Network Formation with Heterogeneous Players

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Abstract

This paper studies network formation in settings where players are heterogeneous with respect to benefits as well as the costs of forming links. If an individual's costs of forming links are the same across players but benefits vary then an equilibrium network is either a single center-sponsored star or a collection of such structures. If, on the other hand, the values are identical but costs of forming links are different across links then any minimal/acyclic network can be sustained in equilibrium.

We then study a society which is composed of distinct groups, values are homogeneous and the cost of links is an increasing function of the distance between the groups. In this framework, we provide a complete characterization of equilibrium networks. We find that an equilibrium network is a generalized center-sponsored star with one of the groups forming the core of the network. This core group is entirely internally linked while all other groups are entirely externally linked and thus completely fragmented. This relative absence of insider links (which are cheaper than outsider links) creates a sharp conflict between individual incentives and social welfare.

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1 Introduction

This paper studies the nature of network formation when individual players differ in their cost of forming links with other players as well as in the benefits derived from other players. There is a variety of settings in which asymmetries between players are natural. For example, some players may be more productive or more informed as compared to others; likewise, some players may have high costs of linking while others have low cost of forming links. We examine the incentives of individual players to form or sever links and the architecture of the strategic stable and efficient networks in such settings.

Our point of departure is the paper by Bala and Goyal (2000).[1] This paper provides a complete characterization of (strict) Nash networks when players are homogeneous. In particular, they show that when a player's payoffs are increasing in the number of other players accessed and decreasing in the number of links formed, the strict Nash network is either a center-sponsored star or the empty network. Thus partially connected networks cannot arise in equilibrium and only a very specific architecture can be sustained within the class of connected networks. It is worthwhile to briefly sketch the argument underlying these two aspects of the result. First, we discuss the absence of partial connectedness. Suppose there are two components in an equilibrium network with one of them being non-singleton. In the non-singleton component, it must be the case that there is a player who forms a link and that this individual's returns to forming the link exceed the costs of doing so. However, for a player external to the component, the returns to forming a link with someone in the component are strictly larger (since he has access to all the members of the component) while the costs are the same as that for the player internal to the component. Hence the player external to the component would like to form a link with the component and this network cannot be sustained in equilibrium. Next we discuss the argument underlying the center-sponsored star. Suppose that a player i forms a link with a player j in the network. Then it cannot be the case that player j is directly connected with any other player. This is because if player j is linked with a player k, then player i is indifferent between linking with player j or with player k. Hence, the network is not a strict Nash network. Thus player j must not have any links with other players; however, the network is connected, and so player j must access every player via player i. Using a variant of the above switching argument it can be shown that player i must form a link with every other player. Hence player i must be the centre of a star and he must sponsor all the links in the network.

In this paper we examine the role of the assumption that players are homogeneous with respect to costs and valuations in deriving this characterization

[1] In the present paper we will be studying a model where links can be formed by individuals independently (they are one-sided) but flow of benefits is two-sided. This model is referred to as the two-way flow model in Bala and Goyal (2001).

result. We shall also study the implications of heterogeneity in shaping the architecture of efficient networks.

We start with a general model of heterogeneous players: the costs of forming links can differ across links and the benefit to a player i from a link with player j is allowed to depend on both i and j. In addition, we assume that the length of the path does not matter in defining the benefits (there is no decay). This assumption implies that any Nash network must be acyclic or minimal. Our first result establishes an equivalence between the set of minimal and equilibrium networks: every equilibrium network is minimal and every minimal network can be sustained as a strict Nash equilibrium for some set of costs and value parameters. This result shows that individual incentives and strategic interaction generate no further restrictions apart from minimality. This result motivates a closer examination of the role of cost and value heterogeneity.

First, we study a setting where costs of forming links for a particular player are the same across the links he forms, while the valuations are allowed to differ. In this setting, we show that an equilibrium network is either a center-sponsored star or a collection of such architectures. The converse result also obtains: every such network can be supported in a Nash equilibrium for some parameters. Moreover, if the benefits of accessing a particular individual are the same for everyone but different players are differently valued by everyone then there is at most one non-singleton component and this is a center-sponsored star. A comparison of this result with the result of Bala and Goyal suggests that heterogeneity in values is important in defining the level of connectedness but does not play any role in defining the (center-sponsored star) architecture of individual components.

We then study the model with cost heterogeneity and homogeneous values. We start by noting that in this setting, the above mentioned correspondence between the set of minimal networks and the set of equilibrium networks still obtains. We therefore need to place some restrictions on the cost parameters of the model in order to obtain any further restrictions on the set of equilibrium networks. This leads us to consider a society with several distinct groups which are spatially arranged. The cost of forming a link between two players is non-decreasing in the distance between the groups to which the two players belong. Thus, the spatial distance among groups represents the degree of heterogeneity across players. As before, Nash networks are minimal due to the no-decay assumption. However, a Nash network, depending on the cost levels, can be either empty, minimal connected or partially connected. Partially connected networks arise if the difference between the cost of links within a group (insider links) and the cost of links across groups (outsider links) is high. Thus a large number of networks can arise in equilibrium and this leads us to study strict Nash networks. We first note that partially connected structures survive the strictness refinement. However, we prove if the equilibrium network has distinct components then each of the non-singleton components comprises of members of only one group and has the center-sponsored star architecture. Figure 1A depicts a partially connected strict Nash network in a society composed by three groups, each of them containing three players, *i.e.* $1a$ denotes player a belonging to group 1. player $1a$ bears all the links with members belonging to his own group (as represented by the filled circles on each link adjacent to this player), and the same holds for player $2a$ and $3a$ that are the centers of groups 2 and 3, respectively. Our main result pertains to connected networks. We prove here that a strict Nash network is a generalized center-sponsored star with the following features: each path in the network is oriented towards a unique player, say i. Therefore, this player plays a central role in the network itself. Furthermore, the group to which player i belongs, say N_l, is the only group to be entirely internally linked: any two players belonging to N_l are either directly linked or indirectly linked via a path that contains only members of their same group. This group represents the *core* of the network. Furthermore, all the remaining groups are entirely externally linked: members of these groups are completely split up in the network structure. From these properties it follows that the diameter of this class of architectures depends only on the number of groups composing the society and not on the number of network participants per se. Figure 1B depicts a generalized center-sponsored star. Each path in this network is oriented toward player $1a$; hence, player $1a$ is the center of this graph and group 1 is the core of the network. Furthermore, it is easy to note that each pair of players both belonging either to groups 2 or 3, say $2a$ and $2b$, access each other through intermediary players, in this case $1b$, that always belong to a different group from their own.

Our final set of results pertain to the architecture of efficient networks in the insider-outsider model. It is clear that an efficient network must minimize on the number of outsider links since they are more costly as compared to insider links. Indeed, an efficient connected network is characterized by having each group entirely internally linked and $m - 1$ outsider links, where m is the number of groups in the network formation game (see Figure 1C). However, our characterization result above states that a (connected) equilibrium network is a generalized centre-sponsored star, with $N - N_l$ outsider links (where N_l is the number of players in the core group). Thus strategic incentives of players potentially generate a significant waste of resources. This conflict disappears when the cost of forming outside links is so high to make connectivity between groups socially inefficient. When this is the case, our model becomes essentially equivalent to a set of homogeneous player models and the general result of Bala and Goyal obtains, sustainable networks are also efficient.

To summarize: our analysis illustrates the scope of the research programme which studies network formation from the perspective of individual incentives.

Our results highlight the robustness of properties such as centrality, center-sponsorship and short network diameter. Moreover, the characterization of Nash networks shows how individual incentives can generate very particular and somewhat unexpected network outcomes, such as the generalized center-sponsored star, where there is a core group which is entirely internally connected while all the other groups are entirely externally linked and hence completely fragmented. This strategic implication has high social costs and can lead to significant waste of resources.

We briefly mention three papers that are most closely related to our analysis. McBride (2001) considers a non-cooperative network formation game in a framework characterized by imperfect information about valuations of players. Haller and Sarangi (2000) exploit the implications of heterogeneous players. In contrast to our paper, in their model links are not fully reliable and heterogeneity is introduced in the reliability of links. Finally, Johnson and Gilles (1999) investigate the presence of heterogeneous cost of linking across players in a two-side network formation game. The kind of player heterogeneity considered in their model is similar to ours, but the equilibrium concept implemented is different. In their setting, for high cost levels only the empty network is pairwise stable, while when costs are low the set of equilibrium architectures becomes richer as compared to the homogeneous case. In particular they show that networks where localities are completely connected play an important role in the set of stable equilibrium architectures.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 presents results on network formation when costs and values are allowed to vary freely. Section 4 analyzes the case of a insider-outsider model of network formation. Section 5 concludes.

2 The Model

Let $N = \{1, ..., n\}$ be a set of players and let i and j be typical members of this set. To avoid trivialities, we shall assume throughout that $n \geq 3$. Each player is assumed to possess some information of value to himself and to other players. He can augment his information by communicating with other people; this communication takes resources, time and effort and is made possible via *pair-wise* links.

A strategy of player $i \in N$ is a (row) vector $g_i = (g_{i,1}, ..., g_{i,i-1}, g_{i,i+1}, ..., g_{i,n})$ where $g_{i,j} \in \{0, 1\}$ for each $j \in N \setminus \{i\}$. We say that player i has a link with j if $g_{i,j} = 1$. A link between player i and j can allow for either one-way (asymmetric) or two-way (symmetric) flow of information. We assume throughout the paper that a link $g_{i,j} = 1$ allows both players to access each other's information. The set of strategies of player i is denoted by G_i. Throughout the paper we restrict our attention to pure strategies. Since player i has the option of forming or not forming a link with each player of the remaining $n-1$ players, the number of strategies of player i is clearly $|G_i| = 2^{n-1}$. The set $G = G_1 \times ... \times G_n$ is the space of pure strategies of all the players.

A strategy profile $g = (g_1, ..., g_n)$ can be represented as a direct network. Let $g \in G$. To describe information flows, it is useful to define the closure of g : this is a non-directed network denoted $\overline{g}$ =cl(g) , and define by $\overline{g}_{i,j} = \max\{g_{i,j}, g_{j,i}\}$ for each i and j in N.[2] Pictorially, the closure of a network simply means replacing every directed edge of g by a non-directed one. We say there is a path in g between i and j if either $\overline{g}_{i,j} = 1$ or there exists players $j_1, ..., j_m$ distinct from each other and i and j such that $\left\{\overline{g}_{i,j_1} = ... = \overline{g}_{j_m,j} = 1\right\}$. We write $i \overset{\overline{g}}{\longleftrightarrow} j$ to indicate a path between i and j in g. Furthermore, a path between i and j is said to be $i-$ *oriented* if either $g_{i,j} = 1$ or there is a sequence of distinct players $i_1, i_2, ..., i_n$ with the property that: $\{g_{i,i_1} = g_{i_1,i_2} = 1, ..., g_{i_n,j} = 1\}$. Define $N^d(i;g) = \{k \in N \,|g_{i,k} = 1\}$ as the set of players with whom i maintains a link and let $\mu_i^d(g) = \left|N^d(i;g)\right|$ be the cardinality of the set. The set $N(i;\overline{g}) = \left\{k \in N \,\middle|\, i \overset{\overline{g}}{\longleftrightarrow} k\right\} \cup \{i\}$ consists of players that i observes in g, while $\mu_i(g) = |N(i;\overline{g})|$ is its cardinality. To complete the definition of a normal-form game of network formation, we specify the payoffs. Let $V_{i,j}$ denote the benefits that player i derives form accessing player j. Similarly, let $c_{i,j}$ denote the cost for player i of forming a link with player j. The payoff function can now be stated as follows:

$$\Pi_i(g) = \sum_{j \in N(i;\overline{g})} V_{i,j} - \sum_{j \in N^d(i;g)} c_{i,j} \tag{1}$$

Throughout we shall assume that $c_{i,j} > 0$ and $V_{i,j} > 0$ for all i, j in N.

Given a network $g \in G$, let g_{-i} denote the network obtained when all of player i's links are removed. Note that the network g_{-i} can be regarded as the strategy profile where i chooses not to form a link with anyone. The network g can be written as $g = g_i \otimes g_{-i}$ where the '$\otimes$' indicates that g is formed as the union of the links in g_i and g_{-i}. The strategy g_i is said to be a *best response* of player i to g_{-i} if:

$$\Pi_i(g_i \otimes g_{-i}) \geqslant \Pi_i(g_i' \otimes g_{-i}) \text{ for all } g_i' \in G_i.$$

The set of all of player $i's$ best responses to g_{-i} is denoted by $BR_i(g_{-i})$. Furthermore, a network $g = (g_1, ..., g_n)$ is said to be a *Nash network* if $g_i \in BR_i(g_{-i})$ for each i, i.e. players are playing a Nash equilibrium. A *strict* Nash network is one where each player gets a strictly higher payoff with his current strategy than he would with any other strategy.

Finally, in order to analyze the efficient architectures we need to introduce a welfare measure. As in Bala and Goyal we define the social welfare of a network

[2] Note that $\overline{g}_{i,j} = \overline{g}_{j,i}$ so that the order of players is irrelevant.

g as the sum of payoffs of all players. Formally, given a network g, its welfare, $W : G \rightarrow R$, can be stated as follows:

$$W(g) = \sum_{i=1}^{n} \Pi_i(g) \text{ for } g \in G. \tag{2}$$

A network is said to be efficient if $W(g) \geqslant W(g')$ for any $g' \in G$. Hence, an efficient architecture can be seen as the one that minimize the cost of providing a certain amount of information to the players.

3 General Model

We begin our analysis with some results that outline the scope of our study. In our analysis, we shall use the idea of minimal networks. Given a network g, we define a component of g, $C(g)$, a set $C(g) \subset N$ such that $\forall i, j \in C(g)$ there exists a path between them and there does not exist a path between $\forall i \in C(g)$ and an player $k \in N \diagdown C(g)$. Given a network g, let $\eta(g)$ be the number of components in g. A network g is said to be minimal if $\eta(g) < \eta(g - g_{i,j})$, $\forall$ $i \neq j$. Moreover a network g is said to be connected if it is composed by only one component, i.e. $\eta(g) = 1$. If this component is minimal, then g is said to be minimal connected. It follows that each link in a minimal connected network is critical in the way that it is enough to delete it, ceteris paribus, to induce some degree of social isolation in the society.[3]

Our first result shows a correspondence between the set of minimal networks and the set of strict Nash equilibrium networks.

Proposition 3.1: *Let the payoffs satisfy (1). Then a strict Nash network is minimal. Given any minimal network g there exist costs and benefits, $\{c_{i,j}, V\}$, such that this network is a strict Nash.*

Proof. We first show that an equilibrium network is minimal. Let g be a Nash network, and suppose that it is not minimal. Then there is a link $g_{i,j} = 1$ such that $N(i; g) = N(i; g - g_{i,j})$, for all $i \in N$. Given the specification of payoffs, and the assumption that $c_{i,j} > 0$, for all i, j, player i can strictly increase his payoff by deleting the link. This violates the assumption that g is Nash.

We now prove the converse. Fix some minimal network g. For any link $g_{i,j} = 1$, set the corresponding cost $c_{i,j} = \epsilon < V$, while for any link $g_{i,j} = 0$, set

[3]Social isolation stands for a situation where the social architecture does not allow each player to observe the whole society.

the corresponding $c_{i,j} > (n-1)V$. The optimality of forming the existing links follows form the cost restrictions and the fact that the network g is minimal. The optimality of not forming the link follows directly form the assumption on the costs. ■

Proposition 3.1 motivates an examination of conditions under which we can derive some restrictions on the strict Nash networks, apart from minimality. We note that in the second part of the result, we show that any minimal network can be sustained in a setting where values are homogeneous. A comparison of this result with the earlier results of Bala and Goyal for the homogeneous players setting suggests that costs homogeneity plays a crucial role in restricting the architecture of equilibrium networks. To get a clearer idea of this issue, we now analyze the case where the cost of link formation is homogeneous across links for any particular individual i.e. $c_{i,j} = c_i$ for all $j \in N \setminus \{i\}$, but it may vary across individuals, $c_i \neq c_j$ is allowed. In addition we allow values to vary freely.

Proposition 3.2: *Suppose that for each $i \in N$, $c_{i,j} = c_i$, for every $j \in N \setminus \{i\}$. Let g be a strict Nash network and suppose that $C(g)$ is a component in g, with $|C(g)| \geqslant 3$. Then $C(g)$ is a center-sponsored star. Let g be a minimal network in which every component with 3 or more players is a center-sponsored star. Then there exist costs and benefits $\{c_i, V_{i,j}\}$, such that this network is strict Nash.*

Proof. We start with the first part. Let players i, j and k belong to $C(g)$. Suppose $g_{i,j} = 1$. We claim that player j cannot have any other link. Suppose not and let $\overline{g}_{j,k} = 1$. Since $c_{i,j} = c_{i,k}$ it then follows form the payoffs (1) that player i is indifferent between forming a link with players j or k. This contradicts strictness of equilibrium. Since $C(g)$ is a component, it follows that player j accesses everyone in $C(g)$ via the link $g_{i,j}$. By analogous reasoning we can infer that no player k forms a link with player i. Hence, player i must form all links and must be the center of a star. This implies that the component must be a center-sponsored star.

We prove the second part now. Fix some minimal network g with the said properties. Let there be m components in this network, $C_1(g), ..., C_m(g)$. Fix some player i and without loss of generality, let $i \in C_1(g)$. For any link $g_{i,j} = 1$, set the corresponding returns $c_{i,j} = c_i < V_{i,j}$, while for every component $C_k(g)$, $k = 2, ..., m$, and any player $j \in C_k(g)$, let $\sum_{j \in C_k(g)} V_{i,j} < c_i$. The optimality of forming the existing links follows from the cost restrictions and the fact that the network g is minimal. The optimality of not forming the link follows directly from the assumption on the costs and benefits. Since i was arbitrary, the proof follows. ■

This result shows that the center-sponsored star architecture plays a prominent role even in the presence of heterogeneous values and differences in cost of forming links across players. A comparison of this result with the earlier result, Proposition 3.1, also suggests that the assumption $c_{i,j} = c_i$, $\forall\ j$, plays a critical role in the analysis. In section 4 we shall explore further the role of heterogeneity in costs of forming links.

The above proposition shows that value and cost heterogeneity permit the existence of more than one non-singleton component in an equilibrium. We now show that a slightly stronger restriction on valuations, $V_{j,i} = V_i$, for all j, implies that this is no longer possible in equilibrium.

Proposition 3.3: *Suppose that for any player i, $c_{i,j} = c_i$, and $V_{j,i} = V_i$ for all $j \neq i$. Then every strict Nash network is minimal and has at most one non-singleton component (which is a center sponsored star). Moreover, any minimal network with these properties is sustainable as a strict Nash network for some value of $\{c_i, V_i\}$.*

Proof. Suppose that g is a strict Nash network and $C_1(g)$ and $C_2(g)$ are two non-singleton components. From earlier results we know that the network is minimal and that in each of the two components only one player forms all the links. Let $i \in C_1(g)$ and let $j \in C_2(g)$ be these players. Since g is strict Nash network, it follows that $\Pi_i(g) > \Pi_i(g - g_{i,k})$, with player $k \in C_1(g)$. In other words,

$$V_s - c_i > 0 \ \forall s \in C_1(g).$$

However, the net payoff to player j from forming a link with player i is given by:

$$\sum_{s \in C_1(g)} V_s - c_j < 0$$

Since player j does not form this link. It must be the case that $c_i < c_j$. Likewise, we can now reason that:

$$V_t - c_j > 0,\ \forall t \in C_2(g)$$

However, the net payoff to player i from forming a link with player j is given by:

$$\sum_{t \in C_2(g)} V_t - c_i < 0$$

Since player i does not form this link, it must be the case that $c_i > c_j$. This leads to a contradiction. Hence, there can be at most one non-singleton component in a strict Nash network.

We now prove that every such network can be sustained as a strict Nash network. The proof is by construction. Take any such network: minimal, with one non-singleton component, where only one player forms all the links. Suppose that $C_1(g)$ is the non-singleton component, and that player $i \in C_1(g)$ forms all the links. Then this network is strict Nash for the following cost/value parameters: (i) $c_i < V_j$, $\forall j \in C_1(g)$; $c_i > V_k$, $\forall k \notin C_1(g)$; (ii) $c_k > \sum_{l \neq k} V_l$. ∎

The results in this section demonstrate several points. *First*, that if we allow for values and costs to vary freely then the only restriction imposed by the equilibrium requirement is minimality. Minimality is a direct consequence of the assumption that the distance between players does not affect the transmission of value. *Second*, if we allow for value heterogeneity but only a moderate amount of cost heterogeneity, then equilibrium has considerable bite. In particular, if any individual i's costs of forming links are the same, i.e. if $c_{i,j} = c_i$, for all $j \in N \setminus \{i\}$, then any equilibrium network is either a center-sponsored star or comprises of smaller center-sponsored stars. Thus, the results demonstrate that the center-sponsored stars continue to be prominent even in settings with considerable value and cost heterogeneity across individuals as long as a player's costs of forming links is independent of the identity of the player being connected to. However, little can be said of equilibrium networks if costs of link formation differs across links for the same individual. The next section presents a model with heterogeneous costs where individual incentives do have strong implications.

4 An insider-outsider model

We consider a society composed by m groups each of them of size $|N_l|$, $l = 1, ..., m$. The set of players playing the network formation game is then defined by $N \equiv \cup_{l=1}^{m} N_l$. We assume perfect symmetry in value across individuals and we normalize it to one, i.e. $V_{i,j} = 1$ for all $i, j \in N$. To allow for cost heterogeneity we consider a spatial cost structure: groups can be ordered in a line according to some well defined characteristics. Hence, the distance between two groups can be interpreted as a measure of the heterogeneity that distinguishes them. Given two players $i \in N_l$ and $j \in N_k$, the cost of forming a link $g_{i,j}$, is:

$$c_{i,j} = c_{j,i} = f(|l - k|) \qquad (3)$$

If i and j belong to the same group we let:

$$c_{i,j} = c_{j,i} = f(0) = c_L$$

Throughout the paper we assume that $f(\cdot)$ is non-decreasing in its argument and $c_L > 0$.

In order to specify the class of payoff functions of this normal-form game of network formation, it is useful to consider the set introduced before, $N^d(i; g)$, as the union of the sub-sets: $N^{d,k}(i;g) = \{j \in N_k \,|g_{i,j} = 1 \ \& \ c_{i,j} = f(|l,k|)\}$, defined for any $k = 1,...,m$, *i.e.* $N^d(i;g) \equiv \cup_{k=1}^m N^{d,k}(i;g)$. Furthermore, let $\mu_i^{d,k}(g)$ be the cardinality of $N^{d,k}(i;g)$. In other words, $\mu_i^{d,k}(g)$ represents the number of links initiated by i with members of group k. Hence, given a network g and an player $i \in N_l$, the payoff function described by (1) can be rewritten as follows:

$$\Pi_i(g) = \mu_i(g) - \sum_{k=1}^{m} \mu_i^{d,k} f(|l,k|) \tag{4}$$

We note two interesting special cases of our specification.

1. Homogeneous Players: This case arises when $f(0) = f(1) = ... = f(m-1) = c$. This implies that player i's payoff is the number of players he observes less the total cost of link formation. Clearly, the distinction between inside and outside links becomes irrelevant and we can consider that the whole society is composed by only one group. In this case, the payoff is given by:

$$\Pi_i(g) = \mu_i(g) - \mu_i^d(g)\, c$$

2. Two-cost levels: The case of two-cost levels arises when we assume that $f(d) = c_H$, $\forall d \geqslant 2$, and $f(0) = c_L < c_H$. Thus, we can rewrite the cost structure as follows:

$$c_{i,j} = \begin{cases} c_L, & \text{if } i,j \in N_l \\ c_H, & \text{if } i \in N_l \text{ and } j \in N_k,\ l \neq k \end{cases} \tag{5}$$

In words, the cost of creating a link across groups, c_H (outside link), is equal or higher than the cost of creating a link within a group, c_L (inside link). However, links formed with different external groups are equally costly. The two-cost levels case will be discussed below to illustrate some of our results.

We extend the former notation. We say that $i,j \in N_l$ are entirely internally linked if either $\overline{g}_{i,j} = 1$ or there is a path between i and j in which all players belong to N_l. We say that $i,j \in N_l$ are externally linked if they are not entirely internally linked. Moreover, let the diameter of a non-singleton component, $C(g)$, be defined as the length of the largest geodesic between any pair of players belonging to it, *i.e.* $D(C(g)) = \max_{i,j \in C(g)} d(i,j;C(g))$.[4] To complete the notation, in the characterization of strict Nash networks we will deal with the class of generalized center-sponsored star architectures. More specifically, a generalized center-sponsored star architecture is a minimally connected network which satisfies the following conditions: (a) $\exists i \in N_l$ *such that* $g_{i,j} = 1$, $\forall j \in N_l \diagdown \{i\}$; (b) Let $i \in N_l$ be the player identified in (a), then any $i \overset{\overline{g}}{\longleftrightarrow} j$, $\forall j \in N \diagdown \{i\}$ is an $i-$*oriented* path; (c) given $g_{j,j'} = 1$, $j \in N_{l'}$ and $j' \in N_{l''}$, $g_{j',j''} = 1$ only if $j'' \in N_k$, where k is such that $|k - l'| > |l' - l''|$; (d) $D(g) \leq 2m$.

[4]Given two players i and j in g, the geodistic distant, $d(i,j;g)$, is defined as the length of the shortest path between them.

4.1 Equilibrium Networks

Our first result describes Nash equilibrium networks.

Proposition 4.1: *Let the cost structure be given by (3) and let the payoffs be given by (4). A Nash network is minimal. In particular, depending on the cost levels, a Nash network can be empty, connected or unconnected where any non-singleton component consists of members of one group, respectively.*

The proofs of all the results stated in this subsection are given in the Appendix. We briefly sketch the intuition here. First we note that a Nash networks are minimal. This follows from the no-decay assumption. Next, we note that in the presence of heterogeneity in cost levels, partially connected networks (with each component being composed of members belonging to the same group) can be sustained in equilibrium. There are two reasons for this: a coordination problem and an incentive problem. First, suppose the inside cost is slightly higher than 1 and the outside cost, $f(1)$, is a bit higher then the size of the smallest group, say N_l. Consider a network where the smallest group is internally linked and all other players are singletons. It is easy to see that such a network can be sustained as Nash equilibrium. However, if the largest group is able to coordinate by generating a minimal connected component, it will create the right incentives to achieve a connected network, and therefore a partially connected structure is not sustainable anymore. Next, consider the case where the inside cost is low, $c_L \in (0,1)$, such that each group will have an incentive to be internally linked. Moreover, consider $f(1)$ sufficiently high, then a (minimally) connected network cannot be sustained as a Nash network. In this situation a partially connected network with m minimal components is the only Nash equilibrium. The above result suggests that a wide range of networks can arise in a Nash equilibrium. Earlier work suggests that strictness has considerable bite in homogeneous player settings. Is this also true in a setting with heterogeneous players? Our next result provides a complete response to this question.

Proposition 4.2: *Assume that* $|N_l| \geqslant 2, \forall l = 1, ..., m$. *Let the cost structure be given by (3) and let the payoffs be given by (4):*

1) *If* $c_L > 1$ *then the only strict Nash network is the empty network.*

2) *Suppose* $c_L \in (0,1)$, *then there are three cases:* 2*a*) *if* $f(1) \in (c_L, 1)$, *then a strict Nash network is a generalized center-sponsored star.* 2*b*) *If* $f(1) \in (1, \max[N_1, .., N_m])$, *then a strict Nash network does not exist.* 2*c*) *If* $f(1) > \max[N_1, .., N_m]$, *then the only strict Nash network is partially connected with each group constituting a center-sponsored star.*

The *first* step of the proof consists of showing that in each non-singleton component there exists one group that is entirely internally linked. We start proving that in each non-singleton component there exists at least one inside link. Suppose g is a strict Nash network. For simplicity assume that it is connected; then there is a path between any two players belonging to the same group, say i,$i' \in N_l$. There are two possible path configurations. First, the two players are directly linked and if this is the case the claim follows. Second, the two players access each other indirectly, through other players. In this case, it can be shown by an application of the switching argument that this path has to have the following pattern of links: $\{g_{j,i} = 1, ..., g_{j',i'} = 1\}$. Next we note that the same property must also hold for any other pair of players belonging to the same group. Since the number of groups is finite and each group is composed by at least two players, we can iterate in respect to j the just mentioned argument ending-up in a situation where there always exist two players belonging to the same group accessing each other via a direct connection. Hence there exists a pair of players of some group that connects directly. We then use network externality effects to argue that if two players of a group are directly linked then all members of this group must belong to the same component. Finally, we use the switching argument to show that given an inside link, i.e. $g_{i,i'} = 1$ with $i, i' \in N_l$, i will bear all the links with members of his own group. Hence, group N_l is entirely internally linked.

The *second* step in the proof shows that if a group is not entirely internally linked then it is entirely externally linked. Consider a connected strict Nash network. Let N_l be the group highlighted in the previous step and let i be the center of this group. Consider a path between i and an arbitrary player j. Using a variant of the argument sketched above, we show that the path must be $i - oriented$. Now it is easy to see that in any path leading away from player i, there can be at most one player of any specific group. Hence it follows that if we take a pair of players in a group $l' \neq l$ there exists a path and along this path there is no player of group l'. Thus all groups apart from l are entirely externally linked.

The *final* step in the proof consists of combining the above observations for different cost parameters.

We discuss some aspects of this characterization result. The *first* remark is about insider and outsider links. Our result shows that there is one group which is entirely internally linked in the connected strict Nash network, while all other groups are entirely externally linked. In other words, the formation of local connections is not allowed in equilibrium. This is an unexpected result and it suggests that incentives for link formation completely undermine the structure that one might have expected: a set of local center-sponsored stars (corresponding to individual groups) linked with each other. The *second* observation concerns with the centrality and center-sponsorship properties. If the strict Nash network is connected, there is a player i such that all paths are oriented toward him. Hence, this player plays a particularly central role in the network. Furthermore, if the strict Nash network is non-empty but unconnected, then each component consists of members of one group and it has the a center-sponsored star structure. *Third*, it is worth noting that the diameter of connected strict Nash network is independent of the number of players, while it only depends on the number of groups. *Fourth*, we consider the two special cases introduced in the specification of the insider-outsider model. When applying Proposition 4.2 to the homogeneous case we obtain the result provided by Bala and Goyal (2000): if $c > 1$ the only strict Nash network is the empty one, while if $c_L \in (0,1)$ then the only strict Nash network is a center-sponsored star. Let's now turn to the two-cost levels case. When $c_H \in (c_L, 1)$, a strict Nash network has a generalized center-sponsored star architecture. More formally, there is an individual, say $i \in N_l$ which is the center of the whole network: each path in the network is oriented to him. Furthermore, group N_l is the only group to be entirely internally linked. Therefore, the members of all the remaining groups will be passively linked with some members belonging to group N_l. In particular, if all the remaining players are passively linked with player i, then the network is a center-sponsored star (see Figure 1D).

Finally, we remark on the assumption that there are at least two members in each group. If we relax this assumption and allow for some groups to have only one member then two substantial changes occur. The first change is that there may exist more than one entirely internally linked group while the second change is that the non-existence result may be ruled out. The following example illustrates these points. Consider a society composed by three groups, where group N_1 and N_3 are composed by two players and group N_2 by only one player. Let g be a connected network depicted in Figure 1E. When $f(1) \in (c_L, 1)$, g is strict Nash. We note that in g all groups are entirely internally linked. Now, suppose that $f(1) = 1 + \epsilon$, where ϵ is positive and small enough. Again, the network g is strict Nash. However, if we assume that also group N_2 is composed by more than one player, a standard switching argument leads to the non-existence result.

4.2 Efficient Networks

We now turn into the issue of efficiency. We first introduce some new terminology that will be used in the proposition below. Let g^{mc} refer to a minimal connected network with each group N_l forming a minimal connected component with $N_l - 1$ inside links respectively and with $(m-1)$ outside links of distance one (see Figure 1C). A network g is said to be partially connected, g^{pc}, if there exists at least one non-singleton component but the network is not connected. Finally, a partially connected network with each group generating a minimal connected component will be denoted as g_m^{pc} (see Figure 1A).

In the further analysis we assume that groups are of equal size. We develop a simple example in order to illustrate how relaxing this assumption can lead to a variety of efficient networks. For sake of clearness, we consider the of two-cost levels case. Let the society be composed by three groups where group N_1 is small while groups N_2 and N_3 are large. Suppose now that $c_L \in (0,1)$ and $c_H < 2N_2N_3$, then an efficient network must have the three groups internally linked and group N_2 and N_3 connected by one outside link. However, if $c_H \in (2N_1(N_2+N_3), 2N_2N_3)$ then it is socially efficient to leave group N_1 isolated. Therefore, the efficient network is one in which the three groups are linked internally and where group N_2 and N_3 are connected by one outside link while group N_1 is left out. Clearly, if $c_L \in (0,1)$ and $c_H < 2N_1(N_2+N_3)$ the efficient network is minimal connected with $m-1$ outside links, while if $c_H > 2N_2N_3$ then only a partially connected network where the three groups are linked internally is efficient. Finally for $c_L > \max\{N_1, N_2, N_3\}$ and c_H sufficiently high the only efficient network is the empty one.

The proposition below shows that when assuming equal group size a partially connected efficient network is characterized by having each group internally linked and no outside links.

Proposition 4.3: *Let the cost structure be given by (3) and let the payoffs be given by (4). Suppose* $N_l = \overline{N}$, $\forall l = 1,..,m$:

1) *There exists* $c = m\overline{N}^2$ *such that if* $c_L \in (0, \overline{N}]$ *and* $c_H \in (c_L, c]$ *the network* g^{mc} *is the only efficient one, while if* $c_L \in (0, \overline{N}]$ *and* $c_H > c$ *the network* g^{pc}_m *is the only efficient one.*

2) *There exists* $c = \left[m\overline{N}\left(m\overline{N}-1\right) - \left(m\overline{N}-m\right)c_L\right](m-1)$ *such that if* $c_L \in (\overline{N}, m\overline{N})$ *and* $c_H \in (c_L, c]$ *the network* g^{mc} *is the only efficient one, while if* $c_L \in (\overline{N}, m\overline{N})$ *and* $c_H > c$ *the empty network is the only efficient one.*

3) *If* $c_L \geqslant m\overline{N}$ *then the empty network is the only efficient one.*

The proof is presented in the Appendix. We briefly sketch the intuition here. As in the homogeneous case, an efficient network is minimal. When c_L is high enough the empty network is efficient for a social point of view; when c_L is relatively low it is beneficial for the society to have each group internally linked. Considerations on $f(1)$ allow us to divide this cost space into two sub-spaces: for $f(1)$ high enough the society is better-off leaving each group isolated by the others, g^{pc}_m, while if c_H is no so high then the minimal connected network arises. However, only a minimal connected network characterized by having a minimal number of outside links $(m-1)$ and all of length one, is efficient, g^{mc}. This last requirement arises by the necessity of minimizing the cost of heterogeneity in the model, given the gross surplus produced by connected structures.[5]

The characterization of efficient networks allows us to state some remarks on the trade-off between efficient and sustainable architectures.[6] We have showed that in each cost space where g^{mc} is efficient, the corresponding set of strict Nash networks does not contain any architectures compatible with the efficient one. This conflict persists until when the level of $f(1)$ is such that any outside link is not beneficial both from an individual and social point of view. When this is the case, the heterogeneity introduced in the model becomes irrelevant and our problem degenerates in a sum of independent homogeneous problems leading to partially connected strict Nash network with each group generating a center-sponsored star component. It follows that the trade-off between efficiency and stability fades.

The conflict between efficient and sustainable connected architectures arises out of a misallocation of links: too many outside links are set-up in order to obtain connectedness. Consider a connected network g and pick two players belonging to a group different from the core, then if g is strict Nash, they will access each other via a sequence of outside links. This does not allow network participants to minimize the costs of connecting with each other and this lowers social welfare.

This result is mitigated if we relax the assumption, used in the characterization of strict Nash networks, that each group is composed of at least two players, *i.e.* $|N_l| \geqslant 2$, $\forall l = 1, ..., m$. Consider a society composed by three groups where groups N_1 and N_3 are composed of two individuals and group N_1 of a single individual player. Suppose $f(1) \in (c_L, 1)$. The network depicted in Figure 1E is strict Nash. Moreover, this network satisfies all the necessary conditions for a connected network to be efficient: the allocation of links is optimal from a societal point of view. In general, the presence of a single player between two heterogeneous groups composed by at least two individuals mitigates substantially the conflict between the notion of efficiency and strategic stability.

5 Conclusion

In this paper we have investigated the implications of heterogeneity for network formation. We have used an extension of the Bala and Goyal (2000) model of one-sided link formation to study this issue. In the general setting, if costs and values are allowed to vary freely there exists a correspondence between

[5]Each minimal connected network produces the same gross social welfare but different minimal connected networks will be characterized by a different total cost depending on the allocation of links.

[6]It is worth noting that the characterization of strict Nash networks presented in Proposition 4.2 is neutral to the assumption of equal group size. This allows us to compare the set of strict Nash networks with the set of efficient ones.

minimal networks and equilibrium networks. Given that minimality follows directly from the no-decay assumption, this result shows that the requirement of equilibrium imposes no additional restriction on networks. This motivates a closer examination of the role of value and cost heterogeneity.

We find that if costs of forming links are similar across links of any particular individual and values vary freely then individual incentives do matter: any equilibrium network is a center-sponsored star or a collection of such structures. However, we find that any minimal network can be sustained in equilibrium if values are identical but costs forming links are allowed to vary freely.

This motivates a study of a model of restricted cost-heterogeneity: an insider-outsider model. In this model there are several spatially organized groups and the cost of linking depends on the distance between the groups. In this setting, we show that strict Nash networks can be partially connected as well as connected. A partially connected equilibrium network is a collection of m center-sponsored stars each star constituted of members of one group. If the equilibrium network is connected then it has a generalized center-sponsored star architecture with the following features: First, there exists one *core* group that is entirely internally linked via a central player, while all the members belonging to the remaining groups are entirely externally linked and thus each of these groups is completely fragmented. Second, each path is oriented to the central player. Third, the maximum diameter of these networks depends only on the number of groups belonging to the society. In the connected equilibrium network all but one group is linked via expensive external links. Hence, strategic incentives are in sharp conflict with what is socially desirable.

Appendix:

Proof of Proposition 4.1. Minimality follows as a direct consequence of the no-decay assumption.

First, we observe that the empty network is Nash if and only if $c_L \geqslant 1$. The proof of this is straightforward and omitted.

Next, we argue that if $c_L < 1$ and $f(1) < \max\{N_1, ..., N_m\}$, then a Nash network is connected. Since $c_L < 1$ there must exist a path between any pair of players in the same group N_l. Moreover, without loss of generality, let $\overline{N}_M$ be the largest group, i.e. $\overline{N}_M = \arg\max\{N_1, .., N_m\}$, since $f(1) < \overline{N}_M$, it follows that each player in all groups other than $\overline{N}_M$ have an incentive to form a link with group $\overline{N}_M$. Hence, a Nash network must involve a path between any players $i \in N_l$ and some $j \in \overline{N}_M$. Then the network is connected.

Finally we note that if $c_L < 1$ but $f(1) > \overline{N}_M = \max\{N_1, ..., N_m\}$ then no player $i \in N_l$ has an incentive to form a link with $j \in N_{l'}$, $l \neq l'$, so long as $N_{l'}$ constitutes a component by itself. Hence, a Nash network has m components, each component consisting of members of one group, respectively. ■

Proof of Proposition 4.2. We recall some definitions that will be used further in the proof. Consider a network g, a path between i and j is said to be $i-$*oriented* if either $g_{i,j} = 1$ or there is a sequence of distinct players $i_1, i_2, ..., i_n$ with the property that: $\{g_{i,i_1} = g_{i_1,i_2} = 1, ..., g_{i_n,j} = 1\}$.

First we introduce Lemmas $1-6$ and then we proceed with the proof of the proposition.

Lemma 1 *Suppose g is a strict Nash network. If $i \in N_l$ and $j \in N_{l'}$, $l \neq l'$, and $g_{i,j} = 1$, then $\overline{g}_{j,j'} = 0$, $\forall j' \in N_k \subset N \setminus \{N_l\}$, where k is such that $|l-k| \leq |l-l'|$.*

Proof. Consider a strict Nash network g. Choose $i \in N_l$ and $j \in N_{l'}$, $l \neq l'$, such that $g_{i,j} = 1$. Let $j' \in N_k \subset N \setminus \{N_l\}$, where k is such that $|l-k| \leq |l-l'|$. For seeking of contradiction, assume $\overline{g}_{j,j'} = 1$. The spatial cost structure implies that i can do at least as well by deleting his link with j and forming a link with j'. This contradicts strict Nash. ■

Lemma 2 *Suppose g is a strict Nash network. If $i \in N_l$ and $j \in N_{l'}$, $l \neq l'$, and $g_{i,j} = 1$, then $g_{j',i} = 0$ $\forall j' \in N_k$ such that $|k-l| \geq |k-l'|$.*

Proof. For the sake of contradiction, suppose $g_{j',i} = 1$. Since the cost of forming links is non-decreasing in its argument, j' can do at least as well by deleting his link with i and forming a link with j. This contradicts strict Nash. ■

Lemma 3 *Assume $|N_l| \geqslant 2$, $\forall\, l = 1, ..., m$. Suppose g is a non empty strict Nash network, then in each non-singleton component there always exists $i, i' \in N_l$ such that $g_{i,i'} = 1$.*

Proof. Consider an arbitrary non-singleton component $C(g)$. There exists $g_{i,j} = 1$, $i \in N_l$ and $j \in N \diagdown \{i\}$. We have two possible configurations. First, suppose $j \in N_l \diagdown \{i\}$. Then the proof follows. Next, suppose $j \in N_{l'}$, $l \neq l'$. We first note that, given $g_{i,j} = 1$, $C(g)$ must contain all the members of group N_l, *i.e.* $N_l \subset C(g)$. Suppose not; that is there exists $i' \in N_l$ such that $i' \notin C(g)$. Since g is strict Nash, given $g_{i,j} = 1$, it follows that $f(|l - l'|) < |C(g - g_{i,j})|$. Since the cost structure is non-decreasing in its argument, it follows that i' is strictly better-off creating a link with player i. Therefore $i \in N_l$ must access any $i' \in N_l \diagdown \{i\}$. There are three possibilities through which i can access i'. First and trivially, the two players may be directly linked and then the proof follows. Second, given $g_{i,j} = 1$, i accesses i' via j. This violates Lemma 1 and therefore this configuration is not sustainable in a strict Nash network. The last possibility is that i accesses i' via an player j', where $g_{j',i} = 1$. Given $g_{i,j} = 1$, Lemma 2 implies that the link $g_{j',i} = 1$ is sustainable in a strict Nash network, only if j' belongs to a new group, *i.e.* j' needs to belong to a group that is not accessed by i before the link $g_{j',i} = 1$ has been formed. If this is the case, then an analogous reasoning provided above implies that j' must access in $C(g)$ any members belonging to his own group, say j''. Again Lemma 1 and Lemma 2 implies that j' accesses any j'' either by being directly linked, and if this is the case the proof trivially follows, or by being passively linked with an player belonging to some group that is not accessed by j'. Since the number of groups is discrete and finite, we can iterate the argument ending-up in a situation where there always exist two players of the same group accessing each other by being directly linked. Hence, the proof follows. ■

Lemma 4 *Assume* $|N_l| \geqslant 2$, $\forall\ l = 1, ..., m$. *Suppose* g *is a non-empty strict Nash network. If* $g_{i,i'} = 1$, $i, i' \in N_l$, *then* $g_{i,i''} = 1$, $\forall i'' \in N_l \diagdown \{i\}$.

Proof. Consider a non-singleton component, $C(g)$. First we note that there always exists $i, i' \in N_l$ such that $g_{i,i'} = 1$ where $i, i' \in C(g)$. This follows by Lemma 3. Furthermore, reasoning used in the proof of Lemma 3, implies that, given $g_{i,i'} = 1$, then $N_l \subset C(g)$. Assume, for seeking of contradiction, that there exists $i'' \in N_l$ such that $g_{i,i''} = 0$. We first note that, given $g_{i,i'} = 1$, then $g_{i'',i} = 0$, $\forall\ i'' \in N_l \diagdown \{i\}$. This follows from the standard switching argument. Hence, $\overline{g}_{i,i''} = 0$. Thus we have two possible configurations. First, suppose that $N_l \equiv C(g)$. Then it must be the case that $g_{i,i''} = 1$. This contradicts our hypothesis that $g_{i,i''} = 0$. Second, suppose $N_l \subsetneqq C(g)$. Since $C(g)$ is connected, there is a path between i and i''. Without loss of generality, suppose $j \in N \diagdown \{N_l\}$ lies along this path and $\overline{g}_{i,j} = 1$. If $g_{j,i} = 1$, then j is indifferent between linking with i and i' . Hence g is not strict Nash. Next, if $g_{i,j} = 1$ then i is strictly better-off by deleting the link with j and creating a new link with i''. Hence g is not Nash. Thus $\overline{g}_{i,j} = 0$ which contradicts our hypothesis that i can access i'' via j. ■

Lemma 5 *Assume* $|N_l| \geqslant 2$, $\forall\ l = 1, ..., m$. *Suppose* g *is a connected strict*

Nash network and let $i \in N_l$ be the player identified by Lemma 4. Then any path $i \overset{\overline{g}}{\longleftrightarrow} j$, $\forall j \in N \diagdown \{i\}$, is $i-oriented$.

Proof. Let $i \in N_l$ be the player such that $g_{i,i'} = 1$, $\forall i' \in N_l \diagdown \{i\}$. Since g is connected i will access $\forall j \in N \diagdown \{i\}$ via g. Suppose, for seeking of contradiction that $i \overset{\overline{g}}{\longleftrightarrow} j$ is not an inward path. Consider two players h and z such that both lie in the path $i \overset{g}{\longleftrightarrow} j$ and z access i via h. Assume that $g_{z,h} = 1$. Since by assumption $\forall N_l$, $|N_l| \geq 2$ and since g is connected, there exists h' belonging to the same group of h and he will access h via g. We have two possible configurations. First, z does not lie in the path between h and h'. If this is the case, given $g_{z,h} = 1$, a standard switching argument implies that z is indifferent between h and h'. Hence, the only possibility is that z lies between h and h'. We first note that this implies that $g_{s,h} = 0$ for any player $s \neq z$. If not, s will be indifferent between h and h'. Furthermore, since h lies in the path $i \overset{\overline{g}}{\longleftrightarrow} j$, either $g_{h,i} = 1$ or there exists another player z' standing in the path between i and h such that $g_{h,z'} = 1$. Consider the first case, since i is the player identified by Lemma 4, h will be indifferent between i and $\forall i' \in N_l \diagdown \{i\}$. This clearly contradicts the strict Nash notion. Consider the second case, $g_{h,z'} = 1$. We can repeat the argument proposed above: since $\forall N_l$, $|N_l| \geq 2$ and since g is connected, there exists z'' belonging to the same group of z' and he will access z' via a path that contains h. Furthermore $g_{s'',z'} = 0$ for any $s'' \neq h$. Since group are finite, we can assume, without loss of generality, that z' is the player closer to i. Then a standard switching argument implies that $g_{i,z'} = 1$ and this contradicts the condition that $g_{s'',z'} = 0$ for any $s'' \neq h$. Since each player has been chosen arbitrarily, the proof follows. ■

Lemma 6 *Assume $|N_l| \geqslant 2$, $\forall\, l = 1, ..., m$. Suppose g is a connected strict Nash network. Then $D(g) \leq 2m$.*

Proof. This follows directly by Lemma 1, 3, 4 and 5 ■

We now complete the proof of Proposition 4.2.

1) Consider a strict Nash network g and suppose $c_L > 1$. We claim that the only strict Nash network is the empty one. Suppose that there exists a non-singleton component $C(g)$. Suppose $C(g)$ is composed only by members belonging to the same group, say N_l. Lemmas 3 and 4 imply that there exists a set of inward end-players. Since $c_L > 1$ an inward end-player cannot be sustained in a strict Nash network. Next, suppose that $C(g)$ is composed by members of different groups. Then, there exists $g_{i,j} = 1$, $i \in N_l$ and $j \in N \diagdown \{N_l\}$. The arguments proposed in Lemma 3 implies that g is a connected network. Therefore, Lemma 5 implies that there exists a set of inward-end players connected to the network through outside links. Since $f(1) \geqslant c_L > 1$, it follows that an inward end-player cannot be sustained in a strict Nash network. Hence, the only empty network can be

strict Nash. Finally, since $1 < c_L \leq f(1)$, it is straightforward to check that g^e is strict Nash

2a) Suppose $c_L \in (0,1)$ and $f(1) \in (c_L, 1)$. Consider a strict Nash network g. It is immediate that a strict Nash network is minimal connected. Lemma 3 and Lemma 4 implies that g satisfies property (a). Since g is minimal connected in these parameter space, Lemma 5 holds leading to property (b). Considering the restrictions imposed by Lemma 1, Property (c) follows by verification. Finally, Lemma 6 implies that the diameter of g satisfies properties (d)

2b) Suppose $c_L \in (0,1)$ and $f(1) \in (1, \max[N_1, .., N_m])$. Consider a strict Nash network g. Nash results imply that if g exists, it is minimal connected. Lemma 5 implies that there exists a set of inward-end players connected to the network through outside links. Since $f(1) > 1$, g cannot be sustained as a strict Nash network, leading to a contradiction. Hence, in this parameter space it follows that a strict Nash network does not exist.

2c) Suppose $c_L \in (0,1)$ and $f(1) > \max[N_1, .., N_m]$. Consider a strict Nash network g. Nash results implies that in this parameter space g is partially connected with each group generating a minimal connected component. Lemma 3 and 4 now implies that each component is a center-sponsored star.

Hence, the proof follows. ■

Proof of Proposition 4.3. In this proposition we assume equal group size, *i.e.* $|N_l| = \overline{N}$ for any $l = 1, ..., m$. We first start with three observations:

$a)$ The no-decay assumption implies that each non-singleton component part of an efficient architecture is minimal;

$b)$ Consider an efficient network g. If g is not empty then it is either minimal connected with $m-1$ outside links of length one and $m\overline{N} - m$ inside links, g^{mc}, or partially connected with each group generating a minimal connected component, g_m^{pc}. This observation follows by the assumption of equal group size and by the definition of efficiency concept. If a link between two members of the same group is socially efficient, then, from a societal point of view, each group should be internally linked, g_m^{pc}. Furthermore, the assumption of equal group sizes implies that each group internally linked contributes equally to the total social welfare produced by the network. It follows that if an outside link is social enhancing, then an efficient network should be minimal connected. Moreover, since the definition of efficiency requires the minimization of the total cost of information flow, a connected efficient network should have $m-1$ outside links of length one.

Using these observations we compare three different architectures:

1) Consider g^{mc}, it will produce a social benefit equal to:

$$W\left(g^{mc}\right) = \left(m\overline{N}\right)^2 - m\left(\overline{N}-1\right)c_L - (m-1)\,f\,(1)$$

2) Consider g_m^{pc}, it will produce a social welfare equals to:

$$W\left(g_m^{pc}\right) = m\left(\overline{N}\right)^2 - m\left(\overline{N}-1\right)c_L$$

3) Consider g^e, it will produce a social welfare equals to:

$$W\left(g^e\right) = m\overline{N}$$

First, we compare g_m^{pc} with g^e. The efficiency definition implies that g_m^{pc} is more efficient than g^e if and only if $W\left(g_m^{pc}\right) \geqslant W\left(g^e\right)$. This happens when $c_L \leq \overline{N}$.

Second, suppose $c_L \in (0, \overline{N}]$ and compare g^{mc} with g_m^{pc}. Simple computations show that $W\left(g^{mc}\right) \geqslant W\left(g^{pc}\right)$ if and only if $f\,(1) \leq m\overline{N}^2 = c_1$. It follows that given $c_L \in (0, \overline{N}]$ if $f\,(1) \in (c_L, c_1]$ the only efficient network is g^{mc}, while if $f\,(1) > c_1$ the only efficient network is g_m^{pc}. This proves point (1).

Third, suppose $c_L > \overline{N}$ and compare g^{mc} with g^e. Again, simple computations show that $W\left(g^{mc}\right) \geqslant W\left(g^e\right)$ if and only if $f\,(1) \leq \frac{m\overline{N}\left(m\overline{N}-1\right)-\left(m\overline{N}-m\right)c_L}{m-1} = c_2$. Since c_2 depends by c_L and since by assumption $f\,(1) > c_L$, we need to impose that $c_2 > c_L$. It is easy to show that when $c_L < m\overline{N}$ then c_2 is at most equal to c_L. It follows that given $c_L \in (\overline{N}, m\overline{N})$, if $f\,(1) \in (c_L, c_2]$ then g^{mc} is the more efficient network, while if $c_H > c_2$ then g^e is the more efficient network. Moreover if $c_L \geqslant m\overline{N}$ then for any value of $f\,(1)$, the empty network is the more efficient one. This proves point (2) and (3). ■

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Figures: 1A-1E

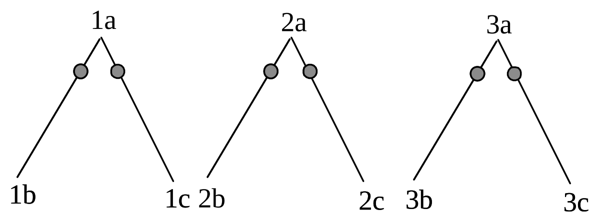

Figure 1A

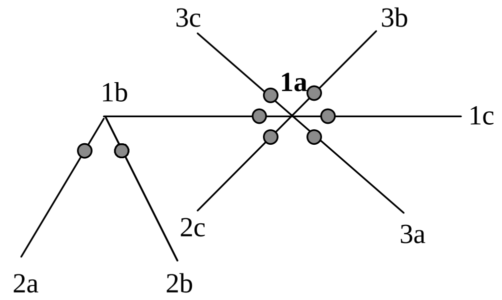

Figure 1B: Generalized Center-Sponsored Star

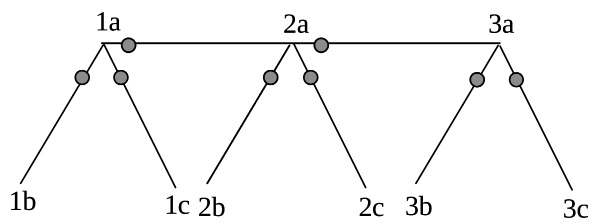

Figure 1C

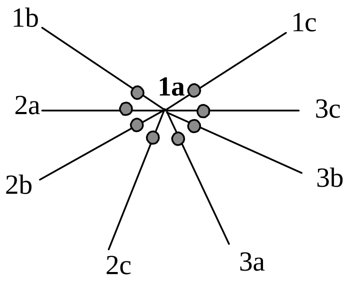

Figure 1D: Center-Sponsored Star

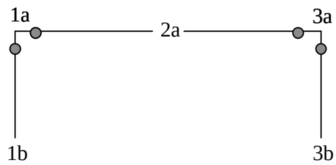

Figure 1E