward to include intercepts in (8).

The restrictions on γ are a bit more complex, since - in addition to the restrictions we want to place on γ - the cointegration vector also imposes restrictions on γ. The restrictions we want to impose on γ can be written as $R_\gamma' \gamma = c_\gamma$, whereas the restrictions imposed by the cointegration vectors can be expressed as $\beta' \gamma = 0$. In a compact notation, these restrictions involve $(\beta, R_\gamma)' \gamma = \left(0', c_\gamma'\right)'$, which equivalently can be written as

$$\gamma = H_\gamma \psi + h_\gamma, \tag{9}$$

where $H_\gamma = (\beta, R_\gamma)_\perp$ and $h_\gamma = \left((\beta, R_\gamma)^+\right)' \cdot \left(0', c_\gamma'\right)'$.

Before we present the log likelihood function, we must define some variables. We first define $Z_t = vec\left(\Delta X_t, \Delta X_{t-1}, ..., \Delta X_{t-p+1}\right)$ and $\Phi = (I_n, -\Gamma_1, -\Gamma_2, ..., -\Gamma_{p-1})$. This log likelihood function (apart form a constant) becomes

$$\begin{aligned} & \log L\left(\alpha, \beta^*, \gamma, \Gamma_1, \Gamma_2, ..., \Gamma_{p-1}, s, \Omega\right) \\ = & -\frac{T}{2} \log |\Omega| \\ & -\frac{1}{2} \sum_{t=1}^{T} \left[\left(\Phi Z_t - \Gamma\gamma - \alpha\beta^{*\prime} X_{t-1}^* - sD_t\right)' \Omega^{-1} \left(\Phi Z_t - \Gamma\gamma - \alpha\beta^{*\prime} X_{t-1}^* - sD_t\right)\right]. \end{aligned} \tag{10}$$

The maximisation problem is to maximize (10) under the restrictions (7) - (9). In Theorem 1 we derive the conditional maximum likelihood estimators for this maximisation problem.

It turns out to be convenient also to use a log likelihood function where we condition on the growth rates. For a given set of growth rates satisfying (9) we can define $Z_{0t}^*(\psi) = \Delta X_t - \gamma$, $Z_{1t}^* = X_{t-1}^*$, $Z_{2t}^*(\psi) = vec\left(\Delta X_{t-1} - \gamma, \Delta X_{t-2} - \gamma, ..., \Delta X_{t-p+1} - \gamma, D_t\right)$ and $\Theta = (\Gamma_1, \Gamma_2, ..., \Gamma_{p-1}, s)$. The log likelihood function conditioned on the growth rates is

$$\begin{aligned} & \log L\left(\alpha, \beta^*, \Gamma_1, \Gamma_2, ..., \Gamma_{p-1}, s, \Omega; \gamma\right) \\ = & -\frac{T}{2} \log |\Omega| \\ & -\frac{1}{2} \sum_{t=1}^{T} \left[\left(Z_{0t}^* - \alpha\beta^{*\prime} Z_{1t}^* - \Theta Z_{2t}^*\right)' \Omega^{-1} \left(Z_{0t}^* - \alpha\beta^{*\prime} Z_{1t}^* - \Theta Z_{2t}^*\right)\right]. \end{aligned} \tag{11}$$

Furthermore, we define

$$M_{ij}^*(\psi) = T^{-1} \sum_{t=1}^{T} Z_{it}^* Z_{jt}^{*\prime}, \quad i, j = 0, 1, 2, \tag{12}$$

and

$$S_{ij}^*(\psi) = T^{-1} \sum_{t=1}^{T} R_{it}^* R_{jt}^{*\prime}, \quad i, j = 0, 1, \tag{13}$$

where $R_{0t}^*(\psi)$ and $R_{1t}^*(\psi)$ are the residuals we obtain by regressing $Z_{0t}^*(\psi)$ and Z_{1t}^* on $Z_{2t}^*(\psi)$ respectively. Finally, $\otimes$ is the Kronecker product.

Theorem 1 (The conditional maximum likelihood estimators) *The conditional maximum likelihood estimators for* Θ, ϕ, φ, Ω *and* ψ *in (10) under the restrictions (7) - (9) are given by*

$$\widehat{\Theta}\left(\psi,\phi,\varphi\right) = M_{02}^{*}\left(M_{22}^{*}\right)^{-1} - \alpha\beta^{*\prime}M_{12}^{*}\left(M_{22}^{*}\right)^{-1}, \tag{14}$$

$$\begin{aligned}\widehat{\phi}\left(\psi,\varphi,\Omega\right) &= \left[H_{\beta}^{\prime}\left(\alpha^{\prime}\Omega^{-1}\alpha\otimes S_{11}^{*}\right)H_{\beta}\right]^{-1} \\ &\quad\times H_{\beta}^{\prime}\left[\left(\alpha^{\prime}\Omega^{-1}\otimes I_{n+1}\right)vecS_{10}^{*} - \left(\alpha^{\prime}\Omega^{-1}\alpha\otimes S_{11}^{*}\right)h_{\beta}\right],\end{aligned} \tag{15}$$

$$\widehat{\varphi}\left(\psi,\phi,\Omega\right) = \left[H_{\alpha}^{\prime}\left(\Omega^{-1}\otimes\beta^{*\prime}S_{11}^{*}\beta^{*}\right)H_{\alpha}\right]^{-1}\left[H_{\alpha}^{\prime}\left(\Omega^{-1}\otimes\beta^{*\prime}\right)vecS_{10}^{*}\right], \tag{16}$$

$$\widehat{\Omega}\left(\psi,\phi,\varphi\right) = S_{00}^{*} - \alpha\beta^{*\prime}S_{10}^{*} - S_{01}^{*}\beta^{*}\alpha^{\prime} + \alpha\beta^{*\prime}S_{11}^{*}\beta^{*}\alpha^{\prime}, \tag{17}$$

$$\begin{aligned}\widehat{\psi}\left(\phi,\varphi,\Theta,\Omega\right) &= \left[H_{\gamma}^{\prime}\Gamma^{\prime}\Omega^{-1}\Gamma H_{\gamma}\right]^{-1} \\ &\quad\times\left[H_{\gamma}^{\prime}\Gamma^{\prime}\Omega^{-1}\left(\Phi\overline{Z} - \alpha\beta^{*\prime}\overline{X^{*}} - s\overline{D} - \Gamma h_{\gamma}\right)\right],\end{aligned} \tag{18}$$

where $\overline{Z} = T^{-1}\sum_{t=1}^{T}Z_t$, $\overline{X^{*}} = T^{-1}\sum Z_{1t}^{*} = T^{-1}\sum X_{t-1}^{*}$ *and* $\overline{D} = T^{-1}\sum D_t$.

See the appendix for the proof.

The term $s\overline{D}$ in (18) equals to zero if we have the same number of observations for each season in the calendar year (and D includes no impulse dummies). If, however, we have an estimation period with more observation from some seasons than from others, this term will generally not equal zero. In the example below the estimation period is 1974Q3-1987Q3, which means that we have one more observation from the third quarter than the others.

Note that $\beta^{*\prime}\overline{X^{*}} = \beta^{\prime}\overline{X} - \mu$ in (18). The first part is the average cointegration mean in the estimation period, and μ is the system cointegration mean. These will not generally be equal.

We now suggest the following estimation procedure:

The maximum likelihood estimators of ψ, ϕ, φ, Θ *and* Ω *may be obtained by the following iterative procedure, starting from a set of initial values* $\{\psi_0,\phi_0,\varphi_0,\Theta_0,\Omega_0\}$:

$$\begin{array}{llll} I & \widehat{\psi}_j = \psi\left(\widehat{\phi}_{j-1},\widehat{\varphi}_{j-1},\widehat{\Theta}_{j-1},\widehat{\Omega}_{j-1}\right) & II & \widehat{\phi}_j = \phi\left(\widehat{\psi}_j,\widehat{\varphi}_{j-1},\widehat{\Omega}_{j-1}\right) \\ III & \widehat{\varphi}_j = \varphi\left(\widehat{\psi}_j,\widehat{\phi}_j,\widehat{\Omega}_{j-1}\right) & IV & \widehat{\Theta}_j = \Theta\left(\widehat{\psi}_j,\widehat{\phi}_j,\widehat{\varphi}_j\right) \\ V & \widehat{\Omega}_j = \Omega\left(\widehat{\psi}_j,\widehat{\phi}_j,\widehat{\varphi}_j\right) & & j = 1,2,... \end{array}$$

The iterative procedure needs a set of starting values. In fact, it only needs starting values for the free growth rates parameters (ψ), the cointegration vectors (ϕ) and the loading parameters (φ) since starting values for the other parameters (Θ and Ω) can be calculated by (14) and (17).

It may be tempting to use the relations in theorem 1 with unrestricted parameters to compute starting values for ψ, ϕ and φ too. However, this is not a good idea when there are more than one cointegration vector. The unrestricted estimator of

β^* is only unique up to a rotation which spans the same space. When restrictions are imposed on β these restrictions may lead to a rotation of this space. To take account of this, we use the method described in Doornik (1995).

Let

$$vec\widehat{b^*} = H_\beta \widehat{\phi} + h_\beta = H_\beta \left(\overline{H_\beta}' \left(vec\widehat{\beta^*}_{unr} - h_\beta \right) \right) + h_\beta,$$

where the subscript unr indicates the parameters are revealed by the unrestricted cointegrated VAR model. Define $[\cdot]$ as dropping those rows which have no restrictions in them; if this yields less than r rows, then add rows back in, so that the $[\cdot]$ matrix is $q \times r$, with $q \geq r$. Then the least square estimator

$$\widehat{A} = \left(\left[\widehat{\beta^*}_{unr} \right]' \left[\widehat{\beta^*}_{unr} \right] \right)^{-1} \left(\left[\widehat{\beta^*}_{unr} \right]' \left[\widehat{b^*} \right] \right)$$

is used to derive

$$\widehat{\alpha}_{-1} = \widehat{\alpha}_{unr} \cdot \widehat{A}'^{-1}.$$

Now the loading matrix α is consistent with the restricted β, and we can use the relations in theorem 1 to calculate starting values for ψ, ϕ and φ;

$$\begin{aligned} \widehat{\phi}_0 &= \phi \left(\widehat{\gamma}_{unr}, \widehat{\alpha}_{-1}, \widehat{\Omega}_{unr} \right), \\ \widehat{\varphi}_0 &= \varphi \left(\widehat{\gamma}_{unr}, \widehat{\phi}_0, \widehat{\Omega}_{unr} \right), \\ \widehat{\psi}_0 &= \psi \left(\widehat{\phi}_0, \widehat{\varphi}_0, \widehat{\Theta}_{unr}, \widehat{\Omega}_{unr} \right). \end{aligned}$$

As discussed in Johansen (1991), the distribution of $\widehat{\beta}$ is mixed normal, i.e. the variance matrix is stochastic. The discussion there also indicates that inference on β may be done as if α were known, and vice versa. Following this result, we compute the 'variance' of $\widehat{\beta}$ as

$$V\widehat{(vec\beta^*)} = \frac{T}{T-k} \left(H_\beta \left[T H_\beta' \left(\alpha' \Omega^{-1} \alpha \otimes S_{11}^* \right) H_\beta \right]^{-1} H_\beta' \right), \tag{19}$$

where the term inside the square brackets is (the negative of) the double derivative of (11) with respect to ϕ. The scale factor $T/(T-k)$ (where k is the integer part of the ratio between the freely estimated parameters in the system and the number of the dependent variables in the system) is used to control for degrees of freedom, see Doornik (1995). The standard deviations of $vec\beta^*$ are the square roots of the diagonal elements in (19).

The distribution of $\widehat{\alpha}$ is normal, and the 'variance' is

$$V\widehat{(vec\alpha')} = \frac{T}{T-k} \left(H_\alpha \left[T H_\alpha' \left(\Omega^{-1} \otimes \beta' S_{11} \beta \right) H_\alpha \right]^{-1} H_\alpha' \right). \tag{20}$$

In (20) we use

$$S_{11} = T^{-1} \sum_{t=1}^{T} R_{1t} R_{1t}',$$

Table 1: Money demand: Cointegration rank

$H_0 : rank = r$	λ	$\lambda-\max$	95%	95%†	*trace*	95%	95%†
$r = 0$	0.4169	28.59*	27.1	27.1	45.67	47.2	48.3
$r \leq 1$	0.1776	10.36	21.0	21.1	17.07	29.7	31.5
$r \leq 2$	0.1125	6.33	14.1	14.9	6.71	15.4	18.0
$r \leq 3$	0.0072	0.38	3.8	8.2	0.38	3.8	8.2

One asterisk denotes significance at the 5 per cent level. The rows labelled '95%' contains the standard critical values, and in the rows labelled '95%†' the critical values for the case where the true model has no deterministic trends are reported. The critical values are taken from Osterwald-Lenum (1992).

where R_{1t} are the residuals we obtain by regressing $Z_{1t} = X_{t-1}$ on $Z_{2t} = \left(1, \Delta X'_{t-1}, \Delta X'_{t-2}, ..., \Delta X'_{t-p+1}, D_t\right)'$. The reason for using $\beta' S_{11} \beta$ in (20) instead of $\beta^{*\prime} S^*_{11} \beta^*$ (which we would obtain if we used the double derivatives of (11) with respect to ψ) is to take account to the covariance between the intercepts in the cointegration relations and the other coefficients outside the cointegration vectors. The expression in (20) is used to compute the 'variance' of $\widehat{\alpha}$ in the standard literature, see e.g. Johansen (1991).

Since the estimator of γ is on expectation orthogonal to the other coefficients (see appendix), we compute this 'variance' as

$$\widehat{V(\gamma)} = \frac{T}{T-k} \left(H_\gamma \left[T H'_\gamma \left(\Gamma' \Omega^{-1} \Gamma \right) H_\gamma \right]^{-1} H'_\gamma \right), \qquad (21)$$

which is the observed information matrix for γ (adjusted for degrees of freedom). The distribution of $\widehat{\gamma}$ is normal, see Johansen (1995, Theorem 13.6 and 13.7).

4 Application: Danish money demand

To illustrate the estimation method we use data for money demand in Denmark. This is the data used by Johansen and Juselius (1990) to illustrate how one can restrict the intercepts to lie in the cointegration space. Restricting the intercepts to lie in the cointegration space implies restricting the variables in the system not to grow over time. This might be realistic for the bond rate (i^b) and the deposit rate (i^d), but not for (the logs of) real money ($m2$) and real income (y). The data are plotted in figure 1.

Centred seasonal dummies are included in the empirical analysis. We use the same estimation period as Johansen and Juselius (1990): 1974Q3-1987Q3. In the VEqCM 2 lags are included. In contrast to Johansen and Juselius (1990) we include the intercepts unrestricted.[5]

Table 3 includes two columns of critical values for each of the two tests. The first row (labelled 95%) contains the standard critical values in a system with the

---

[5] The results are obtained by combining PcFiml 9.2 (see Doornik and Hendry, 1997) and Ox 2.1 (see Doornik, 1996).

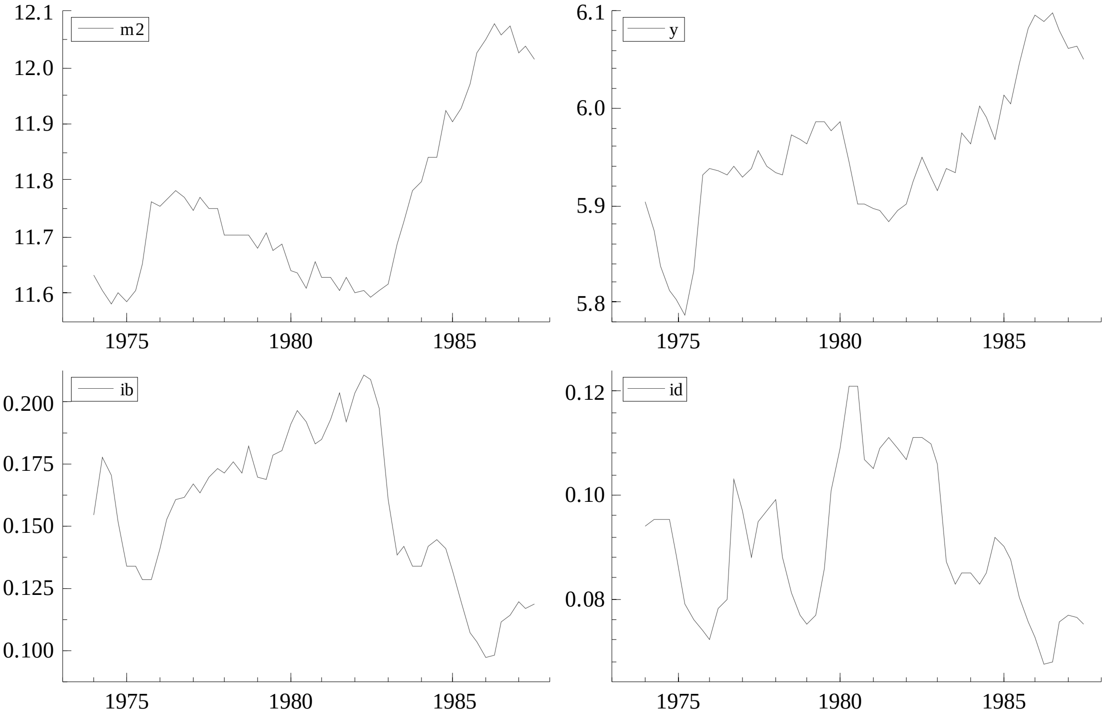

Figure 1: Danish money demand

Plot of real money (m2), real income (y), the bond rate (ib) and the deposit rate (id). Source: Johansen and Juselius (1990).

intercepts unrestricted. In the second row (labelled 95%†) the critical values for the case where the true model has no deterministic trends are reported, see Osterwald-Lenum (1992). The latter set of critical values is reported since we cannot reject the hypothesis that there is no growth in the system.

The rank test indicates that there is one or zero cointegration vectors in the data. The λ-max test supports one cointegration vector at a five per cent significance level and the trace test supports one cointegration vector at a 10 per cent level (independent of which of the two tables of critical values we use).[6] We continue the analysis by assuming that there is one cointegration vector among the variables. Estimating the system with one cointegration vector yields the following equilibrium relation:

$$m2 = \underset{(0.14)}{1.04}y - \underset{(0.56)}{5.22}i^b + \underset{(1.10)}{4.23}i^d - \underset{(0.87)}{6.02} \tag{22}$$

The difference between the bond rate and the deposit rate can be interpreted as the cost of holding money. We therefore restrict the money demand to be homogenous of degree zero in the two interest rates. In addition, we restrict the income elasticity

---

[6] In Johansen and Juselius (1990) the λ-max test is significant at five per cent. However, their trace test is not significant even at the 10 per cent level (though very close to be so).

to equal unity, i.e.

$$\beta^{*} = H_{\beta}\phi + h_{\beta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b \\ -\mu \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This yields

$$\begin{aligned} &\begin{pmatrix} \Delta m2 - \underset{(0.0035)}{0.0081} \\ \Delta y - \underset{(0.0036)}{0.0038} \\ \Delta i^{b} + \underset{(0.0012)}{0.0012} \\ \Delta i^{d} + \underset{(0.0011)}{0.0005} \end{pmatrix}_{t} \\ = &\begin{pmatrix} \underset{(0.058)}{-0.166} \\ \underset{(0.061)}{0.101} \\ \underset{(0.022)}{0.016} \\ \underset{(0.015)}{0.032} \end{pmatrix} \left(m2 - y + \underset{(0.53)}{5.91} \left(i^{b} - i^{d} \right) - \underset{(0.04)}{6.19} \right)_{t-1} \\ &+ \widehat{\Gamma}_{1} \begin{pmatrix} \Delta m2 - 0.0081 \\ \Delta y - 0.0038 \\ \Delta i^{b} + 0.0012 \\ \Delta i^{d} + 0.0005 \end{pmatrix}_{t-1} + \widehat{s}D_{t} + \widehat{\varepsilon}_{t}. \end{aligned} \tag{23}$$

The results indicate a positive growth in money and income. In annual terms these growth rates are 3.3 and 1.5 per cent respectively. The results also indicate a negative growth in the interest rates; a 0.5 percentage points annual decrease in the bond rate and a 0.2 percentage points decrease annually in the deposit rate. However, most of the growth rates parameters are insignificant (measured with the t-value).

We now impose the restriction that there is no underlying growth in the two interest rates. These restrictions imply

$$R_{\gamma}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In addition we have the restriction $\beta'\gamma = 0$. With the restriction we have imposed on the cointegration vector, this restriction involves $(1, -1, b, -b)\,\gamma = 0$. Therefore, the total set of restrictions on γ can be written as

$$\begin{pmatrix} \beta' \\ R_{\gamma}' \end{pmatrix} \gamma = \begin{pmatrix} 1 & -1 & b & -b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_{1} \\ \gamma_{2} \\ \gamma_{3} \\ \gamma_{4} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Table 2: Money demand: Likelihood ratio test of reductions

Equation	$\log L$	$-\log\lvert\Omega\rvert$	p-value	$[d.f.]$
22	970.92	36.6386		
23	970.47	36.6214	0.64	[2]
24	970.08	36.6070	0.80	[4]
25	967.42	36.5065	0.32	[6]

The restrictions can also be expressed as

$$\gamma = H_\gamma \psi = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \psi,$$

where ψ is a scalar.[7]

We see that the restrictions imposed on the growth rates imply that real money and real income grow at the same rate, i.e. $\gamma_1 = \gamma_2$. Imposing the restrictions on the growth rates we get the following results:

$$\begin{pmatrix} \Delta m2 - \underset{(0.0033)}{0.0040} \\ \Delta y - \underset{(0.0033)}{0.0040} \\ \Delta i^b \\ \Delta i^d \end{pmatrix}_t = \begin{pmatrix} \underset{(0.058)}{-0.165} \\ \underset{(0.060)}{0.107} \\ \underset{(0.022)}{0.019} \\ \underset{(0.015)}{0.032} \end{pmatrix} \left(m2 - y + \underset{(0.52)}{5.89} \left(i^b - i^d \right) - \underset{(0.04)}{6.21} \right)_{t-1} + \widehat{\Gamma}_1 \begin{pmatrix} \Delta m2 - 0.0040 \\ \Delta y - 0.0040 \\ \Delta i^b \\ \Delta i^d \end{pmatrix}_{t-1} + \widehat{s} D_t + \widehat{\varepsilon}_t \tag{24}$$

The common estimated growth rate for money and income corresponds to an annual growth rate of 1.6 per cent. From the estimated model we see that the two interest rates may be weakly exogenous. Imposing weak exogeneity yields equation

---

[7] In our example H_γ is independent of b, which means we do not have to update H_γ for each iteration. Generally, however, H_γ will change when the unrestricted parameters in β changes, and H_γ must therefore be updated for each iteration.

Table 3: Cointegration coefficient estimates for different restrictions on α and γ

	$\alpha' = (*, *, *, *)$	$\alpha' = (*, *, 0, 0)$
$\gamma' = (*, *, *, *)$	$\underset{(0.531)}{b = 5.907}$, $\underset{(0.037)}{\mu = 6.193}$	$\underset{(0.560)}{b = 5.808}$, $\underset{(0.038)}{\mu = 6.189}$
$\gamma' = (*, *, 0, 0)$	$\underset{(0.523)}{b = 5.889}$, $\underset{(0.037)}{\mu = 6.209}$	$\underset{(0.559)}{b = 5.805}$, $\underset{(0.039)}{\mu = 6.204}$
$\gamma' = (0, 0, 0, 0)$	$\underset{(0.523)}{b = 5.884}$, $\underset{(0.038)}{\mu = 6.214}$	$\underset{(0.560)}{b = 5.811}$, $\underset{(0.040)}{\mu = 6.207}$

Asterisk denote that the parameter is unrestricted.

(25).

$$\begin{aligned}
&\begin{pmatrix} \Delta m2 - \underset{(0.0034)}{0.0047} \\ \Delta y - \underset{(0.0034)}{0.0047} \\ \Delta i^b \\ \Delta i^d \end{pmatrix}_t \\
&= \begin{pmatrix} \underset{(0.053)}{-0.126} \\ \underset{(0.059)}{0.146} \\ 0 \\ 0 \end{pmatrix} \left(m2 - y + \underset{(0.56)}{5.80} \left(i^b - i^d\right) - \underset{(0.04)}{6.20} \right)_{t-1} \qquad (25) \\
&+ \widehat{\Gamma}_1 \begin{pmatrix} \Delta m2 - 0.0047 \\ \Delta y - 0.0047 \\ \Delta i^b \\ \Delta i^d \end{pmatrix}_{t-1} + \widehat{s} D_t + \widehat{\varepsilon}_t
\end{aligned}$$

Table 4 shows that none of the restrictions imposed are rejected. (The unrestricted system (22) is always the alternative hypothesis.) From equation (25) we see that the growth rates for money and income is probably not significant (t-value of 1.3). Imposing the restriction that there is no growth in the system (the restrictions imposed by Johansen and Juselius, 1990), we get a log likelihood value of 966.56 and a corresponding p-value of 0.27 (with 7 degrees of freedom). We can therefore not reject that all the growth rates equals zero.

In table 5 we see how the estimates of the parameters in the restricted cointegrated vector $\beta^{*\prime} = (1, -1, b, -b, -\mu)$ change with different restrictions on the loading parameters (α) and growth rates (γ). From the table we see that the restrictions on the loading parameters change the estimates of the cointegration vector (and particularly b) more than restrictions on the growth rates do. There can be two reasons for this result. First, the restrictions on the loading vector are more binding, as can be seen from the relatively large drop in the log likelihood value as the restrictions of weak exogeneity are imposed. More binding restrictions will normally change the other parameters more. Second, the multiplicative relationship between α and β may lead to that restrictions on α will be more important than restrictions on γ with respect to the cointegration vector.

5 Conclusions and suggestions for further work

Sometimes it is relevant to estimate and restrict growth rates and cointegration means in VAR models. These parameters may have economic interpretations, and in particular restrictions on the growth rates are interesting to test. We show that this can be achieved by using an iterative procedure.

When restricting growth rates and cointegration means, the degrees of freedom increases. If these restrictions are valid, the estimates of the other parameters in the system will be more precise. On the other hand, the parameters in the cointegration vectors are superconsistent, and the gain may not be large. In the Danish data we see that the estimates hardly changes by including these restrictions. However, this may be so as these restrictions are barely binding. More research will be needed in order to learn how important restrictions on growth rates and cointegration means are for the estimates of the cointegration vectors.

The method presented here can easily be extended to models including more deterministic variables. A deterministic trend is often included in the cointegration vectors. When a trend is included, the growth rates are no longer orthogonal to the cointegration vectors. However, if ρ is the vector of trend coefficients in the cointegration vectors, $\beta'\gamma = \rho$ (or $\gamma = \overline{\beta}\rho$) will capture the restrictions between the coefficients

Sometimes we also want to include step dummies in the system. If included, these will pick up changes in the growth rates as well as changes in the cointegration means. However, we may also want to know how the step dummy influences the system: Does the step dummy change the cointegration means, or the growth rates only? Do all growth rates change when the step dummy is included? Such questions can be answered by applying the same method to the step dummy as for the intercepts.

As an example take testing of purchasing power parity: In many countries the inflation rate was higher in the 1980s than in the 1990s. If we test for purchasing power parity, we may include a shift dummy to take account of the shift in the growth rate for prices. However, the shift dummy may pick up shift in the real exchange rate as well as shift in the growth rates. Utilizing the estimation procedure presented here we can test whether the shift in the real exchange rate is significant or not.

6 References

Boswijk, P. (1995). Identifiability of Cointegrated Systems. Discussion Paper ti 7-95-078, Tinbergen Institute, University of Amsterdam.

Clements, M. P. and D. F. Hendry (1999). *Forecasting Non-stationary Economic Time Series.* The MIT Press.

Doornik, J. A. (1995). Testing General Restrictions on the Cointegrating Space. www.nuff.ox.ac.uk/users/doornik, Nuffield College.

Doornik, J. A. (1996). *Object-oriented Matrix Programming using Ox.* Timberlake Consultants Ltd.

Doornik, J. A. and D. F. Hendry (1997). *Modelling Dynamic Systems Using PcFiml 9.0 for Windows.* International Thomson Business Press.

Doornik, J. A. and D. F. Hendry (2001). *Modelling Dynamic Systems Using PcGive, Volume II.* Timberlake Consultants Ltd.

Engle, R. F. and C. W. J. Granger (1987). Co-integration and Error Correction: Representation, Estimating and Testing, *Econometrica 55*, 251-276.

Hansen, H. and K. Juselius (1994). Manual to Cointegration Analysis of Time Series CATS in RATS. Institute of Economics, University of Copenhagen.

Johansen, S. (1988). Statistical Analysis of Cointegration vectors, *Journal of Economic Dynamics and Control 12*, 231-254.

Johansen, S. (1991). Estimation and Hypothesis Testing of Cointegration Vectors in Gaussian Vector Autoregressive Models, *Econometrica 59*, 1551-1580.

Johansen, S. (1995). *Likelihood-based Inference in Cointegrated Vector Autoregressive Models.* Oxford: Oxford University Press.

Johansen, S. and K. Juselius (1990). Maximum Likelihood Estimation and Inference on Cointegration - With Application to the Demand for Money, *Oxford Bulletin of Economics and Statistics 52*, 169-210.

Magnus, J. and H. Neudecker (1988). *Matrix Differations Calculus with Applications in Statistics and Econometrics.* New York: John Whiley & Sons.

Osterwald-Lenum, M. (1992). A Note with Quantiles of the Asymptotic Distribution of the ML Cointegration Rank Test Statistics, *Oxford Bulletin of Economics and Statistics 54*, 461-472.

Reimers, H.-E. (1992). Comparisons of Tests for Multivariate Cointegration, *Statistical Papers 33*, 335-359.

Theil, H. (1983), "Linear Algebra and Matrix Methods in Econometrics", in Z. Griliches and M. Intriligator, eds., *Handbook of Econometrics, Vol 1.* New York: North-Holland.

7 Appendix

Proof of Theorem 1. To prove the theorem we use $trAB = trBA = (vecA')' \, vecB$ and $vec\,(AXB) = (B' \otimes A)\, vecX = ((A \otimes B')\, vecX')'$, where tr is the trace operator.

Deriving equation (14) is straightforward, see e.g. Johansen (1995, p. 90).

Equations (15) and (16) (see Boswijk, 1995 Theorem 2): The derivatives of (11) with respect to ϕ and φ (under restrictions (8) and (7)) are

$$\begin{aligned} \frac{\partial \log L}{\partial \phi} &= TH_{\beta}' vec\left[S_{10}^{*}\Omega^{-1}\alpha - S_{11}^{*}\beta^{*}\alpha'\Omega^{-1}\alpha\right], && (26) \\ \frac{\partial \log L}{\partial \varphi} &= TH_{\alpha}' vec\left[\beta^{*\prime}S_{10}^{*}\Omega^{-1} - \beta^{*\prime}S_{11}^{*}\beta^{*}\alpha'\Omega^{-1}\right]. && (27) \end{aligned}$$

Setting (26) equal zero, substituting (7) in (26) and solving for ϕ yields (15). Similarly, setting (27) equal to zero and using (27) and (8) leads to (16).

Equation (17): Solving (11) with respect to Ω and using $\partial \log |\Omega| / \partial \Omega = \Omega^{-1\prime} = \Omega^{-1}$ together with (12) and (13) leads to (17).

Equation (18): The derivative of (10) with respect to ψ (under the restriction (9)) is

$$\begin{aligned} & \frac{\partial \log L}{\partial \psi} \\ = \; & TH_{\gamma}' vec\left[\left(\overline{Z}'\Phi' - \overline{X}\beta^{*}\alpha' - \overline{D}'s' - \gamma' E'\Phi'\right)\Omega^{-1}\Phi E\right] && (28) \end{aligned}$$

Setting (28) equal to zero and using (9) lead to (18). ■

Proof of the claim that the estimator for γ is expectationally ortohogonal to the estimators for β and Θ. To prove this claim we must prove that the derivatives of (28) with respect to φ and $vec\Theta$ has expectation zero. The derivative of (28) with respect to φ is

$$\frac{\partial \log L}{\partial\,(vec\Theta')'\,\partial\varphi} = -TH_{\gamma}'\left(E'\Phi'\Omega^{-1} \otimes \overline{X^{*}}'\beta^{*}\right)H_{\alpha}. \tag{29}$$

Since $\overline{X^{*}}'\beta^{*} = \left(\beta^{*\prime}\overline{X^{*}}\right)' = (\beta' X - \mu)'$ has unconditional expectation equal to zero, (29) will have unconditional expectation equal to zero as well.

Since $\Phi = (I_n, -\Theta)$ proving $E\left[\partial^2 \log L / \partial\,(vec\Theta')'\,\partial\psi\right] = 0$ is equivalent to proving $E\left[\partial^2 \log L / \partial\,(vec\Theta')'\,\partial\psi\right] = 0$. From (28) we have

$$\frac{\partial \log L}{\partial \psi} = TH_{\gamma}' vec\left\{\left[\left(\overline{Z}' - 1_{1\times p} \otimes \gamma'\right)\Phi' - \overline{X}\beta^{*}\alpha' - \overline{D}'s'\right]\Omega^{-1}\Phi E\right\}, \tag{30}$$

where $1_{1\times p}$ is a p-dimensional row vector of ones.

Note that the expression inside the square brackets is zero. Therefore we get an expression equal to zero when we take the derivative with respect to the last Φ. Note also that the element within the normal parentheses has expectation zero, so the expectation of the derivative with respect to the first Φ is also equal to zero. ■