

An Extended Constant Conditional Correlation GARCH Model
and Its Fourth-Moment Structure

by

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Abstract. The constant conditional correlation GARCH model is probably the most frequently applied multivariate GARCH model. In this paper we consider an extension to this model and examine its fourth-moment structure. The extension is motivated by the result found and discussed in the paper that the squared observations from the extended model have a rich autocorrelation structure. This means that already the first-order model is capable of reproducing a whole variety of autocorrelation structures observed in financial return series. These autocorrelations are derived for the first and the second-order constant conditional correlation GARCH model. The usefulness of the theoretical results of the paper is demonstrated by reconsidering an empirical example that appeared in the original paper on the constant conditional correlation GARCH model.

Key Words. Autoregressive conditional heteroskedasticity, moment structure of GARCH, multivariate conditional heteroskedasticity, volatility dynamics

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1 Introduction

Univariate models for conditional heteroskedasticity have long been popular in financial econometrics and volatility forecasting, and a large number of applications have been published using general autoregressive conditional heteroskedasticity (GARCH) models. The probability structure of univariate GARCH(p, q) models has recently been under study. Conditions of the existence of moments and, in particular, the fourth-moment structure of these models has been derived; see, for example, He and Teräsvirta (1999a,b) and Karanasos (1999). These results are important as they help the user to find out how well the GARCH model and its extensions are capable of characterizing stylized facts typical of many high-frequency financial time series.

These models have been generalized to the vector case, but the number of applications has remained rather limited compared to univariate models. Multivariate GARCH models were surveyed in Bollerslev, Engle and Nelson (1994) and Gouriéroux (1996, Chapter 6); see also Palm (1996) for a short review. As yet relatively little is known about the moment structure of these models. Engle and Kroner (1995) derived a necessary and sufficient condition for weak stationarity of vector GARCH models, but results for higher-order moments do not seem to exist in the literature. Our starting-point is probably the most frequently applied multivariate GARCH model, the so-called Constant Conditional Correlation Generalized Autoregressive Heteroskedasticity (CCC-GARCH) model of Bollerslev (1990). Bollerslev's model is in turn a generalization of the Constant Conditional Correlation ARCH model that appeared in Cecchetti, Cumby and Figlewski (1988). In this paper we consider an extended version of the CCC-GARCH model. We derive a sufficient condition for the existence of the fourth moments for this model and, furthermore, its complete fourth-moment structure. Because of rather involved calculations we restrict

our considerations to the second-order CCC-GARCH model. As most of the applications seem to rely on first-order models, this does not appear to be a serious restriction.

As to first-order models, we show that the squared observations of the extended first-order CCC-GARCH model can already have a remarkably rich correlation structure able to cover many shapes of autocorrelation functions that have been observed in practice. This in fact motivates our extension of the standard CCC-GARCH model. In particular, the autocorrelations of individual processes do not necessarily decay monotonically from the first lag onwards. By comparison, the autocorrelations of squared observations in the standard CCC-GARCH(1,1) model still have the same properties as they do in the univariate GARCH(1,1) model. This includes the exponential decay of the autocorrelations of squared observations from the first lag for all variables in the model. The CCC-GARCH model we focus on in this paper can thus be viewed as a potentially useful extension to the basic model of Bollerslev (1990). Using an empirical example in Bollerslev's article we demonstrate how the use of the correlation structure of the CCC-GARCH(1,1) model worked out in this paper helps one to enrich the interpretation of the estimated models.

The plan of the paper is as follows. The extended CCC-GARCH model is introduced in Section 2. Sections 3 and 4 contain the main results on the fourth-moment structure of this model. Section 5 briefly takes up a special case, a bivariate first-order model. Section 6 contains an empirical example, and the conclusions can be found in Section 7.

2 The extended constant conditional correlation GARCH model

Consider the following vector stochastic process:

$$\mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t \tag{1}$$

$$\boldsymbol{\varepsilon}_t = \mathbf{D}_t \mathbf{z}_t \tag{2}$$

where $\mathbf{D}_t = \text{diag}\{h_{1t}, \dots, h_{Mt}\}$, and h_{1t} is the conditional standard deviation of ε_{it} , $i = 1, \dots, M$. Furthermore, the stochastic vector $\mathbf{z}_t = (z_{1t}, \dots, z_{Mt})'$ is independent and identically distributed with mean $\mathbf{0}$ and positive definite covariance matrix $\mathbf{R} = [\rho_{ij}]$ such that $\rho_{ii} = 1$, $i = 1, \dots, M$. The main diagonal elements of \mathbf{R} are restricted to unity for identification reasons; compare this with the univariate case, in which customarily $\mathbf{E}z_t^2 = 1$. Furthermore, $\mathbf{h}_t = (h_{1t}, \dots, h_{Mt})'$ is a $M \times 1$ vector of conditional standard deviations of ε_t . Let

$$\boldsymbol{\varepsilon}_t^{(2)} = (\varepsilon_{1t}^2, \dots, \varepsilon_{Mt}^2)' = \mathbf{Z}_t^2 \mathbf{h}_t^{(2)} \quad (3)$$

where $\mathbf{h}_t^{(2)} = (h_{1t}^2, \dots, h_{Mt}^2)'$ and $\mathbf{Z}_t = \text{diag}\{z_{1t}, \dots, z_{Mt}\}$. Define the vector GARCH(p, q) process

$$\mathbf{h}_t^{(2)} = \mathbf{a}_0 + \sum_{i=1}^q \mathbf{A}_i \boldsymbol{\varepsilon}_{t-i}^{(2)} + \sum_{j=1}^p \mathbf{B}_j \mathbf{h}_{t-j}^{(2)} \quad (4)$$

where \mathbf{a}_0 is an $M \times 1$ vector with positive elements, and \mathbf{A}_i , $i = 1, \dots, q$ and \mathbf{B}_j , $j = 1, \dots, p$, are $M \times M$ matrices such that each element of $\mathbf{h}_t^{(2)}$ is positive for every t . Note that (4) defines the diagonal elements of \mathbf{D}_t . From (2) it follows that

$$\mathbf{E}(\boldsymbol{\varepsilon}_t | \mathcal{F}_{t-1}) = \mathbf{0} \quad (5)$$

$$\mathbf{E}(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t' | \mathcal{F}_{t-1}) = \mathbf{D}_t \mathbf{R} \mathbf{D}_t \quad (6)$$

where \mathcal{F}_{t-1} is the σ -field generated by all the available information up through time $t-1$.

Remark 1. A sufficient condition for $\mathbf{h}_t^{(2)} > \mathbf{0}$ for all t is that all elements in \mathbf{a}_0 be positive and all elements in \mathbf{A}_i and \mathbf{B}_j for each i and j be nonnegative. (Note that all vector and matrix inequality signs in this paper represent element-by-element inequality.) From Nelson and Cao (1992) we conjecture that this condition is not necessary, at least not if $p > 1$ or $q > 1$ or both. It follows from $\mathbf{z}_t \sim iid(\mathbf{0}, \mathbf{R})$ that $\mathbf{E}(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t' | \mathcal{F}_{t-1})$ is positive definite.

Remark 2. The vector GARCH process defined by equations (2)- (6) is a multivariate GARCH model with constant conditional correlations. The constant conditional correlation GARCH model of Bollerslev (1990) is obtained by assuming that $\mathbf{A}_i, \mathbf{B}_j, i = 1, \dots, q$ and $j = 1, \dots, p$, are diagonal matrices. In particular, setting $\mathbf{B}_j = \mathbf{0}, j = 1, \dots, p$, yields the constant conditional correlation ARCH model introduced in Cecchetti, Cumby and Figlewski (1988).

3 The fourth moment structure of the second-order extended CCC-GARCH model

In this section we consider the vector GARCH(2,2) model defined in (2)-(6) and set $\mathbf{C}_{it} = [c_{i,jlt}] = \mathbf{A}_i \mathbf{Z}_t^2 + \mathbf{B}_i, i = 1, 2$. Note that $\{\mathbf{C}_{it}\}$ is a sequence of i.i.d. random matrices such that \mathbf{C}_{it} is independent of $\mathbf{h}_t^{(2)}$. By (3) we may rewrite (4) as

$$\mathbf{h}_t^{(2)} = \mathbf{a}_0 + \mathbf{C}_{1,t-1} \mathbf{h}_{t-1}^{(2)} + \mathbf{C}_{2,t-2} \mathbf{h}_{t-2}^{(2)}. \quad (7)$$

Let $\mathbf{\Gamma}_{C_i} = \mathbf{E} \mathbf{C}_{it}$ and $\mathbf{\Gamma}_{C_i \otimes C_j} = \mathbf{E}(\mathbf{C}_{it} \otimes \mathbf{C}_{jt}), i, j = 1, 2$, where \otimes denotes the Kronecker product of two matrices. Let $\lambda(\mathbf{\Gamma})$ denote the modulus of the largest eigenvalue of $\mathbf{\Gamma}$. We can now state the following result:

Theorem 1 *The vector GARCH(2,2) process defined in (2)-(3) and (7) is weakly stationary if and only if*

$$\lambda(\mathbf{\Gamma}_{C_1} + \mathbf{\Gamma}_{C_2}) < 1. \quad (8)$$

Remark 3. Bollerslev and Engle (1993) and Engle and Kroner (1995) derived a necessary and sufficient condition for weak stationarity of a vector GARCH(p, q) model without the assumption of the constant conditional correlation. Condition (8) is a special case of their result. When it holds, the unconditional variances of the elements of $\boldsymbol{\varepsilon}_t$ in the vector

GARCH(2,2) model (2)-(3) and (7) are

$$\boldsymbol{\mu}_2 = \mathbf{E}\boldsymbol{\varepsilon}_t^{(2)} = (\mathbf{I}_M - \boldsymbol{\Gamma}_{C_1} - \boldsymbol{\Gamma}_{C_2})^{-1} \mathbf{a}_0. \quad (9)$$

We are now ready to state our result concerning the fourth-order unconditional moment matrix of model (2)-(3) with (7). Let $\text{vec}(\mathbf{A})$ be a vector in which the columns of the $M \times M$ matrix \mathbf{A} are stacked one underneath the other. Then it is known that $\text{vec}(\mathbf{A}') = \mathbf{K}_{MM} \text{vec}(\mathbf{A})$, where \mathbf{K}_{MM} is the $M^2 \times M^2$ commutation matrix, see, for example, Magnus (1988, pp. 35-37). We have the following theorem:

Theorem 2 *Consider the vector GARCH(2,2) model (2)-(3) and (7). Assume that $\boldsymbol{\mu}_2$ in (9) and $\boldsymbol{\Gamma}_{Z \otimes Z} = \mathbf{E}(\mathbf{Z}_t^2 \otimes \mathbf{Z}_t^2)$ exist. Then the fourth-order moment matrix $\mathbf{E}[\boldsymbol{\varepsilon}_t^{(2)} \boldsymbol{\varepsilon}_t^{(2)'}]$ of $\{\boldsymbol{\varepsilon}_t\}$ exists if*

$$\lambda(\boldsymbol{\Gamma}) < 1 \quad (10)$$

where

$$\begin{aligned} \boldsymbol{\Gamma} = & (\boldsymbol{\Gamma}_{C_1 \otimes C_1} + \boldsymbol{\Gamma}_{C_2 \otimes C_2}) \\ & + (\mathbf{I}_{M^2} + \mathbf{K}_{MM}) [(\boldsymbol{\Gamma}_{C_1} \otimes \mathbf{I}_M)(\mathbf{I}_M - \boldsymbol{\Gamma}_{C_2} \otimes \boldsymbol{\Gamma}_{C_2})^{-1} \boldsymbol{\Gamma}_{C_1 \otimes C_2} \\ & + (\boldsymbol{\Gamma}_{C_1} \otimes \boldsymbol{\Gamma}_{C_2})(\mathbf{I}_M - \boldsymbol{\Gamma}_{C_2} \otimes \boldsymbol{\Gamma}_{C_2})^{-1} \boldsymbol{\Gamma}_{C_2 \otimes C_1}]. \end{aligned}$$

Under (10),

$$\begin{aligned} \text{vec} \mathbf{E}[\boldsymbol{\varepsilon}_t^{(2)} \boldsymbol{\varepsilon}_t^{(2)'}] = & \boldsymbol{\Gamma}_{Z \otimes Z} (\mathbf{I}_{M^2} - \boldsymbol{\Gamma})^{-1} \{ \text{vec}(\mathbf{a}_0 \mathbf{a}_0') + [\sum_{i=1}^2 (\boldsymbol{\Gamma}_{C_i} \otimes \mathbf{a}_0 + \mathbf{a}_0 \otimes \boldsymbol{\Gamma}_{C_i}) \\ & + (\mathbf{I}_{M^2} + \mathbf{K}_{MM})(\boldsymbol{\Gamma}_{C_1} \otimes \mathbf{I}_M) [(\mathbf{I}_M \otimes \boldsymbol{\Gamma}_{C_2})(\mathbf{I}_{M^2} - \boldsymbol{\Gamma}_{C_2} \otimes \boldsymbol{\Gamma}_{C_2})^{-1} (\mathbf{a}_0 \otimes \mathbf{I}_M) \\ & + (\boldsymbol{\Gamma}_{C_2} \otimes \boldsymbol{\Gamma}_{C_2})(\mathbf{I}_{M^2} - \boldsymbol{\Gamma}_{C_2} \otimes \boldsymbol{\Gamma}_{C_2})^{-1} (\mathbf{I}_M \otimes \mathbf{a}_0)] \\ & \times (\mathbf{a}_0' \otimes \mathbf{I}_M) \text{vec}(\mathbf{I}_M - \boldsymbol{\Gamma}_{C_1} - \boldsymbol{\Gamma}_{C_2})^{-1} \}. \end{aligned} \quad (11)$$

Proof. See the Appendix.

Remark 4. Setting $M = 1$ in (10)-(11) yields the condition for the existence of the fourth-order moment for the univariate GARCH(2,2) model

$$\gamma = (\gamma_{c_1 \times c_1} + \gamma_{c_2 \times c_2}) + 2\gamma_{c_1} \gamma_{c_1 \times c_2} (1 - \gamma_{c_2})^{-1} < 1 \quad (12)$$

where $\gamma_{c_i} = a_{i,11} + b_{i,11}$, $\gamma_{c_i \times c_i} = b_{i,11}^2 + 2a_{i,11}b_{i,11} + a_{i,11}^2 \mathbf{E}z_{1t}^4$, $i = 1, 2$, and $\gamma_{c_1 \times c_2} = b_{1,11}b_{2,11} + a_{1,11}b_{2,11} + a_{2,11}b_{1,11} + a_{1,11}a_{2,11} \mathbf{E}z_{1t}^4$, and the expression for the fourth unconditional moment of ε_{1t} :

$$\mathbf{E}\varepsilon_{1t}^4 = \frac{a_0^2(\mathbf{E}z_{1t}^4)[(1 + \gamma_{c_1} + \gamma_{c_2})(1 - \gamma_{c_2}) + 2\gamma_{c_1}\gamma_{c_2}]}{(1 - \gamma_{c_1} - \gamma_{c_2})(1 - \gamma)}.$$

Both were derived in He and Teräsvirta (1999a).

Let $\boldsymbol{\mu}_2 = (\mu_{21}, \dots, \mu_{2M})' = \mathbf{E}\boldsymbol{\varepsilon}_t^{(2)}$ and $\mathbf{M}_2 = \text{diag}\{\mu_{21}, \dots, \mu_{2M}\}$. Define the multivariate “rescaled fourth-moment matrix” of $\{\boldsymbol{\varepsilon}_t\}$ in (2)-(4) as

$$\mathbf{K} = \mathbf{M}_2^{-1}[\mathbf{E}(\boldsymbol{\varepsilon}_t^{(2)} \boldsymbol{\varepsilon}_t^{(2)'})]\mathbf{M}_2^{-1}.$$

The j th diagonal element of \mathbf{K} is the kurtosis of ε_{jt} , $j = 1, \dots, M$.

Corollary 1. Assume that $\Gamma_{Z \otimes Z}$ exists and (10) holds for the vector GARCH(2,2) model (2)-(3) and (7). Then

$$\text{vec}(\mathbf{K}) \geq \text{vec}(\mathbf{K}(\mathbf{z}_t)) \quad (13)$$

where the inequality sign refers to element-by-element inequality and $\mathbf{K}(\mathbf{z}_t)$ is the “rescaled fourth-moment matrix” of $\{\mathbf{z}_t\}$.

Inequality (13) follows from Jensen’s inequality. To illustrate, if \mathbf{z}_t has a multivariate normal distribution, the unconditional distribution for $\boldsymbol{\varepsilon}_t$ is leptokurtic in the sense of (13).

The cross-moment matrix $\boldsymbol{\Gamma}_{C_i \otimes C_j}$, $i \neq j$, is required in the considerations if the vector model of higher-order than one. Expressions in Theorem 2 simplify for the vector GARCH(1,1) model. This is seen from the following result:

Corollary 2. *Let the assumptions of Theorem 2 hold and assume $\mathbf{C}_{2t} = \mathbf{0}$ in (7). Then the fourth-order moment matrix $\mathbb{E}[\boldsymbol{\varepsilon}_t^{(2)} \boldsymbol{\varepsilon}_t^{(2)'}]$ of $\{\boldsymbol{\varepsilon}_t\}$ for the vector GARCH(1,1) model (2)-(3) and (7) exists if*

$$\lambda(\boldsymbol{\Gamma}_{C_1 \otimes C_1}) < 1. \quad (14)$$

Under (14),

$$\begin{aligned} \text{vec} \mathbb{E}[\boldsymbol{\varepsilon}_t^{(2)} \boldsymbol{\varepsilon}_t^{(2)'}] &= \boldsymbol{\Gamma}_{Z \otimes Z} \{(\mathbf{I}_{M^2} - \boldsymbol{\Gamma}_{C_1 \otimes C_1})^{-1} [\text{vec}(\mathbf{a}_0 \mathbf{a}_0')] \\ &\quad + (\boldsymbol{\Gamma}_{C_1} \otimes \mathbf{a}_0 + \mathbf{a}_0 \otimes \boldsymbol{\Gamma}_{C_1})(\mathbf{a}_0' \otimes \mathbf{I}_M) \text{vec}(\mathbf{I}_M - \boldsymbol{\Gamma}_{C_1})^{-1}\}. \end{aligned} \quad (15)$$

4 Autocorrelation function of squares for the extended CCC-GARCH model

In this section we shall derive the multidimensional correlation function of $\{\boldsymbol{\varepsilon}_t^{(2)}\}$ for our vector GARCH(2,2) process. Let $\boldsymbol{\mu}_2 = \mathbb{E}\boldsymbol{\varepsilon}_t^{(2)}$ as before and $\boldsymbol{\Gamma}_M(n) = [\gamma_{ij}(n)] = \mathbb{E}[(\boldsymbol{\varepsilon}_t^{(2)} - \boldsymbol{\mu}_2)(\boldsymbol{\varepsilon}_{t-n}^{(2)} - \boldsymbol{\mu}_2)']$. Furthermore, let $\mathbf{D}_M = \text{diag}\{\gamma_{11}^{1/2}(0), \dots, \gamma_{MM}^{1/2}(0)\}$. To fix notation, write the n th order autocorrelation matrix of $\{\boldsymbol{\varepsilon}_t^{(2)}\}$ for the vector GARCH(2,2) process as

$$\mathbf{R}_M(n) = \mathbf{D}_M^{-1} \boldsymbol{\Gamma}_M(n) \mathbf{D}_M^{-1} \quad (16)$$

for $n \geq 1$.

The i th diagonal element $r_{ii}(n)$, $i = 1, \dots, M$, of $\mathbf{R}_M(n)$ in (16), is the n th autocorrelation of the squared observations for the i th component $\{\varepsilon_{it}\}$ while $r_{ij}(n)$, $i, j = 1, \dots, M$, $i \neq j$, the off-diagonal elements of $\mathbf{R}_M(n)$, represent the cross-correlations between ε_{it}^2 and ε_{jt-n}^2 .

In order to obtain $\mathbf{R}_M(n)$, we must find an expression for $\tilde{\boldsymbol{\Gamma}}_M(n) = \mathbb{E}[\boldsymbol{\varepsilon}_t^{(2)} \boldsymbol{\varepsilon}_{t-n}^{(2)'}]$. This can be done by applying (7) to $\boldsymbol{\varepsilon}_t^{(2)}$ recursively up to the n th step and further applying Theorem 2 to $\mathbb{E}[\boldsymbol{\varepsilon}_t^{(2)} \boldsymbol{\varepsilon}_t^{(2)'}]$. The final result appears in the following theorem:

Theorem 3 Assume that $\Gamma_{Z \otimes Z}$ and $\boldsymbol{\mu}_2$ in (9) exist and condition (10) holds for the vector GARCH(2,2) model (2)-(3) and (7). Then the n th order autocorrelation matrix $\mathbf{R}_M(n)$ of $\{\varepsilon_t^{(2)}\}$ has the stacked form

$$\text{vec}(\mathbf{R}_M(n)) = (\mathbf{D}_M^{-1} \otimes \mathbf{D}_M^{-1}) \text{vec}(\mathbf{\Gamma}_M(n)) \quad (17)$$

where

$$\text{vec}(\mathbf{\Gamma}_M(n)) = \text{vec}(\tilde{\mathbf{\Gamma}}_M(n)) - \text{vec}(\boldsymbol{\mu}_2 \boldsymbol{\mu}_2') \quad (18)$$

such that for $n = 1, 2$,

$$\text{vec}(\tilde{\mathbf{\Gamma}}_M(1)) = (\mathbf{I}_M \otimes \mathbf{a}_0) \boldsymbol{\mu}_2 + \Gamma_{Z \otimes C_1} \text{vec}(\tilde{\mathbf{\Gamma}}_M) + \text{vec}(\tilde{\mathbf{\Gamma}}_M) \quad (19)$$

$$\begin{aligned} \text{vec}(\tilde{\mathbf{\Gamma}}_M(2)) &= (\mathbf{I}_M \otimes \mathbf{a}_0) \boldsymbol{\mu}_2 + \Gamma_{Z \otimes C_2} \text{vec}(\tilde{\mathbf{\Gamma}}_M) \\ &\quad + (\mathbf{I}_M \otimes \Gamma_{C_1}) \text{vec}(\tilde{\mathbf{\Gamma}}_M(1)). \end{aligned} \quad (20)$$

Furthermore, for $n \geq 3$,

$$\begin{aligned} \text{vec}(\tilde{\mathbf{\Gamma}}_M(n)) &= (\mathbf{I}_M \otimes \mathbf{a}_0) \boldsymbol{\mu}_2 + (\mathbf{I}_M \otimes \Gamma_{C_1}) \text{vec}(\tilde{\mathbf{\Gamma}}_M(n-1)) \\ &\quad + (\mathbf{I}_M \otimes \Gamma_{C_2}) \text{vec}(\tilde{\mathbf{\Gamma}}_M(n-2)) \end{aligned} \quad (21)$$

with

$$\text{vec}(\tilde{\mathbf{\Gamma}}_M) = (\Gamma_{Z \otimes Z})^{-1} \text{vec} \mathbf{E}(\varepsilon_t^{(2)} \varepsilon_t^{(2)'}) \quad (22)$$

In (19),

$$\begin{aligned} \text{vec}(\tilde{\mathbf{\Gamma}}_M) &= [(\mathbf{I}_M \otimes \Gamma_{C_2})(\mathbf{I}_{M^2} - \Gamma_{C_2} \otimes \Gamma_{C_2})^{-1}(\mathbf{a}_0 \otimes \mathbf{I}_M) \\ &\quad + (\Gamma_{C_2} \otimes \Gamma_{C_2})(\mathbf{I}_{M^2} - \Gamma_{C_2} \otimes \Gamma_{C_2})^{-1}(\mathbf{I}_M \otimes \mathbf{a}_0)] \\ &\quad \times (\mathbf{a}_0' \otimes \mathbf{I}_M) \text{vec}(\mathbf{I}_M - \Gamma_{C_1} - \Gamma_{C_2})^{-1} \\ &\quad + [(\mathbf{I}_{M^2} - \Gamma_{C_2} \otimes \Gamma_{C_2})^{-1} \Gamma_{C_1 \otimes C_2} \\ &\quad + (\Gamma_Z \otimes \Gamma_{C_2})(\mathbf{I}_{M^2} - \Gamma_{C_2} \otimes \Gamma_{C_2})^{-1} \Gamma_{C_2 \otimes C_1}] \text{vec}(\tilde{\mathbf{\Gamma}}_M) \end{aligned} \quad (23)$$

where $\mathbf{\Gamma}_{Z \otimes C_i} = \mathbf{E}(\mathbf{Z}_i^2 \otimes \mathbf{C}_{it})$, $i = 1, 2$.

Proof. See the Appendix.

If the assumptions of Theorem 3 are satisfied and the first two autocorrelation matrices $\mathbf{R}_M(1)$ and $\mathbf{R}_M(2)$ are known we can compute the n th autocorrelation matrix $\mathbf{R}_M(n)$ recursively through equations (17)-(23). The autocorrelation structure simplifies for the first-order model. We state this result in the following corollary:

Corollary 3. *Assume that assumptions of Theorem 3 hold and $\mathbf{C}_{2t} = \mathbf{0}$ in (7). Then the vectors (19)-(21) in the definition of the autocorrelation matrix $\mathbf{R}_M(n)$ of $\{\boldsymbol{\varepsilon}_t^{(2)}\}$ for the CCC- GARCH(1,1) model (2)-(3) and (7) are*

$$\text{vec}(\tilde{\mathbf{\Gamma}}_M(1)) = (\mathbf{I}_M \otimes \mathbf{a}_0)\boldsymbol{\mu}_2 + \mathbf{\Gamma}_{Z \otimes C_1} \text{vec}(\tilde{\mathbf{\Gamma}}_M) \quad (24)$$

$$\text{vec}(\tilde{\mathbf{\Gamma}}_M(n)) = (\mathbf{I}_M \otimes \mathbf{a}_0)\boldsymbol{\mu}_2 + (\mathbf{I}_M \otimes \mathbf{\Gamma}_{C_1}) \text{vec}(\tilde{\mathbf{\Gamma}}_M(n-1)), \quad n \geq 2 \quad (25)$$

with

$$\begin{aligned} \text{vec}(\tilde{\mathbf{\Gamma}}_M) &= (\mathbf{I}_{M^2} - \mathbf{\Gamma}_{C_1 \otimes C_1})^{-1} [\text{vec}(\mathbf{a}_0 \mathbf{a}'_0) \\ &\quad + (\mathbf{\Gamma}_{C_1} \otimes \mathbf{a}_0 + \mathbf{a}_0 \otimes \mathbf{\Gamma}_{C_1})(\mathbf{a}'_0 \otimes \mathbf{I}_M) \text{vec}(\mathbf{I}_M - \mathbf{\Gamma}_{C_1})^{-1}]. \end{aligned} \quad (26)$$

A number of theoretical properties of the autocorrelation matrix $\{\mathbf{R}_M(n)\}$, $n = 1, 2, \dots$, for the first-order model, obtained through Corollary 3, are listed below:

1. $\mathbf{R}_M(n) \rightarrow \mathbf{0}$ as $n \rightarrow \infty$.

It follows from Corollary 3 that we can write

$$\begin{aligned} \text{vec}(\tilde{\mathbf{\Gamma}}_M(n)) &= \left[\sum_{i=0}^{n-2} (\mathbf{I}_M \otimes \mathbf{\Gamma}_{C_1})^i \right] (\mathbf{I}_M \otimes \mathbf{a}_0)\boldsymbol{\mu}_2 \\ &\quad + (\mathbf{I}_M \otimes \mathbf{\Gamma}_{C_1})^{n-1} \text{vec}(\tilde{\mathbf{\Gamma}}_M(1)). \end{aligned} \quad (27)$$

Under condition (10), $\lim_{n \rightarrow \infty} \text{vec}(\tilde{\mathbf{\Gamma}}_M(n)) = [(\mathbf{I}_{M^2} - (\mathbf{I}_M \otimes \mathbf{\Gamma}_{C_1}))^{-1} (\mathbf{I}_M \otimes \mathbf{a}_0)\boldsymbol{\mu}_2$. This implies

$$\text{vec}(\mathbf{\Gamma}_M(n)) \rightarrow \mathbf{0}, \text{ as } n \rightarrow \infty.$$

2. The autocorrelation matrices $\mathbf{R}_M(n)$ satisfy the Yule-Walker equations.

Suppose that $\text{vec}(\tilde{\mathbf{\Gamma}}_M(1))$ is known so that $\text{vec}(\mathbf{R}_M(1))$ is known, see (17) and (18).

It follows from equation (25) that $\text{vec}(\mathbf{R}_M(n))$, for any $n \geq 2$, can be solved by $\mathbf{R}_M(1)$ and $(\mathbf{I}_M \otimes \mathbf{\Gamma}_{C_1})$ through

$$\begin{aligned} \text{vec}(\mathbf{R}_M(n)) &= (\mathbf{D}_M^{-1} \otimes \mathbf{D}_M^{-1}) \left\{ \left[\sum_{i=0}^{n-2} (\mathbf{I}_M \otimes \mathbf{\Gamma}_{C_1})^i \right] (\mathbf{I}_M \otimes \mathbf{a}_0) \boldsymbol{\mu}_2 \right\} \\ &\quad + (\mathbf{D}_M^{-1} \otimes \mathbf{D}_M^{-1}) \left\{ (\mathbf{I}_M \otimes \mathbf{\Gamma}_{C_1})^{n-1} \text{vec}(\tilde{\mathbf{\Gamma}}_M(1)) - \text{vec}(\boldsymbol{\mu}_2 \boldsymbol{\mu}_2') \right\}. \end{aligned} \tag{28}$$

In particular,

$$\text{vec}(\mathbf{R}_M(2)) = (\mathbf{D}_M^{-1} \otimes \mathbf{D}_M^{-1}) \left\{ (\mathbf{I}_M \otimes \mathbf{a}_0) \boldsymbol{\mu}_2 \right\} + \text{vec}(\mathbf{R}_M(1)).$$

3. The first-order auto- and cross correlations $r_{ij}(1)$ for $i, j = 1, \dots, M$, in $\mathbf{R}_M(1)$ are positive if the positivity restrictions $a_{0i} > 0$, $a_{l,ij}$ and $b_{l,ij} \geq 0$, $i, j = 1, \dots, M$, $l = 1, \dots, \max\{p, q\}$, mentioned in Section 2, are satisfied.

4. The decay rate of $\mathbf{R}_M(n)$ as a function of n depends on the eigenvalues of $(\mathbf{I}_M \otimes \mathbf{\Gamma}_{C_1})$. When $M = 2$ the autocorrelations in $\mathbf{R}_2(n)$ will exhibit a mixture of decaying exponential decay, because $(\mathbf{I}_M \otimes \mathbf{\Gamma}_{C_1})$ only has real roots. When $M \geq 3$ the autocorrelations display a mixture of exponential decay if there exists a dominant real root in $(\mathbf{I}_M \otimes \mathbf{\Gamma}_{C_1})$ and dampening sinusoidal behaviour if the moduli of the complex conjugate pairs of eigenvalues are sufficiently large. An example of the latter case is depicted in Figure 1. When $M = 1$, the decay rate of the autocorrelation function of the squared observations for the univariate GARCH(1,1) model is exactly $(\mathbf{I}_1 \otimes \mathbf{\Gamma}_{C_1}) = \gamma_{e_1} = \mathbb{E}(a_{1,11} z_t^2 + b_{1,11})$. In this case the decay is exponential as $r(n) = \gamma_{e_1}^{n-1} r(1)$, $n \geq 2$. This property also holds when $M > 1$ and \mathbf{A}_1 and \mathbf{B}_1 are diagonal matrices as in Bollerslev (1990). Thus our extension of the CCC-GARCH model allows a considerably richer autocorrelation structure than the original

CCC-GARCH model.

5 Bivariate GARCH(1,1) model

In order to illustrate the general correlation results we consider the bivariate GARCH(1,1) process (2)-(3) and (7). The correlation matrix of $\{\varepsilon_t^{(2)}\}$ for this process is obtained as a special case of Corollary 3:

Corollary 4. *Let $M=2$, $\mathbf{C}_{2t} = \mathbf{0}$ and $c_{1,12t} = c_{1,21t} = 0$ in (7). Furthermore, let $\mathbf{\Gamma}_{C_1} = [\gamma_{ij}]$ where $\gamma_{ij} = \mathbf{E}c_{1,ij}$ for $i, j = 1, 2$, $\mathbf{\Gamma}_{C_1 \otimes C_1} = (\gamma_{ij \times kl})$ where $\gamma_{ij \times kl} = \mathbf{E}(c_{1,ijt}c_{1,klt})$ for $i, j, k, l = 1, 2$, and, finally, $\mathbf{\Gamma}_{Z \otimes C_1} = [\gamma_{z_i c_{1,jk}}]$ where $\gamma_{z_i c_{1,jk}} = E(z_{it}^2 c_{1,jkt})$, for $i, j, k = 1, 2$. Assume that $\mathbf{z} = (z_{1t}, z_{2t})' \sim \text{NID}(0, \mathbf{R})$ and, furthermore, that condition (10) holds for the bivariate GARCH(1,1) model (2)-(3) with (7). Then $\mathbf{R}_2(n) = [r_{ij}(n)]$ in (17) for $i, j = 1, 2$ has the simplified form*

$$\begin{aligned} r_{11}(n) &= \frac{\gamma_{11}^{n-1}[\gamma_{z_1 c_{11}}(1 - \gamma_{11}^2) - \gamma_{11}(1 - \gamma_{11,11})]}{3(1 - \gamma_{11}^2) - (1 - \gamma_{11,11})}, \\ r_{21}(n) &= \frac{\gamma_{22}^{n-1}[(1 - \gamma_{11,11})(1 - \gamma_{22,22})]^{1/2}[\gamma_{z_1 c_{22}}(1 - \gamma_{11}\gamma_{22}) - \gamma_{22}(1 - \gamma_{11,22})]}{\{[3(1 - \gamma_{11}^2) - (1 - \gamma_{11,11})][3(1 - \gamma_{22}^2) - (1 - \gamma_{22,22})]\}^{1/2}}, \\ r_{12}(n) &= \frac{\gamma_{11}^{n-1}[(1 - \gamma_{11,11})(1 - \gamma_{22,22})]^{1/2}[\gamma_{z_2 c_{11}}(1 - \gamma_{11}\gamma_{22}) - \gamma_{11}(1 - \gamma_{11,22})]}{\{[3(1 - \gamma_{11}^2) - (1 - \gamma_{11,11})][3(1 - \gamma_{22}^2) - (1 - \gamma_{22,22})]\}^{1/2}}, \\ r_{22}(n) &= \frac{\gamma_{22}^{n-1}[\gamma_{z_2 c_{22}}(1 - \gamma_{22}^2) - \gamma_{22}(1 - \gamma_{22,22})]}{3(1 - \gamma_{22}^2) - (1 - \gamma_{22,22})}. \end{aligned}$$

According to Corollary 4, the elements in $\mathbf{R}_2(n)$, $n \geq 1$, decay exponentially. The autocorrelations $r_{11}(n)$ and cross-correlations $r_{12}(n)$, $n \geq 1$, have the decay rate γ_{11} , while $r_{21}(n)$ and $r_{22}(n)$ also have a common decay rate γ_{22} . The exponential decay rates generalize to the case $M > 2$ and are a characteristic feature of the standard CCC-GARCH model.

6 An empirical example

The purpose of this section is to illustrate practical usefulness of our theoretical results. Bollerslev (1990) fitted two CCC-GARCH models with nonzero constant conditional correlations to a set of five weekly exchange rate return series. His purpose was to analyze the dynamic behaviour of these returns before the introduction of the European Monetary System (EMS) and thereafter. The first period ran from July 1973 to the second week of March 1979 (299 observations) and the second, EMS, one from the third week of March 1979 till the second week of August 1985 (333 observations). The estimated constant correlations from the pre-EMS and EMS models appear in Table 1. The parameter estimates are not reproduced and can be found in Tables 1 and 2 of Bollerslev (1990). The exchange rates that are rates against the US dollar are indexed as follows: 1 = “DEM” (the Deutschmark), 2 = “FRF”, (the French franc) 3 = “ITL” (the Italian lira), 4 = “CHF” (the Swiss franc) and 5 = “GBP” (the British pound). Bollerslev concluded, among other things, that the conditional correlations were significantly higher for the EMS period than for the period preceding it.

Using plug-in auto- and cross correlation estimates based on the definitions of the true quantities derived in the previous sections we are able to complete Bollerslev’s analysis. However, in the pre-EMS model, $\hat{\gamma}_{33} > 1$, so that the ITL process does not have a finite variance. As this would invalidate our small example, we shrink the estimate of a_{33} to 0.287 in order to satisfy the fourth moment existence condition for the pre-EMS system. Estimated auto- and cross correlations of the squared observations from the two models after this adjustment can be found in Table 2. The individual autocorrelations are relatively high and persistent for FRF and CHF and, of course, very high and persistent for ITL (the adjusted $\hat{\gamma}_{33}$ remains just below unity). On the other hand, they are low for

DEM. It may also be noted that most cross-correlations are negligible.

This can be compared with the results from the model for the EMS period. The autocorrelation structure of DEM is practically unaffected by EMS. But then, the three rates with previously large autocorrelations of squared returns are now much more weakly autocorrelated than before. The autocorrelations in the GBP not included in the EMS increase slightly. Thus, while the conditional correlations between the returns of EMS currencies increase as a result of the EMS, the autocorrelations of squared returns decrease. The anchor currency of the monetary system, DEM, constitutes an exception. On the other hand, cross-correlations appear where none were observed before the EMS: note the ones between DEM and ITL and FRF and ITL. It seems that in the EMS period, the changes in the volatility of DEM and FRF have dynamic effects on the volatility of ITL. This may not be unexpected as these currencies belong to the EMS during the observation period. Finally, about the only non-negligible pre-EMS cross-correlations, the ones between GBP and DEM, practically vanish in the EMS period. As GBP has not been a part of the EMS during the observation period, this may not be an unexpected result either.

The above results should be viewed with caution because one parameter estimate was adjusted in order to allow the estimated pre-EMS process to have finite unconditional fourth moments. Besides, the uncertainty in the parameter estimates is not accounted for in the discussion. We would like to emphasize, however, that the main purpose of this example is to demonstrate practical uses of the theoretical results of the paper. The example shows that the fourth-moment results are useful already in the case of the standard first-order CCC-GARCH model, and the same can be said about more general situations as well.

7 Conclusions

In this work we have derived the fourth-moment structure for a constant-correlation GARCH model that contains as a special case the CCC-GARCH model of Bollerslev (1990). We demonstrate the fact that already the first-order version of this model is capable of characterizing processes with rather general autocorrelation structures. This extended model could then be a viable alternative to the standard CCC-GARCH model.

Despite the appealing theoretical properties of the extended model, more work is needed to find out whether or not the model is useful in practice. As a first step, a test of the standard CCC-GARCH model against the extension, in which it is nested, should be developed. Such a test should precede any application of the more general model, since one of the virtues of the standard CCC-GARCH model is that the estimation of parameters is quite straightforward. Maximum likelihood estimation of parameters in the extended model and the asymptotic properties of the estimators have to be considered as well as the evaluation of the estimated model. All this, however, is beyond the scope of the present paper and is left for further research.

References

- Bollerslev, T. (1990) Modelling the coherence in short-run nominal exchange rates: A multivariate generalized ARCH model. *Review of Economics and Statistics* **72**, 498-505.
- Bollerslev, T. & R.F. Engle (1993) Common persistence in conditional variances. *Econometrica* **61**, 167-186.
- Bollerslev, T., R.F. Engle & D.B. Nelson (1994) ARCH models. In: R.F. Engle and D.L. McFadden, eds. *Handbook of Econometrics*, Vol. **4**, pp. 2959-3038. Amsterdam: Elsevier.
- Cecchetti, S.G., R.E. Cumby & S. Figlewski (1988) Estimation of the optimal futures hedge. *Review of Economics and Statistics* **70**, 623-630.
- Engle, R.F. & Kroner (1995) Multivariate simultaneous generalized ARCH. *Econometric Theory* **11**, 122-150.
- Gouriéroux, C. (1997) *ARCH Models and Financial Applications*. New York: Springer.
- He, C. & T. Teräsvirta (1999a) Fourth moment structure of the GARCH(p, q) process. *Econometric Theory* **15**, 824-846.
- He, C. & T. Teräsvirta (1999b) Properties of moments of a family of GARCH processes. *Journal of Econometrics* **92**, 173-192.
- Karanasos, M. (1999) The second moment and the autocovariance function of the squared errors of the GARCH model. *Journal of Econometrics* **90**, 63-76.
- Magnus, J.R. (1988) *Linear Structures*. London: Griffin.
- Palm, F.C. (1996) GARCH models of volatility. In G.S. Maddala & C.R. Rao (eds.), *Handbook of Statistics*, Vol. **14**, pp. 209-240. Amsterdam: Elsevier.

Nelson, D.B. & C.Q. Cao (1992) Inequality constraints in the univariate GARCH model.
Journal of Business and Economic Statistics **10**, 229-235.

Appendix.

Let \mathbb{I}_i be an indicator function defined for non-negative integers i such that

$$\mathbb{I}_i = \begin{cases} 1, & \text{if } i = 0, 2, \dots \\ 0, & \text{if } i = 1, 3, \dots \end{cases}$$

Lemma 1 *Let $k \geq 1$. For the vector GARCH(2,2) model (2)-(3) and (7), $vec(\mathbf{C}_{2,t-2}\mathbf{h}_{t-2}^{(2)}\mathbf{h}_{t-1}^{(2)'}\mathbf{C}'_{1,t-1})$ can be expressed in terms of $\mathbf{h}_{t-i}^{(2)}$, $vec(\mathbf{h}_{t-j}^{(2)}\mathbf{h}_{t-j}^{(2)'})$ and $vec(\mathbf{h}_{t-i}^{(2)}\mathbf{h}_{t-j}^{(2)'})$ as follows:*

$$\begin{aligned} & vec(\mathbf{C}_{2,t-2}\mathbf{h}_{t-2}^{(2)}\mathbf{h}_{t-1}^{(2)'}\mathbf{C}'_{1,t-1}) \\ = & \sum_{j=1}^k \{(\mathbf{C}_{1,t-1} \otimes \mathbf{I}_M) [\prod_{i=1}^j (\mathbb{I}_{i-1}(\mathbf{I}_M \otimes \mathbf{C}_{2,t-(i+1)}) + \mathbb{I}_i(\mathbf{C}_{2,t-(i+1)} \otimes \mathbf{I}_M))]\} \\ & \times [\mathbb{I}_{j-1}(\mathbf{a}_0 \otimes \mathbf{I}_M) + \mathbb{I}_j(\mathbf{I}_M \otimes \mathbf{a}_0)] \mathbf{h}_{t-(1+j)}^{(2)} \\ & + \sum_{j=1}^k \{(\mathbf{C}_{1,t-1} \otimes \mathbf{I}_M) [\prod_{i=1}^j (\mathbb{I}_{i-1}(\mathbf{I}_M \otimes \mathbf{C}_{2,t-(i+1)}) + \mathbb{I}_i(\mathbf{C}_{2,t-(i+1)} \otimes \mathbf{I}_M))]\} \\ & \times [\mathbb{I}_{j-1}(\mathbf{C}_{1,t-(1+j)} \otimes \mathbf{I}_M) + \mathbb{I}_j(\mathbf{I}_M \otimes \mathbf{C}_{1,t-(1+j)})] vec(\mathbf{h}_{t-(1+j)}^{(2)}\mathbf{h}_{t-(1+j)}^{(2)'}) \\ & + (\mathbf{C}_{1,t-1} \otimes \mathbf{I}_M) \prod_{j=0}^k [\mathbb{I}_j(\mathbf{I}_M \otimes \mathbf{C}_{2,t-(2+j)}) + \mathbb{I}_{j+1}(\mathbf{C}_{2,t-(2+j)} \otimes \mathbf{I}_M)] \\ & \times [\mathbb{I}_{k+1}vec(\mathbf{h}_{t-(1+k)}^{(2)}\mathbf{h}_{t-(2+k)}^{(2)'}) + \mathbb{I}_kvec(\mathbf{h}_{t-(2+k)}^{(2)}\mathbf{h}_{t-(1+k)}^{(2)'})]. \end{aligned} \quad (\text{A.1})$$

Proof. Applying (7) to $\mathbf{h}_{t-1}^{(2)}$ in $\mathbf{h}_{t-2}^{(2)}\mathbf{h}_{t-1}^{(2)'}$ yields

$$\begin{aligned} vec(\mathbf{C}_{2,t-2}\mathbf{h}_{t-2}^{(2)}\mathbf{h}_{t-1}^{(2)'}\mathbf{C}'_{1,t-1}) & = (\mathbf{C}_{1,t-1} \otimes \mathbf{C}_{2,t-2})(\mathbf{a}_0 \otimes \mathbf{I}_M)\mathbf{h}_{t-2}^{(2)} \\ & + (\mathbf{C}_{1,t-1} \otimes \mathbf{C}_{2,t-2})(\mathbf{C}_{1,t-2} \otimes \mathbf{I}_M)vec(\mathbf{h}_{t-2}^{(2)}\mathbf{h}_{t-2}^{(2)'}) \\ & + (\mathbf{C}_{1,t-1} \otimes \mathbf{C}_{2,t-2})(\mathbf{C}_{2,t-3} \otimes \mathbf{I}_M)vec(\mathbf{h}_{t-2}^{(2)}\mathbf{h}_{t-3}^{(2)'}). \end{aligned} \quad (\text{A.2})$$

Applying (7) to $\mathbf{h}_{t-2}^{(2)}$ in $\mathbf{h}_{t-2}^{(2)}\mathbf{h}_{t-3}^{(2)'}$ on the right-hand side of (A.2) and continuing the iteration until the k -th step gives (A.1). ■

Lemma 2 Assume that $\{\varepsilon_t\}$ defined in (2)-(3) and (7) is strictly stationary, $\Gamma_{Z \otimes Z}$ exists and condition (8) holds. Then

$$\mathbf{S}_{t-k} \xrightarrow{a.s.} \mathbf{0} \text{ as } k \rightarrow \infty \quad (\text{A.3})$$

where

$$\begin{aligned} \mathbf{S}_{t-k} &= (\mathbf{C}_{1,t-1} \otimes \mathbf{I}_M) \prod_{j=0}^k [\mathbb{I}_j(\mathbf{I}_M \otimes \mathbf{C}_{2,t-(2+j)}) + \mathbb{I}_{j+1}(\mathbf{C}_{2,t-(2+j)} \otimes \mathbf{I}_M)] \\ &\quad \times [\mathbb{I}_{k+1} \text{vec}(\mathbf{h}_{t-(1+k)}^{(2)} \mathbf{h}_{t-(2+k)}^{(2)'}) + \mathbb{I}_k \text{vec}(\mathbf{h}_{t-(2+k)}^{(2)} \mathbf{h}_{t-(1+k)}^{(2)'})]. \end{aligned}$$

Proof. $\{\mathbb{I}_j(\mathbf{I}_M \otimes \mathbf{C}_{2,t-(2+j)}) + \mathbb{I}_{j+1}(\mathbf{C}_{2,t-(2+j)} \otimes \mathbf{I}_M)\}$ is a sequence of i.i.d. random matrices. This and the fact that $\lambda(\mathbb{E}(\mathbf{I}_M \otimes \mathbf{C}_{2,t-j})) < 1$ for any j lead to $\lim_{k \rightarrow \infty} \mathbf{S}_{t-k} = \mathbf{0}$ almost surely when the assumptions of the lemma hold. Then

$$\begin{aligned} &\mathbb{E}\left\{(\mathbf{C}_{1,t-1} \otimes \mathbf{I}_M) \prod_{j=0}^{\infty} [\mathbb{I}_j(\mathbf{I}_M \otimes \mathbf{C}_{2,t-(2+j)}) + \mathbb{I}_{j+1}(\mathbf{C}_{2,t-(2+j)} \otimes \mathbf{I}_M)]\right\} \\ &= \Gamma_{C_1} \prod_{j=1}^{\infty} \mathbb{E}(\mathbf{I}_M \otimes \mathbf{C}_{2,t-2j}) \prod_{j=1}^{\infty} \mathbb{E}(\mathbf{I}_M \otimes \mathbf{C}_{2,t-(2j+1)}) \\ &= \mathbf{0}. \end{aligned}$$

On the other hand, as $\{\varepsilon_t\}$ is strictly stationary we can assume that sequence $\{\varepsilon_t\}$ started at a finite value in the infinite past. Then (A.3) holds. ■

Proof. [Theorem 2] From (7) we have

$$\begin{aligned} \text{vec}(\mathbf{h}_t^{(2)} \mathbf{h}_t^{(2)'}) &= \text{vec}(\mathbf{a}_0 \mathbf{a}_0') + (\mathbf{C}_{1,t-1} \otimes \mathbf{a}_0 + \mathbf{a}_0 \otimes \mathbf{C}_{1,t-1}) \mathbf{h}_{t-1}^{(2)} \\ &\quad + (\mathbf{C}_{2,t-2} \otimes \mathbf{a}_0 + \mathbf{a}_0 \otimes \mathbf{C}_{2,t-2}) \mathbf{h}_{t-2}^{(2)} \\ &\quad + (\mathbf{C}_{1,t-1} \otimes \mathbf{C}_{1,t-1}) \text{vec}(\mathbf{h}_{t-1}^{(2)} \mathbf{h}_{t-1}^{(2)'}) + (\mathbf{C}_{2,t-2} \otimes \mathbf{C}_{2,t-2}) \text{vec}(\mathbf{h}_{t-2}^{(2)} \mathbf{h}_{t-2}^{(2)'}) \\ &\quad + (\mathbf{I}_{M^2} + \mathbf{K}_{M^2})(\mathbf{C}_{1,t-1} \otimes \mathbf{C}_{2,t-2}) \text{vec}(\mathbf{h}_{t-2}^{(2)} \mathbf{h}_{t-1}^{(2)'}). \end{aligned} \quad (\text{A.4})$$

Noting that $(\mathbf{C}_{1,t-1} \otimes \mathbf{C}_{2,t-2}) \text{vec}(\mathbf{h}_{t-2}^{(2)} \mathbf{h}_{t-1}^{(2)'}) = \text{vec}(\mathbf{C}_{2,t-2} \mathbf{h}_{t-2}^{(2)} \mathbf{h}_{t-1}^{(2)'}) \mathbf{C}_{1,t-1}'$ and applying Lemmas 1 and 2 yields, after rewriting (A.4), as $k \rightarrow \infty$, gives

$$\Phi(L) \text{vec}(\mathbf{h}_t^{(2)} \mathbf{h}_t^{(2)'}) = \text{vec}(\mathbf{a}_0 \mathbf{a}_0') + \Theta(L) \mathbf{h}_t^{(2)} \quad (\text{A.5})$$

where $\Phi(L)$ ($M^2 \times M^2$) and $\Theta(L)$ ($M^2 \times M$) are infinite-order matrix polynomials in the lag operator L such that $\Phi(L) = \sum_{i=0}^{\infty} \Phi_i L^i$ and $\Theta(L) = \sum_{i=1}^{\infty} \Theta_i L^i$. In particular,

$$\begin{aligned}
\Phi_0 &= \mathbf{I}_{M^2} \\
\Phi_1 &= -(\mathbf{C}_{1,t-1} \otimes \mathbf{C}_{1,t-1}) \\
\Phi_2 &= -[(\mathbf{I}_{M^2} + \mathbf{K}_{M^2})(\mathbf{C}_{1,t-1} \otimes \mathbf{C}_{2,t-2})(\mathbf{C}_{2,t-2} \otimes \mathbf{I}_M) \\
&\quad + (\mathbf{C}_{2,t-2} \otimes \mathbf{C}_{2,t-2})] \\
\Phi_{k+1} &= -(\mathbf{I}_{M^2} + \mathbf{K}_{M^2}) \sum_{j=1}^k \{(\mathbf{C}_{1,t-1} \otimes \mathbf{I}_M) [\prod_{i=1}^j (\mathbb{I}_{i-1}(\mathbf{I}_M \otimes \mathbf{C}_{2,t-(i+1)}) \\
&\quad + \mathbb{I}_i(\mathbf{C}_{2,t-(i+1)} \otimes \mathbf{I}_M))] \times [\mathbb{I}_{j-1}(\mathbf{C}_{1,t-(1+j)} \otimes \mathbf{I}_M) \\
&\quad + \mathbb{I}_j(\mathbf{I}_M \otimes \mathbf{C}_{1,t-(1+j)})]\}
\end{aligned}$$

for $k = 2, 3, \dots$, and

$$\begin{aligned}
\Theta_1 &= \mathbf{C}_{1,t-1} \otimes \mathbf{a}_0 + \mathbf{a}_0 \otimes \mathbf{C}_{1,t-1} \\
\Theta_2 &= \mathbf{C}_{2,t-2} \otimes \mathbf{a}_0 + \mathbf{a}_0 \otimes \mathbf{C}_{2,t-2} \\
&\quad + (\mathbf{I}_{M^2} + \mathbf{K}_{MM})(\mathbf{C}_{1,t-1} \otimes \mathbf{C}_{2,t-2})(\mathbf{a}_0 \otimes \mathbf{I}_M) \\
\Theta_{k+1} &= (\mathbf{I}_{M^2} + \mathbf{K}_{MM}) \sum_{j=1}^k \{(\mathbf{C}_{1,t-1} \otimes \mathbf{I}_M) [\prod_{i=1}^j (\mathbb{I}_{i-1}(\mathbf{I}_M \otimes \mathbf{C}_{2,t-(i+1)}) \\
&\quad + \mathbb{I}_i(\mathbf{C}_{2,t-(i+1)} \otimes \mathbf{I}_M))] [\mathbb{I}_{j-1}(\mathbf{a}_0 \otimes \mathbf{I}_M) + \mathbb{I}_j(\mathbf{I}_M \otimes \mathbf{a}_0)]\}
\end{aligned}$$

for $k = 2, 3, \dots$.

Since $\{\mathbf{C}_{it}\}$, $i = 1, 2$, is a sequence of i.i.d random matrices and $\lambda(\mathbf{\Gamma}_{C_1} + \mathbf{\Gamma}_{C_2}) < 1$, it follows that

$$\begin{aligned}
&\mathbf{E}(\Phi(1)) \\
&= \lim_{k \rightarrow \infty} \sum_{i=1}^k \mathbf{E}(\Phi_i) \\
&= (\mathbf{\Gamma}_{C_1 \otimes C_1} + \mathbf{\Gamma}_{C_2 \otimes C_2})
\end{aligned}$$

$$\begin{aligned}
& +(\mathbf{I}_{M^2} + \mathbf{K}_{MM})[(\boldsymbol{\Gamma}_{C_1} \otimes \mathbf{I}_M)(\mathbf{I}_M - \boldsymbol{\Gamma}_{C_2} \otimes \boldsymbol{\Gamma}_{C_2})^{-1}\boldsymbol{\Gamma}_{C_1 \otimes C_2} \\
& +(\boldsymbol{\Gamma}_{C_1} \otimes \boldsymbol{\Gamma}_{C_2})(\mathbf{I}_M - \boldsymbol{\Gamma}_{C_2} \otimes \boldsymbol{\Gamma}_{C_2})^{-1}\boldsymbol{\Gamma}_{C_2 \otimes C_1}] \\
& = \boldsymbol{\Gamma}.
\end{aligned} \tag{A.6}$$

Condition (10) implies that $\mathbf{E}(\boldsymbol{\Phi}(1)) = (\mathbf{I}_{M^2} - \boldsymbol{\Gamma})$ is finite and that $\mathbf{E}(\boldsymbol{\Phi}(1))$ is invertible.

Similarly,

$$\begin{aligned}
& \mathbf{E}(\boldsymbol{\Theta}(1)) \\
& = \lim_{k \rightarrow \infty} \sum_{i=1}^k \mathbf{E}(\boldsymbol{\Theta}_i) \\
& = \sum_{i=1}^2 (\boldsymbol{\Gamma}_{C_i} \otimes \mathbf{a}_0 + \mathbf{a}_0 \otimes \boldsymbol{\Gamma}_{C_i}) \\
& \quad +(\mathbf{I}_{M^2} + \mathbf{K}_{M^2})(\boldsymbol{\Gamma}_{C_1} \otimes \mathbf{I}_M)[(\mathbf{I}_M \otimes \boldsymbol{\Gamma}_{C_2})(\mathbf{I}_{M^2} - \boldsymbol{\Gamma}_{C_2} \otimes \boldsymbol{\Gamma}_{C_2})^{-1}(\mathbf{a}_0 \otimes \mathbf{I}_M) \\
& \quad +(\boldsymbol{\Gamma}_{C_2} \otimes \boldsymbol{\Gamma}_{C_2})(\mathbf{I}_{M^2} - \boldsymbol{\Gamma}_{C_2} \otimes \boldsymbol{\Gamma}_{C_2})^{-1}(\mathbf{I}_M \otimes \mathbf{a}_0)] < \infty
\end{aligned} \tag{A.7}$$

Assume that $\mathbf{E}(\boldsymbol{\Phi}(1))$ and $\mathbf{E}(\boldsymbol{\Theta}(1))$ have no common eigenvalues. It follows from (A.6) and (A.7) that $\text{vec}(\mathbf{h}_t^{(2)} \mathbf{h}_t^{(2)'})$ in (A.5) has a fourth-order weakly stationary solution and $\mathbf{E}(\text{vec}(\mathbf{h}_t^{(2)} \mathbf{h}_t^{(2)'}))$ is finite.

Under condition (10), taking expectations on both sides of (A.5) gives

$$\mathbf{E}(\text{vec}(\mathbf{h}_t^{(2)} \mathbf{h}_t^{(2)'})) = [\mathbf{E}\boldsymbol{\Phi}(1)]^{-1}\{\text{vec}(\mathbf{a}_0 \mathbf{a}_0') + [\mathbf{E}\boldsymbol{\Theta}(1)][\mathbf{E}(\mathbf{h}_t^{(2)})]\}.$$

As $\mathbf{E}(\text{vec}(\boldsymbol{\varepsilon}_t^{(2)} \boldsymbol{\varepsilon}_t^{(2)'})) = \mathbf{E}(\mathbf{Z}_t^2 \otimes \mathbf{Z}_t^2) \mathbf{E}(\text{vec}(\mathbf{h}_t^{(2)} \mathbf{h}_t^{(2)'}))$, we have seen that equation (11) holds.

■

Lemma 3 *Assume that $\Gamma_{Z \otimes Z}$ exists and condition (10) holds for the vector GARCH(2,2) model (2)-(3) and (7). Then*

$$\begin{aligned}
& \text{Evec}(\mathbf{C}_{2,t-2} \mathbf{h}_{t-2}^{(2)} \mathbf{h}_{t-1}^{(2)' } \mathbf{Z}_{t-1}^2) \\
& = (\boldsymbol{\Gamma}_Z \otimes \mathbf{I}_M)[(\mathbf{I}_M \otimes \boldsymbol{\Gamma}_{C_2})(\mathbf{I}_{M^2} - \boldsymbol{\Gamma}_{C_2} \otimes \boldsymbol{\Gamma}_{C_2})^{-1}(\mathbf{a}_0 \otimes \mathbf{I}_M)
\end{aligned}$$

$$\begin{aligned}
& +(\boldsymbol{\Gamma}_{C_2} \otimes \boldsymbol{\Gamma}_{C_2})(\mathbf{I}_{M^2} - \boldsymbol{\Gamma}_{C_2} \otimes \boldsymbol{\Gamma}_{C_2})^{-1}(\mathbf{I}_M \otimes \mathbf{a}_0)] \\
& \times (\mathbf{a}'_0 \otimes \mathbf{I}_M) \text{vec}(\mathbf{I}_M - \boldsymbol{\Gamma}_{C_1} - \boldsymbol{\Gamma}_{C_2})^{-1} \\
& + [(\boldsymbol{\Gamma}_Z \otimes \mathbf{I}_M)(\mathbf{I}_{M^2} - \boldsymbol{\Gamma}_{C_2} \otimes \boldsymbol{\Gamma}_{C_2})^{-1} \boldsymbol{\Gamma}_{C_1 \otimes C_2} \\
& + (\boldsymbol{\Gamma}_Z \otimes \boldsymbol{\Gamma}_{C_2})(\mathbf{I}_{M^2} - \boldsymbol{\Gamma}_{C_2} \otimes \boldsymbol{\Gamma}_{C_2})^{-1} \boldsymbol{\Gamma}_{C_2 \otimes C_1}] \text{Evec}(\mathbf{h}_{t-2}^{(2)} \mathbf{h}_{t-1}^{(2)'}). \tag{A.8}
\end{aligned}$$

Proof. Equation (A.8) follows from Lemma 1 and Theorem 2. ■

Lemma 4 Assume that $\boldsymbol{\Gamma}_{Z \otimes Z}$ exists and condition (10) holds for the vector GARCH(2,2) model (2)-(3) and (7). Let $k \geq 3$. Then

$$\mathbb{E}(\mathbf{h}_t^{(2)} \mathbf{h}_{t-k}^{(2)'}) = \begin{pmatrix} \mathbf{a}_0 & \boldsymbol{\Gamma}_{C_1} & \boldsymbol{\Gamma}_{C_2} \end{pmatrix} \boldsymbol{\Gamma}_*^{k-1} \mathbb{E} \mathbf{h}_{t-k}^* \tag{A.9}$$

$$\text{where } \boldsymbol{\Gamma}_* = \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{a}_0 & \boldsymbol{\Gamma}_{C_1} & \boldsymbol{\Gamma}_{C_2} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{pmatrix} \text{ and } \mathbf{h}_{t-k}^* = \begin{pmatrix} \mathbf{h}_{t-k}^{(2)'} \\ \mathbf{h}_{t-k}^{(2)} \mathbf{h}_{t-k}^{(2)'} \\ \mathbf{h}_{t-(k+1)}^{(2)} \mathbf{h}_{t-k}^{(2)'} \end{pmatrix}.$$

Proof. Rewrite (7) as

$$\mathbf{h}_t^{(2)} = \begin{pmatrix} \mathbf{a}_0 & \mathbf{C}_{1,t-1} & \mathbf{C}_{2,t-2} \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{h}_{t-1}^{(2)} \\ \mathbf{h}_{t-2}^{(2)} \end{pmatrix}. \tag{A.10}$$

Applying equation (7) to $\mathbf{h}_{t-1}^{(2)}$ on the right-hand side of (A.10) and continuing the iteration until the k -th step yields

$$\mathbf{h}_t^{(2)} = \begin{pmatrix} \mathbf{a}_0 & \mathbf{C}_{1,t-1} & \mathbf{C}_{2,t-2} \end{pmatrix} \mathbf{C}_{t-2}^* \mathbf{C}_{t-3}^* \cdots \mathbf{C}_{t-k}^* \begin{pmatrix} 1 \\ \mathbf{h}_{t-k}^{(2)} \\ \mathbf{h}_{t-(k+1)}^{(2)} \end{pmatrix} \tag{A.11}$$

$$\text{where } \mathbf{C}_{t-i}^* = \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{a}_0 & \mathbf{C}_{1,t-i} & \mathbf{C}_{2,t-(i+1)} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{pmatrix}. \text{ Note that } \mathbb{E}(\mathbf{C}_{t-i}^* \mathbf{C}_{t-j}^*) = \boldsymbol{\Gamma}_*^2 \text{ for } i \neq j.$$

Left-multiplying (A.11) by $\mathbf{h}_{t-k}^{(2) \prime}$ and taking expectations yields (A.9). \blacksquare

Proof. [Theorem 3] We prove Theorem 3 by induction for n .

(i) We shall show that (17)-(21) hold for $n = 3$.

First we show that (17)-(19) hold for $n = 1$. Consider

$$\boldsymbol{\varepsilon}_t^{(2)} \boldsymbol{\varepsilon}_{t-1}^{(2) \prime} = \mathbf{Z}_t^2 (\mathbf{a}_0 + \mathbf{C}_{1,t-1} \mathbf{h}_{t-1}^{(2)} + \mathbf{C}_{2,t-2} \mathbf{h}_{t-2}^{(2)}) \mathbf{h}_{t-1}^{(2) \prime} \mathbf{Z}_{t-1}^2. \quad (\text{A.12})$$

Taking expectations on both sides of (A.12) and writing them in the *vec* form yields

$$\begin{aligned} \text{vec} \mathbf{E}[\boldsymbol{\varepsilon}_t^{(2)} \boldsymbol{\varepsilon}_{t-1}^{(2) \prime}] &= (\mathbf{I}_M \otimes \mathbf{a}_0) \boldsymbol{\mu}_2 + \boldsymbol{\Gamma}_{Z \otimes C_1} \text{vec} \mathbf{E}(\mathbf{h}_t^{(2)} \mathbf{h}_t^{(2) \prime}) \\ &\quad + \text{vec} \mathbf{E}(\mathbf{C}_{2,t-2} \mathbf{h}_{t-2}^{(2)} \mathbf{h}_{t-1}^{(2) \prime} \mathbf{Z}_{t-1}^2). \end{aligned} \quad (\text{A.13})$$

Set $\text{vec} \tilde{\boldsymbol{\Gamma}}_M = \text{vec} \mathbf{E}(\mathbf{h}_t^{(2)} \mathbf{h}_t^{(2) \prime})$ and $\text{vec} \tilde{\tilde{\boldsymbol{\Gamma}}}_M = \text{vec} \mathbf{E}(\mathbf{C}_{2,t-2} \mathbf{h}_{t-2}^{(2)} \mathbf{h}_{t-1}^{(2) \prime} \mathbf{Z}_{t-1}^2)$. Rewriting (A.13)

using these definitions and applying Theorem 2 and Lemma 3 yields

$$\begin{aligned} \text{vec} \mathbf{E}[\boldsymbol{\varepsilon}_t^{(2)} \boldsymbol{\varepsilon}_{t-1}^{(2) \prime}] &= (\mathbf{I}_M \otimes \mathbf{a}_0) \boldsymbol{\mu}_2 + \boldsymbol{\Gamma}_{Z \otimes C_1} \text{vec} \tilde{\boldsymbol{\Gamma}}_M + \text{vec} \tilde{\tilde{\boldsymbol{\Gamma}}}_M \\ &= \text{vec} \tilde{\boldsymbol{\Gamma}}_M(1). \end{aligned} \quad (\text{A.14})$$

Thus, (17)-(19) hold for $n = 1$.

Similarly, for $n = 2$,

$$\begin{aligned} \text{vec} \mathbf{E}[\boldsymbol{\varepsilon}_t^{(2)} \boldsymbol{\varepsilon}_{t-2}^{(2) \prime}] &= (\mathbf{I}_M \otimes \mathbf{a}_0) \boldsymbol{\mu}_2 + \boldsymbol{\Gamma}_{Z \otimes C_2} \text{vec} \mathbf{E}(\mathbf{h}_t^{(2)} \mathbf{h}_t^{(2) \prime}) \\ &\quad + (\mathbf{I}_M \otimes \boldsymbol{\Gamma}_{C_1}) [(\mathbf{I}_M \otimes \mathbf{a}_0) \boldsymbol{\mu}_2 + \boldsymbol{\Gamma}_{Z \otimes C_1} \text{vec} \mathbf{E}(\mathbf{h}_t^{(2)} \mathbf{h}_t^{(2) \prime}) \\ &\quad + \text{vec} \mathbf{E}(\mathbf{C}_{2,t-3} \mathbf{h}_{t-3}^{(2)} \mathbf{h}_{t-2}^{(2) \prime} \mathbf{Z}_{t-2}^2)] \\ &= (\mathbf{I}_M \otimes \mathbf{a}_0) \boldsymbol{\mu}_2 + \boldsymbol{\Gamma}_{Z \otimes C_2} \text{vec} \tilde{\boldsymbol{\Gamma}}_M + (\mathbf{I}_M \otimes \boldsymbol{\Gamma}_{C_1}) \text{vec} \tilde{\boldsymbol{\Gamma}}_M(1). \\ &= \text{vec} \tilde{\boldsymbol{\Gamma}}_M(2) \end{aligned} \quad (\text{A.15})$$

so that (17)-(20) are valid for $n = 2$.

We shall now show that (21) holds for $n = 3$. We have

$$\begin{aligned}
\text{vecE}[\boldsymbol{\varepsilon}_t^{(2)} \boldsymbol{\varepsilon}_{t-3}^{(2)'}] &= (\mathbf{I}_M \otimes \mathbf{a}_0) \boldsymbol{\mu}_2 + (\mathbf{I}_M \otimes \boldsymbol{\Gamma}_{C_1}) \{ (\mathbf{I}_M \otimes \mathbf{a}_0) \boldsymbol{\mu}_2 \\
&\quad + \boldsymbol{\Gamma}_{Z \otimes C_2} \text{vecE}(\mathbf{h}_t^{(2)} \mathbf{h}_t^{(2)'}) + (\mathbf{I}_M \otimes \boldsymbol{\Gamma}_{C_1}) [(\mathbf{I}_M \otimes \mathbf{a}_0) \boldsymbol{\mu}_2 \\
&\quad + \boldsymbol{\Gamma}_{Z \otimes C_1} \text{vecE}(\mathbf{h}_t^{(2)} \mathbf{h}_t^{(2)'}) + \boldsymbol{\Gamma}_{Z \otimes C_2} \text{vecE}(\mathbf{C}_{2,t-4} \mathbf{h}_{t-4}^{(2)} \mathbf{h}_{t-3}^{(2)' \prime} \mathbf{Z}_{t-3}^2)] \\
&\quad + (\mathbf{I}_M \otimes \boldsymbol{\Gamma}_{C_2}) [(\mathbf{I}_M \otimes \mathbf{a}_0) \boldsymbol{\mu}_2 + \boldsymbol{\Gamma}_{Z \otimes C_1} \text{vecE}(\mathbf{h}_t^{(2)} \mathbf{h}_t^{(2)'}) \\
&\quad + \text{vecE}(\mathbf{C}_{2,t-4} \mathbf{h}_{t-4}^{(2)} \mathbf{h}_{t-3}^{(2)' \prime} \mathbf{Z}_{t-3}^2)] \}. \tag{A.16}
\end{aligned}$$

Applying (A.14) and (A.15) to (A.16) gives

$$\begin{aligned}
\text{vec} \tilde{\boldsymbol{\Gamma}}_M(3) &= (\mathbf{I}_M \otimes \mathbf{a}_0) \boldsymbol{\mu}_2 + (\mathbf{I}_M \otimes \boldsymbol{\Gamma}_{C_1}) \text{vec} \tilde{\boldsymbol{\Gamma}}_M(2) \\
&\quad + (\mathbf{I}_M \otimes \boldsymbol{\Gamma}_{C_2}) \text{vec} \tilde{\boldsymbol{\Gamma}}_M(1).
\end{aligned}$$

(ii) Assume that (17)-(21) hold for all $n \leq k$. We shall show that they hold for $n = k+1$.

From this assumption one obtains

$$\begin{aligned}
\tilde{\boldsymbol{\Gamma}}_M(k) &= \mathbf{a}_0 \boldsymbol{\mu}'_2 + \boldsymbol{\Gamma}_{C_1} \tilde{\boldsymbol{\Gamma}}(k-1) + \boldsymbol{\Gamma}_{C_2} \tilde{\boldsymbol{\Gamma}}_M(k-2) \\
&= \begin{pmatrix} \mathbf{a}_0 & \boldsymbol{\Gamma}_{C_1} & \boldsymbol{\Gamma}_{C_2} \end{pmatrix} \begin{pmatrix} \boldsymbol{\mu}'_2 \\ \tilde{\boldsymbol{\Gamma}}(k-1) \\ \tilde{\boldsymbol{\Gamma}}_M(k-2) \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{a}_0 & \boldsymbol{\Gamma}_{C_1} & \boldsymbol{\Gamma}_{C_2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{a}_0 & \boldsymbol{\Gamma}_{C_1} & \boldsymbol{\Gamma}_{C_2} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\mu}'_2 \\ \tilde{\boldsymbol{\Gamma}}(k-2) \\ \tilde{\boldsymbol{\Gamma}}_M(k-3) \end{pmatrix} \\
&= \dots \\
&= \begin{pmatrix} \mathbf{a}_0 & \boldsymbol{\Gamma}_{C_1} & \boldsymbol{\Gamma}_{C_2} \end{pmatrix} \boldsymbol{\Gamma}_*^{k-3} \begin{pmatrix} \boldsymbol{\mu}'_2 \\ \tilde{\boldsymbol{\Gamma}}(2) \\ \tilde{\boldsymbol{\Gamma}}_M(1) \end{pmatrix}. \tag{A.17}
\end{aligned}$$

On the other hand, using Lemma 4 one obtains

$$\begin{aligned}
\mathbb{E}[\boldsymbol{\varepsilon}_t^{(2)} \boldsymbol{\varepsilon}_{t-(k+1)}^{(2)'}] &= \boldsymbol{\Gamma}_Z \mathbb{E}(\mathbf{h}_t^{(2)} \mathbf{h}_{t-(k+1)}^{(2)'} \mathbf{Z}_{t-(k+1)}^2) \\
&= \boldsymbol{\Gamma}_Z \begin{pmatrix} \mathbf{a}_0 & \boldsymbol{\Gamma}_{C_1} & \boldsymbol{\Gamma}_{C_2} \end{pmatrix} \boldsymbol{\Gamma}_*^{k-2} \\
&\quad \times \mathbb{E} \left[\mathbf{C}_{t-k}^* \mathbf{C}_{t-(k+1)}^* \begin{pmatrix} 1 \\ \mathbf{h}_{t-(k+1)}^{(2)} \\ \mathbf{h}_{t-(k+2)}^{(2)} \end{pmatrix} \mathbf{h}_{t-(k+1)}^{(2)'} \mathbf{Z}_{t-(k+1)}^2 \right]. \quad (\text{A.18})
\end{aligned}$$

Applying (A.14)-(A.15) to the right-hand side of (A.18) gives

$$\mathbb{E} \left[\mathbf{C}_{t-k}^* \mathbf{C}_{t-(k+1)}^* \begin{pmatrix} 1 \\ \mathbf{h}_{t-(k+1)}^{(2)} \\ \mathbf{h}_{t-(k+2)}^{(2)} \end{pmatrix} \mathbf{h}_{t-(k+1)}^{(2)'} \mathbf{Z}_{t-(k+1)}^2 \right] = \begin{pmatrix} \boldsymbol{\mu}'_2 \\ \tilde{\boldsymbol{\Gamma}}_M(2) \\ \tilde{\boldsymbol{\Gamma}}_M(1) \end{pmatrix}. \quad (\text{A.19})$$

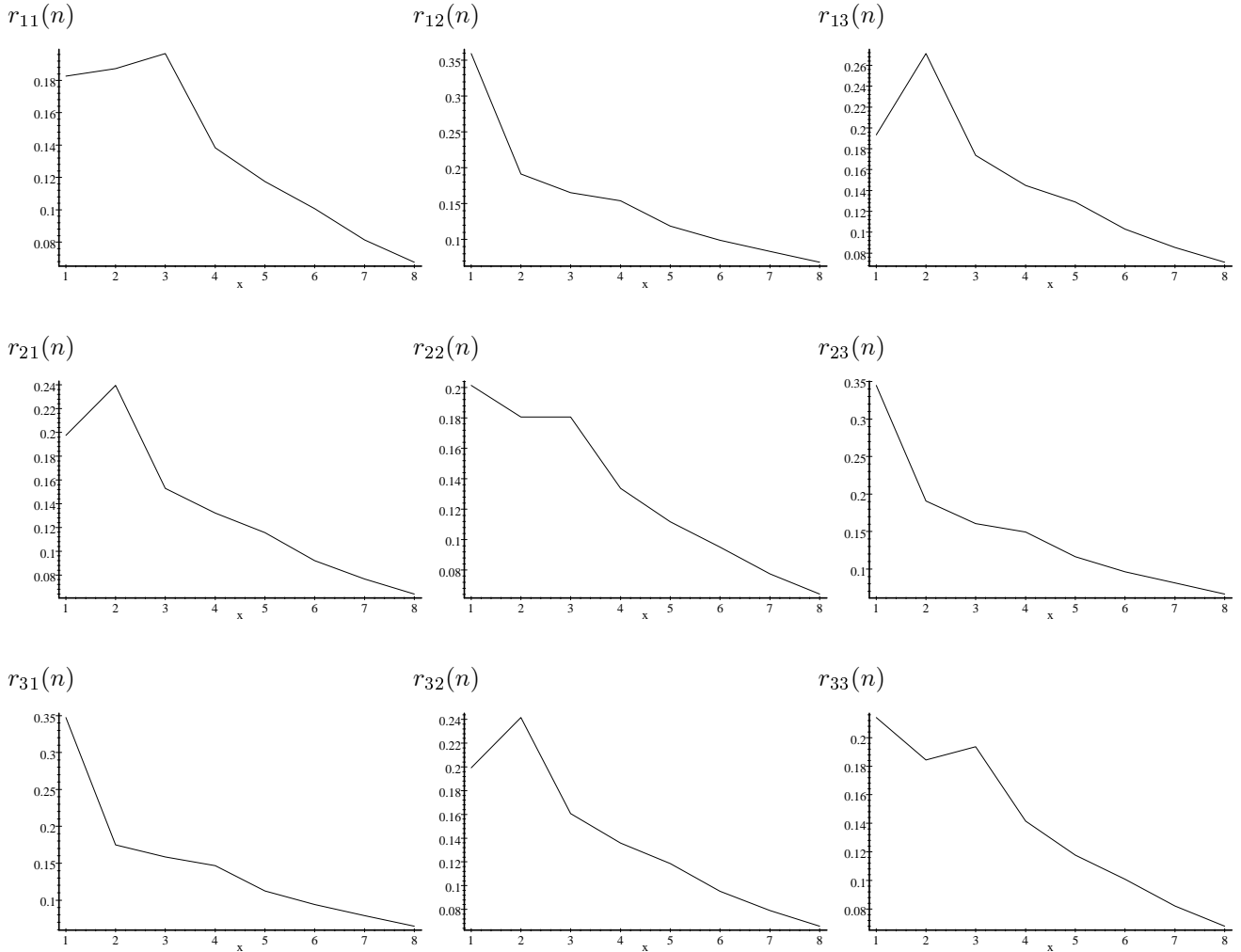
It follows from (A.17) and (A.19) that

$$\begin{aligned}
\boldsymbol{\Gamma}_*^{k-2} \begin{pmatrix} \boldsymbol{\mu}'_2 \\ \tilde{\boldsymbol{\Gamma}}_M(2) \\ \tilde{\boldsymbol{\Gamma}}_M(1) \end{pmatrix} &= \boldsymbol{\Gamma}_*^{k-3} \begin{pmatrix} \boldsymbol{\mu}'_2 \\ \tilde{\boldsymbol{\Gamma}}_M(3) \\ \tilde{\boldsymbol{\Gamma}}_M(2) \end{pmatrix} \\
&= \dots \\
&= \begin{pmatrix} \boldsymbol{\mu}'_2 \\ \tilde{\boldsymbol{\Gamma}}_M(k) \\ \tilde{\boldsymbol{\Gamma}}_M(k-1) \end{pmatrix}. \quad (\text{A.20})
\end{aligned}$$

Applying (A.19) to (A.20) and rewriting it in the *vec* form completes the proof of equation

(21) when $n = k + 1$. ■

Figure 1. Auto and cross correlations of squared observations of a three-dimensional extended CCC-GARCH(1,1) process, lags 1-8.



Note: Γ_C has a pair of complex conjugate eigenvalues with modulus =0.432 and a real root =0.858.

The parameter values are $a_{01} = 0.1, a_{02} = 0.2, a_{03} = 0.25,$

$a_{11} = 0.05, a_{12} = 0.25, a_{13} = 0.06, a_{21} = 0.07, a_{22} = 0.08, a_{23} = 0.25, a_{31} = 0.25, a_{32} = 0.08, a_{33} = 0.08,$

$b_{11} = 0.05, b_{12} = 0.4, b_{13} = 0.05, b_{21} = 0.05, b_{22} = 0.07, b_{23} = 0.3, b_{31} = 0.25, b_{32} = 0.08, b_{33} = 0.08.$

The correlation matrix of the standard normal error process $\{\mathbf{z}_t\}$ has the following nondiagonal elements:

$\rho_{12} = 0.2, \rho_{13} = 0.3, \rho_{23} = 0.23$

Table 1 The estimated constant correlations $\hat{\rho}_{ij}$, $i, j = \text{DM, FF, IL, SF, BP}$, from the two estimated CCC-GARCH models (EMS period and Pre-EMS period) of Bollerslev (1990)

i	EMS					Pre-EMS				
	DM	FF	IL	SF	BP	DM	FF	IL	SF	BP
$\hat{\rho}_{\text{DM}i}$	1	0.932	0.886	0.917	0.674	1	0.607	0.425	0.714	0.443
$\hat{\rho}_{\text{FF}i}$	-	1	0.876	0.866	0.676	-	1	0.441	0.517	0.488
$\hat{\rho}_{\text{IL}i}$	-	-	1	0.816	0.622	-	-	1	0.336	0.345
$\hat{\rho}_{\text{SF}i}$	-	-	-	1	0.635	-	-	-	1	0.305
$\hat{\rho}_{\text{BP}i}$	-	-	-	-	1	-	-	-	-	1

Source: Tables 1 and 2 of Bollerslev (1990)

Table 2. Auto- and cross correlations $r_{ij}(n)$, $i, j = \text{DM, FF, IL, SF, BP}$, $n = 1, 2, 5, 10$, are computed from the two estimated CCC-GARCH models (EMS period and Pre-EMS period) of Bollerslev (1990)

	EMS				Pre-EMS ¹			
	1	2	5	10	1	2	5	10
$r_{11}(n)$	0.0462	0.0441	0.0384	0.0305	0.0584	0.0562	0.0502	0.0416
$r_{12}(n)$	0.0207	0.0205	0.0201	0.0193	0.0577	0.0491	0.0301	0.0133
$r_{13}(n)$	0.1167	0.0977	0.0573	0.0235	0.0059	0.0054	0.0041	0.0026
$r_{14}(n)$	0.0638	0.0633	0.0616	0.0588	0.0519	0.0507	0.0475	0.0425
$r_{15}(n)$	0.0461	0.0444	0.0399	0.0334	0.0217	0.0170	0.0083	0.0025
$r_{21}(n)$	0.0370	0.0353	0.0307	0.0244	0.0234	0.0226	0.0201	0.0167
$r_{22}(n)$	0.0150	0.0148	0.0144	0.0138	0.2039	0.1733	0.1064	0.0472
$r_{23}(n)$	0.1088	0.0911	0.0534	0.0219	0.0074	0.0068	0.0051	0.0032
$r_{24}(n)$	0.0554	0.0549	0.0535	0.0511	0.0258	0.0252	0.0236	0.0211
$r_{25}(n)$	0.0407	0.0392	0.0353	0.0295	0.0309	0.0243	0.0118	0.0036
$r_{31}(n)$	0.0401	0.0383	0.0338	0.0265	0.0020	0.0019	0.0017	0.0014
$r_{32}(n)$	0.0206	0.0204	0.0199	0.0191	0.0050	0.0042	0.0026	0.0011
$r_{33}(n)$	0.0206	0.1821	0.1067	0.0438	0.4939	0.4510	0.3432	0.2177
$r_{34}(n)$	0.0481	0.0477	0.0464	0.0443	0.0022	0.0021	0.0020	0.0018
$r_{35}(n)$	0.0422	0.0408	0.0366	0.0306	0.0020	0.0016	0.0007	0.0002
$r_{41}(n)$	0.0470	0.0449	0.0391	0.0311	0.0307	0.0295	0.0264	0.0218
$r_{42}(n)$	0.0304	0.0301	0.0294	0.0282	0.0270	0.0229	0.0014	0.0062
$r_{43}(n)$	0.0841	0.0704	0.0412	0.0169	0.0029	0.0026	0.0020	0.0012
$r_{44}(n)$	0.2115	0.2096	0.2040	0.1949	0.3532	0.3454	0.3231	0.2891
$r_{45}(n)$	0.0497	0.0479	0.0431	0.0361	0.0060	0.0047	0.0023	0.0007
$r_{51}(n)$	0.0265	0.0253	0.0221	0.0175	0.0894	0.0861	0.0769	0.0637
$r_{52}(n)$	0.0138	0.0137	0.0133	0.0127	0.0393	0.0334	0.0205	0.0091
$r_{53}(n)$	0.0590	0.0494	0.0289	0.0119	0.0040	0.0036	0.0028	0.0017
$r_{54}(n)$	0.0437	0.0434	0.0422	0.0403	0.0071	0.0069	0.0065	0.0058
$r_{55}(n)$	0.1712	0.1652	0.1484	0.1242	0.1331	0.1104	0.0511	0.0154

¹ $r_{ij}(n)$, $n = 1, 2, 5, 10$, are computed from the estimated pre-EMS model by reducing the estimate of a_{33} to 0.287 in order to satisfy condition (10) for this pre-EMS system