Imitators and Optimizers in Symmetric n-Firm Cournot Oligopoly

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December, 2001

Abstract

I present a formal model of symmetric n-firm Cournot oligopoly. Instead of assuming a homogeneous population of profit maximizing firms, the set of firms is divided into two subpopulations of imitators and optimizers respectively. Imitators mimic the output decision of the most successful firms of the previous round. Optimizers are myopic best response players to the previous opponents' output. The dynamics of the decision rules induce a Markov chain. As expression of bounded rationality, firms are allowed to make mistakes and deviate from the decision rules with a small probability. Applying stochastic stability analysis I characterize the long run behavior of the oligopoly. It is known that under certain stability conditions a homogeneous population of optimizers converges to a Cournot Nash equilibrium. Vega-Redondo (1997) showed that a homogeneous population of imitators converges to the competitive solution. I find that a heterogeneous population converges to a recurrent set of states where imitators are better off than optimizers. This outcome is reminiscent of Stackelberg behavior where imitators take here the role of "independent" firms and optimizers are "dependent".

JEL-Classifications: C72, D21, D43, L13

Keywords: imitation, myopic best reply, stochastic stability, learning, Stackelberg

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†Financial support by the Bonn Graduate School of Economics and the DAAD is gratefully acknowledged. I wish to thank Joerg Oechssler and Carlos Alós-Ferrer as well as the seminar audience in Bonn, WEHIA-Maastricht, and at the 2001 Far Eastern Econometric Society Meeting in Kobe.

1 Introduction

"For convenience in discussion, suppose that a man finds himself proprietor of a mineral spring which has just been found to possess salutary properties possessed by no other. He could doubtless fix the price of a liter of this water at 100 francs; but he would soon see by the scant demand, that this is not the way to make the most of his property. He will therefore successively reduce the price of the liter to the point which will give him the greatest possible profit."([12] Cournot, 1838, p. 56). What happens if his neighbors impressed by his sudden fortune also tap the spring on their land and supply the market? Why should they not only mimic the commercial idea of the first but also his output or pricing decision?

This introductory discussion à la Cournot outlines the spirit of this paper. Namely, it analyzes the dynamic interaction between imitators and optimizers in a Cournot market. The main idea is to contrast existing results for homogeneous populations of either imitators or optimizers with results on a heterogeneous population of imitators and optimizers. [67] Vega-Redondo (1997) showed that in a quantity setting n-firm symmetric oligopoly with a homogeneous population of imitators the long run outcome converges to the competitive output if small mistakes are allowed. This is in contrast to a homogeneous population of optimizers which converges with the Cournot tatonnement under certain conditions to the Cournot Nash equilibrium. I was curious to know what happens if imitators and optimizers are mixed together in a heterogeneous population. In such heterogeneous population I found that in the average imitators are better off than optimizers, which is at the first glance a rather surprising result given that imitators' rationality appears to be less sophisticated than optimizers' one. It also makes the outcome reminiscent of Stackelberg behavior where imitators take the role of "independent" firms and optimizers are "dependent". Apart from the theoretical interest, such analysis is of empirical relevance given that imitation in form of "benchmarking" and "best practices" is widely used in today's management.

The results of this paper are related directly to a recurrent discussion of the assumption of profit maximizing behavior in economics. Imitators compare their position relative to others. Already [1] Alchian (1950) suggests that firms might maximize relative profits in the long run rather than absolute profits. In contrast, [19] Friedman (1953) argues that evolutionary selection forces favor absolute profit maximization. Although firms may not know about their profit functions we can take them "as if" they maximize profits. [30] Koopmans (1957), p. 140 argues that if selection does lead to profit maximization then such an evolutionary process should be part of economic modelling. [70] Winter (1971) establishes that such selection-argument does not hold in general.[1] As mentioned before I find that imitators are weakly better off than optimizers. This implies that Friedman's argument may not hold in this model.

Research directly related to [67] Vega-Redondo's (1997) oligopoly analysis has been done by [52] Schaffer (1989) who showed that the competitive solution is evolutionary stable in a symmetric Cournot duopoly with constant marginal costs. [48] Rhode/Stegeman (2001) showed similar result in a stochastic evolutionary framework. [53] Schenk-Hoppé (2000) extends [67] Vega-Redondo's (1997) result to continuum strategy sets. [45] Possajennikov (1998) studies the Cournot oligopoly with imperfect imitation and random matching. [2] Alós-Ferrer/Ania/Vega-Redondo (1999) allow for market entry and exit and study implications for the market structure. Under decreasing returns they obtain competitive solution whereas with increasing returns the unique long run outcome involves a profit maximizing monopolist. [3] Alós-Ferrer/Ania/Schenk-Hoppe (1997) study the stochastic stable outcome in Bertrand competition under imitation and experimentation finding also support for the competitive solution. [61] Tanaka (2000) studies imitation and experimentation in oligopolies with differentiated goods finding an equivalence of price and quantity strategies. In a slightly different approach [46] Qin/Stuart (1997), [23] Hehenkamp/Leininger (1999) and [22] Hehenkamp/Qin/Stuart (1999) study the evolutionary stability of outcomes in oligopoly.

Oligopoly games have been subject of experimental studies closely related to above theoretical research. For example [51] Sauermann/Selten (1959) find support for the convergence to Cournot Nash equilibrium. Contrary [64], [63] Todt (1972, 1975) provides some experimental evidence for imitation of the more successful firm. [25], [24] Huck/Normann/Oechssler (1999, 2000) find support for [67] Vega-Redondo's (1997) result (see also [43] Offerman/Potters/Sonnemans, 1997). Support for the Cournot Nash equilibrium or the competitive solution can be explained by the informational differences of treatments (see discussion later in the text). [58] Selten/Ostmann (2001) provide a general concept of imitation equilibrium in oligopoly setting which is motivated by comparison of earlier and recent experimental findings. Finally, imitation behavior has generally received great attention in literature on learning and evolutionary game theory (e.g. [55], [56], [54] Schlag, 1998a, 1998b, 1999, [16], [17] Ellison/Fudenberg, 1993, 1995).

[1] The explicit evolutionary modelling of decision "routines" and rules has rudimentarily been taken up by evolutionary economics, e.g. by [37] Nelson/Winter (1982). Recently there has been also some research on the profit maximization hypothesis in a general equilibrium context by [8] Blume/Easley (1999).

Studies on populations with heterogeneous decision rules such as imitation as well as optimization are rather rare since the dynamics are often complex and results appear difficult to obtain. [11] Conlisk (1980) studies a dynamic model with imitators and optimizers, taking costs of optimizing into account. [32] Matsuyama (1992) and [44] Pesendorfer (1995) are examples of studies with heterogeneous populations of conformists and nonconformist in a theory of fashion. More recently [29] Kaniovski/Kryazhimskii/Young (2000) study heterogeneous populations of best responders, conformists, and nonconformists in 2x2 games. [26] Kaarbøe/Tieman (1999) analyze equilibrium selection in supermodular games when different learning modes, including imitation and myopic optimization, coexist in the population. [14] Droste/Tuinstra (1998) obtain complicated dynamical phenomena in Cournot competition when firms can choose among different behavioral rules.

In section 4 and 5 this paper makes extensive use of is stochastic stability analysis. Based on mathematical results by [18] Freidlin/Wentzell (1984), these tools were first extended to games by [28] Kandori/Rob/Mailath (1993) and [71] Young (1993). The technique was subsequently further developed and applied to different games, by e.g [40] Nöldeke/Samuelson (1993), [50] Samuelson (1994), [27] Kandori/Rob (1995), [7] Blume (1995), [39] Nöldeke/Samuelson (1997), [15] Ellison (2000) and others. The basic idea is that persistent random perturbation due to mistakes, experimentation or replacement of players may lead to equilibrium selection as the noise goes to zero. In particular the technique characterizes the long run distribution in terms of the number of mutations required to move from (sets of) state(s) to (sets of) state(s). I employ mainly a necessary condition of stochastic stability based on (a sequence of) single mutation(s) only developed by [40] Nöldeke/Samuelson (1993), [50] Samuelson (1994) and [39] Nöldeke/Samuelson (1997).

The paper is organized as follows: After outlining the basic model of a symmetric n-firm Cournot oligopoly in section 2, I introduce decision rules and noise in section 3. This is followed in section 4 by the analysis of homogeneous populations of either imitators or optimizers. The analysis of a heterogeneous population of imitators and optimizers in section 5 contains the main result of the paper. Section 6 concludes the paper. The proofs are contained in the text rather than in a separate appendix since they are quite instructive. All mathematical tools required are introduced along the way.

2 The Basic Model

This section outlines the basic model in the spirit of Cournot ([12] Cournot, 1838, pp. 79). Consider a set of firms $N = \{1, 2, ..., n\}$ and a market for a homogeneous product. Inverse demand is given by $p : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$. For every total output quantity $Q \in \mathbb{R}_+$ this function specifies the market clearing price $p(Q)$. By the assumption of symmetry, every firm $i \in N$ posses the same production technology inducing identical cost functions $c : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ among all firms. For every output q_i of firm i it determines the costs of production. Thus c specifies the total costs associated with all inputs in order to produce a certain quantity of output. Let the total output be $Q := \sum_{i \in N} q_i$. Profits are given by

$$\pi_i(q_i, Q) := q_i p(Q) - c(q_i), \forall i \in N. \tag{1}$$

Definition 1 (Cournot Nash equilibrium) *A combination of output strategies* $(q_1^\circ, q_2^\circ, ..., q_n^\circ) \in \mathbb{R}_+^n$ *is a Cournot Nash equilibrium if* $\forall i \in N$,

$$q_i^\circ p(Q^\circ) - c(q_i^\circ) \geq q_i p(Q^\circ - q_i^\circ + q_i) - c(q_i), \forall q_i \geq 0. \tag{2}$$

Definition 2 (Competitive solution) *A combination of output-strategies* $(q_1^*, q_2^*, ..., q_n^*) \in \mathbb{R}_+^n$ *is a competitive solution if* $\forall i \in N$,

$$q_i^* p(Q^*) - c(q_i^*) \geq q_i p(Q^*) - c(q_i), \forall q_i \geq 0. \tag{3}$$

For technical reasons I assume that firms choose output from a common finite grid $\Gamma = \{0, \delta, 2\delta, ..., \nu\delta\}$, where both $\delta > 0$ and $\nu \in \mathbb{N}$ are arbitrary. This turns the strategic situation in a game with finite action space and allows me to focus on finite Markov chains later in the dynamic analysis.

In the next sections I am going to analyze how the outcomes in the oligopoly depend on decision rules. That is I focus on a variation of behaviorial assumptions but do not go into technical assumptions required for the existence and the uniqueness of the Cournot Nash equilibrium and the competitive solution. Therefore I simply assume existence and uniqueness of competitive solution and Cournot Nash equilibrium.

Assumption 1 $\exists$ *unique* $q^*, q^\circ \in \Gamma$.

Note, that the existence of Cournot Nash equilibrium in a symmetric n-firm Cournot oligopoly does not require concave profit functions. It is provided under very mild conditions such as nonincreasing upper-hemi-continuous inverse demand, nondecreasing, weakly convex lower-hemi-continuous costs and existence of a level of output where loss making is possible. Then the existence of a symmetric Cournot Nash equilibrium can be proven in the spirit of existence theorems by [33] McManus (1964) and [47] Roberts/Sonnenschein (1976) using an alternative proof based on lattice theoretical arguments and the application of Tarski's fixed point theorem ([62] Tarski, 1955, theorem 1, pp. 286 and generalization for correspondences by [72] Zhou, 1994) as well as results by [66] Topkis (1978) on the monotonicity of optimal solutions (refer [66] Topkis, 1978 and [65] Topkis, 1998, pp. 74). Sketches of this proof can be found in [69] Vives (1990), pp. 317, [4] Amir (1996), p. 146 and [68] Vives (2000), pp. 95. A condition for uniqueness of Cournot Nash equilibrium is mentioned later in remark 14.[2]

Note, that the existence of a Cournot Nash equilibrium can be shown also in an asymmetric n-firm Cournot oligopoly (non-identical costs). What is needed is either twice continuous differentiable strictly decreasing inverse demand and twice continuous differentiable weakly increasing costs ([41] Novshek, 1985, [6] Bamon/Frayseé, 1985) or non-increasing log-concave inverse demand and strictly increasing left-continuous costs ([4] Amir, 1996) as well as some boundary condition. Then one finds a fixed point of a decreasing "backwards" reaction correspondence that maps the total market output into the individual best response, a correspondence first defined by [57] Selten (1970) and apparent independently connected to a special fixed point argument for decreasing correspondences by [41], [42] Novshek (1985, 1984) and [6] Bamon/Frayseé (1985). However, I restrict my analysis to a symmetric oligopoly since imitation makes more sense if firms face similar conditions of production. There is also some experimental support for imitation of a reference group characterized by features similar to those of the imitator ([64] [63] Todt, 1972, 1975). [58] Selten/Ostmann (2001) explore these findings theoretically and define the notion of an imitation equilibrium.

The dynamics of the system is assumed to proceed in discrete time, indexed by $t = 0, 1, 2, ...$ At each t the state of the system is identified by the current output schedule

$$\omega(t) = (q_1(t), q_2(t), ..., q_n(t)).$$

Thus, the state space of the system is identical to Γ^n. Associated with any such $\omega(t) \in \Gamma^n$ is the induced profit profile $\pi(t) = (\pi_1(t), \pi_2(t), ..., \pi_n(t))$ at t, defined as follows:

$$\pi_i(t) := q_i(t)p(Q(t)) - c(q_i(t)), \forall i \in N. \tag{4}$$

[2] The uniqueness condition in remark 14 implies also that the unique Cournot Nash equilibrium is symmetric and that the best reply correspondence is in fact a function (see [68] Vives, 2000, p. 43).

3 Decision Rules and Noise

The finite population N is partitioned into two subpopulations of imitators and optimizers respectively. Let I be the subset of N that contains all imitators. The fraction of imitators in the population is denoted by $\theta = \frac{\sharp I}{\sharp N}$. The firms in the two subpopulations are characterized by different decision rules. The idea of decision rule goes apparently back to [37] Nelson/Winter (1982) who write that *"...at any time, firms in an industry can be viewed as operating with a set of techniques and decision rules (routines), keyed to conditions external to the firm ... and to various internal state conditions..."* (p. 165). Conventional economics focuses mainly on profit maximization. However, given that "benchmarking", "best practices" and other imitation rules are found in today's management practice, it is very natural for theorists not only to analyze those rules but also the diversity of decision rules.

At every time $t = 1, 2, ...$, each firm $i \in N$ is assumed to enjoy a common and independent probability $\rho \in (0, 1)$ of being able to revise its former output $q_i(t-1)$ according to its decision rule. Note, since $\rho < 1$ the process has inertia. That is not every period all firms adjust output. Each firm follows one of the two decision rules defined below, depending on which subpopulation it belongs to. I do not allow for firms switching decision rules.

Definition 3 (Imitator) [3] *An imitator $i \in I$ chooses with full support from the set*

$$D_I(t-1) := \{q \in \Gamma : \exists j \in N \ s.t. \ q = q_j(t-1) \ and \\ \forall k \in N, \pi_j(t-1) \geq \pi_k(t-1)\} \tag{5}$$

Definition 4 (Optimizer) *An optimizer $i \in N \backslash I$ chooses from the set*

$$D_O(t-1) := \{q \in \Gamma : q \in b(q_{-i}(t-1))\}, \tag{6}$$

where b is defined as firm i's reaction correspondence $b : \Gamma \longrightarrow \Gamma$ by

$$b(q_{-i}) := \{q_i' \in \Gamma : q_i' p(q_{-i} + q_i') - c(q_i') \\ \geq q_i p(q_{-i} + q_i) - c(q_i), \forall q_i \in \Gamma\}. \tag{7}$$

As usual q_{-i} is defined by $q_{-i} = \sum_{j \neq i, j \in N} q_j$. It is assumed that initially in $t = 0$ both type of firms start with any arbitrary output within the admissible domain Γ.

Definition 3 says that every period there exists a firm j which had the highest profit in previous period. An imitator imitates the previous period's

[3] See [67] Vega-Redondo (1997), p. 378.

quantity of firm j. Definition 4 just says that an optimizer sets best reply to the opponents' output in previous period. Imitators and optimizers differ in respect to the knowledge required to take their decisions. Whereas imitators just need to know the output schedule $\omega(t-1)$ and the associated profit profile $\pi(t-1)$, for optimizers it is sufficient to know the output schedule $\omega(t-1)$ (more specifically $q_i(t-1)$) as well as their own profit function, which implies knowing inverse demand and costs in order to calculate the myopic best response. Note, that imitation of successful behavior can be a rational rule of thumb ([67] Vega-Redondo, 1997) when firms and decision makers have difficulties in perceiving their profit functions. However, they can easily judge their performance relative to other firms in the industry. This might be a reason why in business practice executive remuneration is often based on the firm's stock outperforming the market index or similar.

Older experimental studies on Cournot oligopoly like the one by [51] Sauermann/Selten (1959) found some support of the convergence to Cournot-Nash equilibrium. Recent studies by [25] [24] Huck/Normann/Oechssler (1999, 2000) found support of imitative behavior in experimental Cournot settings. Whereas in former experiments subjects had profit tables for easy calculation of the best reply available, in later studies subjects received feedback about the competitors' profits and output levels. The informational framework of these experimental settings corresponds closely to the information required for each of the two decision rules.

The process induced by the decision rules is a discrete time n-vector finite Markov chain with stationary transition probabilities. Finiteness is provided by the finite state space Γ^n. It is a vector process since each ω is a vector in Γ^n. The process has the Markov property since $prob\{\omega(t+1)|\omega(t), \omega(t-1), ..., \omega(t-k)\} = prob\{\omega(t+1)|\omega(t)\}$. That is the conditional probability of being in state $\omega(t+1)$ given states $\omega(t), \omega(t-1), ..., \omega(t-k)$ is equal to the conditional probability of being in state $\omega(t+1)$ given state $\omega(t)$. Since the decision rules themselves do not change over time it has stationary transition probabilities $prob\{\omega'(t+1)|\omega(t)\} = prob\{\omega'(t+k+1)|\omega(t+k)\}$, $k = 0, 1, ...$ Imitators just mimic the output decision of the firm(s) with highest profits in the previous period whereas every optimizer chooses a best reply against the output by all other firms in the previous round. The assumption of reference to the previous period only shows that firms operate somehow myopically. There are no learning effects based on information of more distant past. The firms' knowledge of the present state is assumed to be complete to the extent it enters the decision rule. Both assumptions are justified by the consideration that (a) the impact of yesterday's state of the world on today's decision may be much greater than the impact of more distant history, and (b) decision makers may be forgetful.

We define a Markov operator, the transition probability matrix $P : \Gamma^n \longrightarrow \Delta(\Gamma^n)$ as arrangement of transition probabilities $prob\{\omega'|\omega\} = p_{\omega\omega'}$ in a $\sharp\Gamma^n \times \sharp\Gamma^n$- matrix $P = (p_{\omega\omega'})_{\omega,\omega'\in\Gamma^n}$ where $p_{\omega\omega'} \geq 0$, $\omega, \omega' \in \Gamma^n$ and $\sum_{\omega'\in\Gamma^n} p_{\omega\omega'} = 1$, $\forall\omega \in \Gamma^n$.[4] That is the element $p_{\omega\omega'}$ in the transition probability matrix P is the conditional probability that the state is in ω' at $t + 1$ given that it is in ω at t.

To complete the model of the decision process I make:

Assumption 2 (Noise) *At every output revision opportunity t, each firm follows its decision rule with independent common probability $(1 - \varepsilon)$, $\varepsilon \in (0, a]$, where a is sufficiently small, and with probability ε it randomizes with full support Γ.*

There are several interpretations of the noise. (1) Firms are assumed to innovate in a sense of experimenting with various output levels. (2) Decision makers of the firms are assumed to be boundedly rational in the sense that there is always a small positive probability of making mistakes in output decisions. (3) Every period, a small fraction of firms is replaced by newcomers who choose their output from tabula rasa. Notice that the noise is just a stylized notion of innovation, bounded rationality or population dynamics but not an explicit model. Nevertheless it adds some realistic feature to the model. As convention I call a firm mutating if it randomizes with full support.

The noise has a convenient technical property because by assumption 2, $P(\varepsilon)$ is regularly perturbed[5] and an ergodic Markov chain on Γ^n. This implies that there exists a unique invariant distribution $\varphi(\varepsilon)$ on Γ^n (see [31] Masaaki (1997) for standard results). To put it more intuitively, the noise makes any state accessible from any other state in finite time. This is sufficient for the existence of the unique invariant distribution.

The following analysis focuses on the unique limiting invariant distribution φ^* of P defined by $\varphi(\varepsilon)P(\varepsilon) = \varphi(\varepsilon)$, $\varphi^* := \lim_{\varepsilon\to 0} \varphi(\varepsilon)$ and $\varphi^* P = \varphi^*$.

[4]According to the definition of a Markov transition matrix, probability distributions over states are represented by row vectors.

[5]P is regularly perturbed by ε if following conditions are satisfied for all $\omega, \omega' \in \Gamma^n$:

(i) $P(\varepsilon)$ is ergodic for all $\varepsilon \in (0, a]$,

(ii) $\lim_{\varepsilon\to 0} P(\varepsilon) = P$,

(iii) $p_{\omega\omega'}(\varepsilon) > 0$ for some $\varepsilon \in (0, a] \Rightarrow \exists r \geq 0$ s.t. $0 < \lim_{\varepsilon\to 0} \varepsilon^{-r} p_{\omega\omega'}(\varepsilon) < \infty$.

The last condition means that either in the perturbed process $P(\varepsilon)$ a transition from ω to ω' is impossible for all $\varepsilon \in (0, a]$, or $p_{\omega\omega'}(\varepsilon)$ is of order ε^r for some unique real number $r \geq 0$ as $\varepsilon \to 0$ (see [71] Young, 1993, p. 70).

In particular, the focus is on how to characterize this probability vector since it provides a description of the long run output behavior of the market when the noise goes to zero. For that reason I will refer to it also as the long run distribution. It determines the average proportion of time spent in each state of the state space in the long run.

4 Homogeneous Populations

This section analyzes two special cases where either $\theta = 1$ or $\theta = 0$. That is, I consider either a homogeneous population of imitators or optimizers respectively. Some technical preliminaries required in the subsequent analysis are introduced beforehand.

4.1 Preliminaries

The proof of the following results relies mainly on two concepts: the notions of submodular game[6] and recurrent sets.

In the quantity setting oligopoly, each firm's quantity and the total output are strategic substitutes.[7] Games with strategic substitutes are essentially submodular games.

A lattice is defined as a poset[8] $(X, \leq)$ whose least upper bound and greatest lower bound are given by $x' \vee x'' = \sup\{x', x''\}$ and $x' \wedge x'' = \inf\{x', x''\}$,[9] $\forall x', x'' \in X$ respectively. Note that $\Gamma = \{0, \delta, 2\delta, ..., \nu\delta\}$ and products thereof are lattices. A real valued function $f : X \longrightarrow \mathbb{R}$ on a lattice X is called submodular on X if $\forall x', x'' \in X,\ f(x' \wedge x'') + f(x' \vee x'') \leq f(x') + f(x'')$. The

[6]For the concept of submodularity see [65] Topkis (1998), pp. 43. For submodularity and supermodularity in the context of a Cournot oligopoly see [69] Vives (1990), [4] Amir (1996) and [68] Vives (2000), chapter 4. Essentially, supermodularity is equivalent to single-crossing property and submodularity to dual single-crossing property ([65] Topkis, 1998, [34] Milgrom/Shannon, 1994).

[7]See [10] Bulow/Geanakoplos/Klemperer (1985) for a discussion of the concept of strategic complements and substitutes.

[8]A partially ordered set (poset) $(X, \leq)$ is defined as a set X with a binary relation $\leq$ such that $\forall x, y, z \in X$ following conditions hold:

(i) reflexivity: $\forall x \in X,\ x \leq x$,

(ii) antisymmetry: if $x \leq y$ and $y \leq x \Longrightarrow x = y$,

(iii) transitivity: if $x \leq y$ and $y \leq z \Longrightarrow x \leq z$.

[9]Note, that if X is a product set, then for example $x' \vee x''$ means the component-wise maximum.

function f is called strictly submodular if the inequality holds strictly ("$<$") for all unordered $x', x'' \in X$.[10]

Assumption 3 (Strictly Decreasing Demand) $\forall Q, Q' \in \{0, \delta, 2\delta, ..., n\nu\delta\}$, $Q' > Q \Longrightarrow p(Q') < p(Q)$.

Lemma 1 *By assumption 3, $\pi_i(q_i, Q)$ is strictly submodular in (q_i, Q) on $\Gamma \times \{0, \delta, 2\delta, ..., n\nu\delta\}$, $\forall i \in N$.*

Proof Lemma 1. To show[11] $\forall (q, Q) \in \Gamma \times \{0, \delta, 2\delta, ..., n\nu\delta\}$, we have to show following inequality

$$\pi((q', Q') \wedge (q'', Q'')) + \pi((q', Q') \vee (q'', Q'')) \leq \pi(q', Q') + \pi(q'', Q'') \qquad (8)$$

for the cases (i) $q' \geq q''$ and $Q' \geq Q''$, (ii) $q' < q''$ and $Q' \geq Q''$, (iii) $q' \geq q''$ and $Q' < Q''$ as well as (iv) $q' < q''$ and $Q' < Q''$. In particular strict submodularity requires us to show a strict inequality for the cases (ii) and (iii) since then neither $(q', Q') \geq (q'', Q'')$ nor $(q'', Q'') \geq (q', Q')$.
Case (i) and (iv):

$$\pi((q', Q') \wedge (q'', Q'')) + \pi((q', Q') \vee (q'', Q'')) = \pi(q', Q') + \pi(q'', Q'')$$

Case (iii): Since by assumption 3, p is strictly decreasing

$$\begin{aligned}
p(Q'') &< p(Q') \\
p(Q'')(q' - q'') &< p(Q')(q' - q'') \\
p(Q'')(q' - q'') - c(q') + c(q'') &< p(Q')(q' - q'') - c(q') + c(q'') \\
\pi(q', Q'') - \pi(q'', Q'') &< \pi(q', Q') - \pi(q'', Q') \\
\pi(q'', Q') + \pi(q', Q'') &< \pi(q', Q') + \pi(q'', Q'') \\
\pi((q', Q') \wedge (q'', Q'')) + \pi((q', Q') \vee (q'', Q'')) &< \pi(q', Q') + \pi(q'', Q'').
\end{aligned}$$

Case (ii): By assumption 3, p is strictly decreasing. Thus we have analogous to previous steps

$$\begin{aligned}
p(Q'') &> p(Q') \\
p(Q'')(q' - q'') &< p(Q')(q' - q'') \\
\pi((q', Q') \wedge (q'', Q'')) + \pi((q', Q') \vee (q'', Q'')) &< \pi(q', Q') + \pi(q'', Q'').
\end{aligned}$$

This completes the proof of lemma 1. **q.e.d.**

[10] A pair $x', x'' \in X$ is unordered iff none of the two statements holds: $x' \leq x''$, $x'' \leq x'$.
[11] For notational ease I drop subscripts and superscripts if no ambiguity seems to arise.

Remark 1 *If assumption 3 is modified such that p is weakly decreasing then $\pi(q, Q)$ is submodular in (q, Q) on $\Gamma \times \{0, \delta, 2\delta, ..., n\nu\delta\}$.*

The second concept required for our analysis is that of stochastic stability. Instead making use of the basic graph theoretic arguments developed by [18] Freidlin/Wentzel (1984) as well as [28] Kandori/Rob/Mailath (1993) and [71] Young (1993), I refer to a simpler necessary condition for a state being in the unique invariant limiting distribution developed by [40] Noeldeke/Samuelson (1993), [50] Samuelson (1994) and [39] Noeldeke/Samuelson (1997). The basic idea is that a necessary condition for being contained in the unique invariant limiting distribution is that a state be contained in the set that is robust to a single mutation.

Consider $\varepsilon = 0$ and define an absorbing set $A \subseteq \Gamma^n$ by

(i) $\forall \omega \in A$, $\forall \omega' \notin A$, $p_{\omega\omega'} = 0$ and

(ii) $\forall \omega, \omega' \in A$, $\exists m \in \mathbb{N}$, m finite, s.t. $p^{(m)}_{\omega\omega'} > 0$, $p^{(m)}_{\omega\omega'}$ being the m-step transition probability from ω to ω'.

Let Z be the collection of all A in Γ^n.

We call states ω and ω' adjacent if exactly one mutation can change the state from ω to ω'. The set of all states adjacent to states ω is the single mutation neighborhood of ω denoted by $M(\omega)$. The basin of attraction of an absorbing set A is the set $B(A) = \{\omega | \exists m, \exists \omega' \in A \text{ s.t. } p^{(m)}_{\omega\omega'} > 0\}$. It is the collection of all states from which there is positive probability that the (unperturbed) dynamics leads to the absorbing set A. A recurrent set R is a minimal collection of absorbing sets with the property that there do not exist absorbing sets $A \in R$ and $A' \notin R$ such that $M(A) \cap B(A') \neq \emptyset$. That is a recurrent set R is a collection of absorbing sets for which it is impossible that a single mutation followed by the dynamics leads to an absorbing set not contained in R. We will make use of following lemma.

Lemma 2 (Samuelson, 1997) *Given a regularly perturbed finite Markov chain, then at least one recurrent set exists. Recurrent sets are disjoint. Let the state ω be contained in the support of the unique limiting invariant distribution φ^*. Then $\omega \in R$, R being a recurrent set. Moreover, $\forall \omega' \in R$, $\varphi^*(\omega') > 0$.*

Proof Lemma. See [49] Samuelson, 1997, lemma 7.1 and proposition 7.7., proof pp. 236-238. **q.e.d.**

Since every finite Markov chain has at least one absorbing set we have by the definition of recurrent set immediately the existence of recurrent set. The same applies for recurrent sets being disjoint. The intuition for the third and forth part of the result is that a necessary condition for being contained in the support of the limiting invariant distribution is that a state is contained in a set that is robust to single mutations. Behind this result is the general idea that mutations select among absorbing sets in the sense that only the most robust absorbing sets remain in the support of the limiting invariant distribution.

4.2 Imitators

This subsection focuses on a homogeneous population of imitators, that is $\theta = 1$. It is the same setting studied by [67] Vega-Redondo (1997). We outline the analysis in order to introduce the techniques used also in later sections and to provide a slightly different but simpler proof. In particular we make use of the observation that the payoff functions are submodular and that the convergence to the long run outcome requires just a single suitable mutation.

Definition 5 (Competitive State) *The state* $\omega^* = (q_1^*, ..., q_n^*) \in \Gamma^n$ *is called the competitive state.*

I show that the competitive state is the unique stochastically stable outcome in the symmetric n-firm Cournot oligopoly with a homogeneous population of imitators.

Theorem 1 (Vega-Redondo, 1997) *If* $\theta = 1$ *then by previous assumptions we have* $\varphi^*(\omega^*) = 1$.

The proof consists of three lemmatas. The first lemma works out the absorbing sets of the unperturbed dynamics. In general the long run distribution can only assign positive probability to states in absorbing sets. In this case we have that each state in which every firm sets identical output is an absorbing state. The set of absorbing states is the set of all state with identical output among firms

The second lemma shows that any absorbing state except the competitive one is not robust against a single mutation. The competitive state can be reached by any absorbing state by just a single suitable mutation.

The third lemma shows that more than one mutation is required to leave the competitive state. Altogether these lemmatas imply that the competitive state is the recurrent set. Hence it is the unique support of the long run distribution.

Proof Theorem. Recall that Z is the collection of absorbing sets.

Lemma 3 (Vega-Redondo, 1997) *Given D_I then $Z = \{\{\omega\} : \omega = (q, ..., q) \in \Gamma^n$ for some $q \in \Gamma\}$.*[12]

Proof Lemma. By D_I for every $\omega = (q, ..., q) \in \Gamma^n$ we have $p_{\omega\omega} = 1$ and $p_{\omega\omega'} = 0$, $\forall \omega' \neq \omega$. Conversely, since at any t, common independent $\rho > 0$, there is positive probability that all firms adjust towards the same output in $D_I(t-1)$ given any arbitrary $\omega(t-1)$. **q.e.d.**

Lemma 4 *$M(\omega) \cap B(\{\omega^*\}) \neq \emptyset$, $\forall \{\omega\} = A \subset Z \backslash \{\omega^*\}$.*[13]

Proof Lemma. Since by assumption 1, q^* is unique, ω^* is unique and by lemma 3 A_{ω^*} consists of the single unique state ω^*. Consider any $A \neq A_{\omega^*}$. We claim that starting in any $A \neq \{\omega^*\}$ a single (suitable) mutation can lead the dynamics to $B(\{\omega^*\})$. It is sufficient to show that $\forall q \in \Gamma$, $q \neq q^*$, and $k \in \mathbb{N}$

$$q^* p((n-k)q + kq^*) - c(q^*) > qp((n-k) + kq^*)q - c(q) \tag{9}$$

for $k = 1$. By lemma 1, $\pi(q, Q)$ is strictly submodular in (q, Q) on $\Gamma \times \{0, \delta, 2\delta, ..., n\nu\delta\}$. Set $q^* \equiv q'$ and $q \equiv q''$. Then $Q' = (n-k)q + kq^*$ and $Q'' = nq^*$. Inequality 8 implies strict quasisubmodularity[14] of π

$$\pi((q', Q') \vee (q'', Q'')) \geq \pi(q'', Q'') \Longrightarrow \pi(q', Q') > \pi((q', Q') \wedge (q'', Q'')) \tag{10}$$
$$\pi((q', Q') \wedge (q'', Q'')) \geq \pi(q'', Q'') \Longrightarrow \pi(q', Q') > \pi((q', Q') \vee (q'', Q'')) \tag{11}$$

Note that the left hand side of $\Longrightarrow$ is the definition of the competitive solution in the inequality 3 whereas the right hand side is the above inequality 9 (for $q < q^*$ the upper formula 10 and for $q > q^*$ the lower formula 11). Setting $k = 1$ yields the desired claim and completes the proof of the lemma. **q.e.d.**

Lemma 5 *$M(\omega^*) \cap B(A) = \emptyset$, $\forall A \neq \{\omega^*\}$.*[15]

Proof Lemma. From setting $k = n - 1$ in inequality 9, it follows that more than one mutation is needed to escape A_{ω^*} since firms producing q^* are still better off after just one mutation. **q.e.d.**

From previous lemmata follows that $R = \{\omega^*\}$. This completes the proof of the theorem 1. **q.e.d.**

[12] [67] Vega-Redondo, 1997, proposition 1.

[13] The analogous result in [67] Vega-Redondo (1997) is lemma 1.

[14] For quasisubmodularity see [34] Milgrom/Shannon (1994), p. 162 and [65] Topkis (1998), pp. 58. It is the ordinal notion of submodularity.

[15] The analogous result in [67] Vega-Redondo (1997) is lemma 2.

The number of mutations required to switch from a non-competitive absorbing state to the competitive state is exactly one, whereas it takes more than one mutation to switch from the competitive state to the non-competitive state. The intuition of theorem 1 is that if a firm deviates from a non-competitive output to competitive output she may decrease her own payoff but decreases the opponents' payoffs even more. This is exactly what lemma 4 says making use of the observation of submodularity in lemma 1. The submodularity argument gives some insights why the result holds indeed. In Cournot oligopoly, a firm's output and the total output are strategic substitutes ([10] Bulow/Geanakoplos/Klemperer, 1985). In other words, the increase in total output decreases the return to an increase in own output. Lemma 5 means that no firm can deviate from a competitive output level and become relatively better off than others.

4.3 Optimizers

Now I turn to a homogeneous population containing only optimizers, i.e. $\theta = 0$. This corresponds essentially to the setting that Cournot had in mind.

Definition 6 (Cournot State) *The state* $\omega^\circ = (q_1^\circ, ..., q_n^\circ) \in \Gamma^n$ *is called the Cournot state.*

A Cournot state is a state where every firm sets its Cournot Nash output. Since I assumed that q° is unique and $q^\circ \in \Gamma, \forall i \in N$, it follows that ω° is unique.

Assumption 4 (Potential) *The Cournot game has a generalized ordinal potential.*[16]

A potential function monitors the drift towards the Nash equilibrium. In the context of a Cournot oligopoly, a potential function is generally difficult to interpret because what do the firms consciously or unconsciously try to jointly optimize? Important is that such a potential function implies the existence of a finite improvement path towards the unique Nash equilibrium ([35] Monderer/Shapley, 1996, lemma 2.5). [59] Slade (1994) proves the existence of an exact potential, a stronger version than the generalized ordinal

[16] A generalized ordinal potential (refer [35] Mondere/Shapley, 1996) is a function $P : \Gamma^n \longrightarrow \mathbb{R}$ for the Cournot game such that for every $i \in N$ and for every $q_{-i} \in \Gamma^{n-1}$ and every $q, q' \in \Gamma$ holds that

$$\pi_i(q, q_{-i}) - \pi_i(q', q_{-i}) > 0 \Longrightarrow P(q, q_{-i}) - P(q', q_{-i}) > 0.$$

potential function, for the case of Cournot oligopoly with linear demand. Another example for the existence of an ordinal potential is given for a Cournot oligopoly with linear costs by [35] Monderer/Shapley (1996), p. 124. The existence of a finite improvement path implies that the sequential best response dynamics converges to the Nash equilibrium. [25] Huck/Normann/Oechssler (1999) make use of this result to show that the best reply process with inertia converges to the Cournot Nash equilibrium in a linear quantity setting symmetric quadropoly.[17]

The theorem states that in a homogeneous population of optimizers the long run outcome converges to the Cournot Nash equilibrium.

Theorem 2 *If $\theta = 0$ then by the previous assumptions we have $\varphi^*(\omega^\circ) = 1$.*

Proof Theorem. Recall that Z was the collection of all absorbing sets $A \subseteq \Gamma^n$. Given D_O and assumptions 1 and 4 then $Z = \{\omega^\circ\}$. To see this note that from assumption 1 we know that ω° is unique. From assumption 4 follows by lemma 2.5 in [35] Monderer/Shapley (1996) that from any initial state sequential best reply converges to ω°. Since every period each firm enjoys a common and independent probability $\rho \in (0, 1)$ to adjust her output, there is positive probability that in each period exactly one firm gets to adjust her output. Hence, $Z = A = \{\omega^\circ\}$. It follows trivially by lemma 2 that $\{\omega^\circ\} \in R$ and hence $\varphi^*(\omega^\circ) = 1$. **q.e.d.**

4.4 Comparative Discussion

Remark 2 *For homogeneous populations (either $\theta = 1$ or $\theta = 0$) at most one suitable mutation is required to trigger the convergence to the long run outcome.*

Both results on homogeneous populations of either imitators or optimizers appear to be robust in the sense that in the former case essentially just one suitable mutation is required to induce the result, whereas in the latter case no mutation is needed. In other words, the results do not depend on a multiplicity of highly unlikely mutations. Thus the convergence to the long run outcome is rather fast.

An important difference between the two results concerns the number of absorbing sets. Whereas in a homogeneous population of imitators we face several absorbing states, the absorbing state in a homogeneous population of optimizers is unique.

[17] There are of course other assumptions that would guaranty convergence to the Cournot Nash equilibrium, for instance dominance solvability (see [36] Moulin, 1984).

The behavior of homogeneous populations is similar with respect to the number of states surviving persistent randomness. In both cases the long run outcome is unique but different. The unique long run outcome of a homogeneous population of imitators and optimizers converges to the competitive solution and the Cournot Nash equilibrium respectively.

5 Heterogeneous Population

This section states the main results. Following the previous section's result on stochastically stable solutions for homogeneous populations of either $\theta = 1$ or $\theta = 0$, I now focus now on stochastically stable outcomes for heterogeneous populations $\theta \in (0, 1)$. In particular I am interested in the question of how robust the previous result on imitators is with regard to an arbitrary fraction of optimizers and vice versa.

5.1 Introductory Discussion

In 1934, von Stackelberg [60] put forward a model of two firms with different individual positions. In particular he assumed that one firm is the leader who aims for independence from others and the other firm is a "hanger on"([60] Stackelberg, 1934, p. 18) or follower. In game theory a similar setting is used to demonstrate backwards induction and subgame-perfect equilibrium. However, the sequential representation is not due to Stackelberg but was introduced in a different context by [38] von Neumann/Morgenstern (1944), pp. 100, as the "majorant game".[18] They noted already the first mover advantage. In the common sequential setting of the Stackelberg game in literature, individual differences among firms are not explicitly considered. Instead it is assumed that all players are fully rational, have complete information about demand and cost etc. The only difference among firms is that one is the leader and the other one is the follower in a two-stage game. The leader maximizes her absolute profit subject to the reaction function of the follower. This usually leads to higher output and profits for the leader and lower output and profits for the follower as well as lower total profit but higher market output than in the Cournot Nash equilibrium and higher total profit and lower market output than in the competitive solution. Generally, it is hard to explain why somebody plays the role of a leader whereas the other is a follower. Indeed, if one assumes homogeneous rationality among all firms where should the leadership come from other than the exogenously

[18] I thank Prof. Selten for pointing out to me the "majorant game".

given stages of the model.[19] Stackelberg himself argued that his description of the problem is incomplete and that an empirical input is necessary to decide on the different positions.[20] However, one should not leave out an important aspect of Stackelberg's model. He explicitly labeled his model as a "special case" ([60] Stackelberg, 1934, p. 22) which "could appear only if the individual circumstances, especially those related to costs and production, differ to a great extent among the duopolists."[21] Since most of the game theoretic models abstract from any significant differences related to costs and production (apart from assigning ad hoc leader and follower positions in a two-stage game), it is not surprising that those models are inadequate to explain why some firm is a leader and another is a follower.[22] Therefore I suggest a re-orientation of Stackelberg's approach whereby I define the different "individual circumstances" as different decision rules of the firms.

5.2 Candidates for Solution

Before I go into the dynamic analysis I want to reflect what could be a possible static solution with imitators and optimizers. Consider first the competitive state that is a state in which every firm - no matter whether imitator or optimizer - sets the individual competitive output. Clearly imitators do not have any reason to deviate from this state since all firms set identical outputs. However, optimizers would deviate since the competitive output is not generally the best reply to all opponents setting the competitive output. Thus contrary to a homogeneous population of imitators the competitive state can not be a solution in a heterogeneous population. Second, consider the Cournot state where each firm sets the best response. In a

[19] The original work by Stackelberg is contained in [60] Stackelberg (1934), pp. 16-24 (especially p. 22) as well as pp. 44-53. See [5] Amir/Grilo (1999) for a discussion of Stackelberg vs. Cournot Equilibrium in the context of supermodular games with endogenous timing.

[20] [60] Stackelberg (1934), p. 18: "In dieser merkwürdigen Verschiedenheit der tatsächlichen Positionen der beiden Anbieter liegt das eigentliche Problem des Dyopols und darüber hinaus des Oligopols beschlossen. Da nämlich jeder Anbieter jede der beiden Positionsarten haben könnte, so ist die oben beschriebene Preisbildungsform unvollständig."

[21] Translation by the author based on the original German text: "Dieser 'Ausnahmefall' kann offenbar nur dann eintreten, wenn die individuellen Verhältnisse, insbesondere die Kosten- und Produktionsverhältnisse der beiden Dyopolisten sich voneinander erheblich unterscheiden." ([60] Stackelberg, 1934, pp. 22).

[22] An exception is [13] van Damme/Hurkens (1999) who show in a linear model that differences in costs can lead to endogenous emergence of a leader and a follower. Commitment can be more risky for high cost firms and hence by [21] Harsanyi/Selten's (1988) concept of risk dominance the low cost firm will choose to commit and emerge endogenously as leader.

static setting imitators do not deviate since no firm earns higher profits than others. Optimizers have no incentive to change output either because they play best reply to opponents' output. Thus the Cournot Nash equilibrium is a solution to the static symmetric Cournot oligopoly with a heterogeneous population. However, the Cournot state is not the unique static outcome. Consider the following solution

Definition 7 (Pseudo-Stackelberg Solution) *Suppose $\theta \in (0,1)$. The Pseudo-Stackelberg solution is a state $\omega^S = (q_1, ..., q_{\theta n}, q_{\theta n+1}, ..., q_n)$ that satisfies the following conditions:*

(i) $\forall i \in I$ $q_i = q^I \equiv q^S$ *s.t.*

$$q^S p(\theta n q^S + (1-\theta) n q^D) - c(q^S) \geq \\ qp(\theta n q^S + (1-\theta) n q^D) - c(q), \forall q \in \Gamma, \tag{12}$$

(ii) $\forall i \in (N \backslash I)$ $q_i = q^D$,[23]

$$q^D := b(\theta n q^S + ((1-\theta)n - 1) q^D), \tag{13}$$

The outcome I call *Pseudo-Stackelberg Solution* because of its similarities and differences to the notion of Stackelberg solution in literature. First, imitators and optimizers play roles analogous to those of the "independent" and the "dependent" firms in Stackelberg's work respectively. By definition every optimizer takes the output decisions by other firms into account when deciding on her output (ii). Roughly any change of output by its opponents will lead to adjustments in its output. In contrast, an imitator just mimics some firms and is happy with her output decision as long as nobody makes higher profits (i). It follows that[24]

$$\pi_i(q^S, q^D, n, \theta) \geq \pi_j(q^S, q^D, n, \theta), \forall i \in I, j \in N \backslash I.$$

Every imitator is weakly better off than any optimizer. According to condition (ii) imitators (independent firms, leaders) maximize profits taking prices as given, which departs from the Stackelberg solution in literature. Moreover, note that I deal with more than one leader.[25] Despite these differences above

[23] The uniqueness condition in remark 14 ensures that the best reply is a singleton.

[24] For notational convenience I write $\pi_i(q, q', n, \theta)$ for $\pi_i(q, \theta n q + (1-\theta) n q')$ if $i \in I$ or $\pi_i(q', \theta n q + (1-\theta) n q')$ if $i \in N \backslash I$.

[25] Stackelberg's graphical analysis remains in the two player case. In the literature we find mainly Stackelberg models with one leader and a finite number of followers. An exception is [9] Bulavsky/Kalashnikov (1996) who provide a generalized version of Stackelberg oligopoly with several leaders. They assume that the leaders have the Cournot conjecture against their fellow leaders and the Stackelberg conjecture against their followers, and that the followers have the Cournot conjecture against leaders and followers.

concept retains the important aspect of Stackelberg's idea: the modelling of asymmetries and behavioral heterogeneity of firms and particularly the modelling of independent (indicated by superscript "I") and dependent firms (indicated by superscript "D"). It is interesting to note that von Stackelberg himself never used the word "leader" in his book but spoke of "independent" and "dependent" firms. In the following text I will reserve q^S to denote the identical output of any imitator in the Pseudo-Stackelberg solution and q^I to denote the identical output of any imitator in some states. q^D is reserved for the identical best reply by any optimizer in some states. The analogous notation applies of the profit functions π^I and π^D. Generally, superscript indicates identical values for all firms with a subpopulation whereas subscript indicates individual values.

Analogous to assumption 1 we assume existence and uniqueness of the Pseudo-Stackelberg solution.

Assumption 5 *$\exists$ unique $\omega^S \in \Gamma^n$.*

Following simple numerical example helps building some intuition for the Pseudo-Stackelberg solution.

Example 1. Suppose $n = 3$ and $\theta = \frac{2}{3}$ such that we have two imitators and one optimizer. Inverse demand is given by $p(q_D + q_1 + q_2) = 10 - q_D - q_1 - q_2$ where q_D is the output of the optimizer and q_1, q_2 are the outputs of the first and second imitator respectively. Costs are strictly convex and identically for all firms $c(q_i) = \frac{1}{2}q_i^2, i \in \{1, 2, D\}$. The optimizer maximizes profits according to the Cournot conjecture as follows (since the setting is symmetric q_I denotes the identical output of either imitator):

$$\pi_D(q_D, 2q_I) = (10 - 2q_I - q_D)q_D - \frac{1}{2}q_D^2 \longrightarrow \max_{q_D},$$

which leads to

$$\Longrightarrow q_D = \frac{10 - 2q_I}{3}.$$

Each imitator maximizes profits taking prices $\bar{p}$ fixed

$$\pi_I(q_I) = \bar{p}q_I - \frac{1}{2}q_I^2 \longrightarrow \max_{q_I},$$

which leads to price equals marginal cost

$$\Longrightarrow \bar{p} = q_I.$$

Thus, substituting inverse demand for $\bar{p}$ under consideration of the optimizer's best reply,

$$10 - \frac{10 - 2q_I}{3} - 2q_I = q_I \Longleftrightarrow q_I = \frac{20}{7}.$$

It follows that $q_D = \frac{10}{7}$, $\pi_I = \frac{200}{49}$ and $\pi_D = \frac{150}{49}$. Note that imitators set each higher output than the optimizer. Moreover, they are better off than the optimizer.

Previous arguments indicate that the Cournot state and the Pseudo-Stackelberg solution may not be the only solutions. To facilitate the analysis of stochastically stable outcomes in heterogeneous populations I define the following set of states:

Definition 8 (Pseudo-Stackelberg States) *Given $\theta \in (0,1)$. The set of Pseudo-Stackelberg states H comprises of all states $\omega = (q_1, ..., q_{\theta n}, q_{\theta n+1}, ..., q_n)$ that satisfy the following properties:*

(i) $q_i = q^I$, $\forall i \in I$ *and some* $q^I \in \Gamma$,

(ii) $q_i = q^D$, $\forall i \in N \backslash I$, $q^D := b(\theta n q^I + ((1-\theta)n - 1)q^D)$,

(iii) $\pi^I(q^I, q^D, n, \theta) \geq \pi^D(q^I, q^D, n, \theta)$,

(iv) $\pi^I(q^I, q^D, n, \theta) = \pi^D(q^I, q^D, n, \theta)$ *iff* $q^I = q^D$.

If condition (i) is not satisfied then an imitator may mimic a different output decision from another imitator. If condition (ii) is not satisfied then not all optimizers are playing the best reply. If condition (iii) is not satisfied imitators will mimic optimizers. To understand the motivation of (iv) note that by identical costs $q^I = q^D$ implies $\pi^I(q^D, q^I, n, \theta) = \pi^D(q^D, q^I, n, \theta)$. To see the purpose of the other direction note that if $\pi^I(q^D, q^I, n, \theta) = \pi^D(q^D, q^I, n, \theta)$ and $q^I \neq q^D$ then imitators would be indifferent between q^I and q^D thus adding a source of instability.

The name "Pseudo-Stackelberg states" shall draw attention to the fact that first, we consider a heterogeneous population of firms, and second that in each Pseudo-Stackelberg state imitators are weakly better of than optimizers.

The following proposition provides a first characterization of the set of Pseudo-Stackelberg states in comparison to previous discussed outcomes.

Proposition 1 *Given $\theta \in (0,1)$ and assumptions 1 and 5. Then*

(i) $\omega^* \notin H$,

(ii) $\omega^\circ \in H$,

(iii) if c strictly convex then $\omega^S \in H$.

Proof Proposition. (i) $\omega^* = (q_1^*, ..., q_{\theta n}^*, q_{\theta n+1}^*, ..., q_n^*)$. Suppose (i), (ii) and (iv) of the definition of Pseudo-Stackelberg states 8 hold (note that (iv) $\Longrightarrow$ (iii)). Then since $q^* \neq b((n-1)q^*)$ property (ii) of the definition of Pseudo-Stackelberg states 8 is violated. Thus $\omega^* \notin H$.

(ii) $\omega^\circ = (q_1^\circ, ..., q_{\theta n}^\circ, q_{\theta n+1}^\circ, ..., q_n^\circ)$. Thus (i) of the definition of Pseudo-Stackelberg states 8 holds. Moreover, $q^\circ = b(\theta n q^I + ((1-\theta)n - 1)q^D) = b((n-1)q^\circ)$ and (ii) of the definition of Pseudo-Stackelberg states 8 holds. This implies (iv) and (iii) of the definition of Pseudo-Stackelberg states hold too. Hence $\omega^\circ \in H$.

(iii) $\omega^S = (q_1^S, ..., q_{\theta n}^S, q_{\theta n+1}^D, ..., q_n^D)$. c being strictly convex $\Longrightarrow \exists q^S, q^I$ s.t. $\pi^I(q^S, q^D, \theta, n) > \pi^D(q^S, q^D, \theta, n)$. Thus properties (i) to (iv) of definition 8 are satisfied. **q.e.d.**

Remark 3 *If c is not strictly convex then* ω^S *may fail to qualify as Pseudo-Stackelberg state.*

Proof Remark. Assume c is linear. Then $q^S \geq 0$ and $\pi^I(q^S, q^D, n, \theta) = 0$. $b(\theta n q^S + ((1-\theta n) - 1)q^D) = q^D = 0$. Since $\pi^D(q^S, 0, n, \theta) = 0$ it follows that property (iv) of the definition of Pseudo-Stackelberg states is violated. Thus $\omega^S \notin H$. **q.e.d.**

By the proposition we have that the set of Pseudo-Stackelberg states is nonempty.

Corollary 1 *Given* $\theta \in (0,1)$ *and assumption 1. Then* $H \neq \emptyset$.

Above proposition shows that the Cournot state qualifies as a Pseudo-Stackelberg state whereas the competitive state does not. If c is strictly convex then the Pseudo-Stackelberg solution is a Pseudo-Stackelberg state. If c is not strictly convex condition (iv) might be violated. To see this assume that costs are linear (weakly convex) then imitators make zero profits when price equals marginal costs. The optimizers' best response is zero output. Then imitators are indifferent between zero output and q^S. If costs are strictly convex then imitators make strict positive profits and optimizers set a positive output level which is lower than q^S. Thus each optimizer makes less profit than any imitator.

5.3 Preliminaries

Before we can turn to the main result of the paper we have to consider some technical preliminaries analogous to subsection 4.1.

Remark 4 *Suppose* $\exists q^\circ \in \Gamma$. *Then uniqueness of* q° *is obtained if for* $q'_{-i} < q_{-i}$, $q' \in b(q'_{-i})$, $q \in b(q_{-i})$,

$$\frac{q' - q}{q'_{-i} - q_{-i}} > -1 \tag{14}$$

A simple proof of the remark can be found in [68] Vives (2000), theorem 2.8. Note that condition 14 is equivalent to $q'_{-i} + q' < q_{-i} + q$. The requirement is that the best reply correspondence must have slopes strictly above -1. It means that total output is strictly increasing in the opponents' output. This uniqueness condition will play a role in the proof of below lemma.

5.4 The Main Result

The following result provides a first characterization of the support of the long run distribution for a heterogeneous population of imitators and optimizers. I show that the support is a subset of the set of Pseudo-Stackelberg states. This implies immediately that in the average imitators are weakly better off than optimizers. Moreover, I show that the Pseudo-Stackelberg solution is in the set of Pseudo-Stackelberg states if and only if it is in the support of the long run distribution. This does not mean of course, that the Pseudo-Stackelberg solution is the unique long run outcome as the example 2 following the proof shows.

Theorem 3 *Suppose* $\theta \in (0, 1)$. *Let* S *denote the support of the long run distribution* φ^*. *Then under the previous assumptions*

(i) $S \subseteq H$,

(ii) $\omega^S \in H \Longleftrightarrow \omega^S \in S$.

The proof is divided into two lemmata. The first lemma characterizes the set of absorbing sets. We show that each absorbing set is a singleton and each Pseudo-Stackelberg state is an absorbing state. Since the support of the long run distribution can only be a subset of the set of absorbing sets, part (i) of above theorem follows. The second lemma shows that if the Pseudo-Stackelberg solution is in the set of Pseudo-Stackelberg states I can connect any Pseudo-Stackelberg state to the Pseudo-Stackelberg solution by one suitable mutation. This implies that the Pseudo-Stackelberg solution is in the support of the long run distribution. The other direction is obvious since every state in the support of the long run distribution must be also in the set of absorbing sets, which is the set of Pseudo-Stackelberg states.

Proof Theorem. Recall that Z is the collection of all absorbing sets $A_1, ..., A_k$. Since the support S of the long run distribution φ^* is a subset of Z, result (i) is obtained by following lemma:

Lemma 6 *Given* $\theta \in (0, 1)$. $Z = H$, *that is* $Z = \{\{\omega\} : \omega \in H\}$.

Proof Lemma. First, suppose that some state $\omega \notin H$ is element of an absorbing set A. At least one condition of (i) to (iv) of the definition 8 of Pseudo-Stackelberg states is not satisfied. Thus there will be an incentive either for imitators or optimizers to deviate from their output levels in ω leading in the subsequent periods to a state $\omega' \in H$. Hence $\omega \notin A$, $\forall \omega \notin H$.

Second, we show that every absorbing set $A \subseteq H$ is a singleton. Suppose $\exists \omega', \omega \in A \subseteq H$ such that $\omega' \neq \omega$. $\implies \exists\ m \in \mathbb{N}$, m finite s.t. $p^{(m)}_{\omega\omega'} > 0$. Consider any $i \in I$. Since in $\omega \in A \subseteq H$ by definition 8 (iii) $\pi^I(q^I, q^D, n, \theta) > \pi^D(q^I, q^D, n, \theta)$ or by definition 8 (iv) $\pi^I(q^I, q^D, n, \theta) = \pi^D(q^I, q^D, n, \theta)$ iff $q^I = q^D$ and by (i) of the same definition 8, $q_i = q^I$, $\forall i \in I$, for some $q^I \in \Gamma$, it is true that by D_I no imitator $i \in I$ has an opportunity to deviate from her output level in ω. Now consider any $i \in N \backslash I$. Since in $\omega \in A \subseteq H$ by (ii) of aforesaid definition 8, $q_i = q^D = b(\theta n q^I + ((1-\theta)n - 1)q^D), \forall i \in N \backslash I$, no optimizer has by D_O an incentive to deviate from her best reply output level in ω. Since both types of firms do not deviate in $\omega \in A \subseteq H$, no firm $i \in N$ deviates in any of the following periods. Thus either $p^{(m)}_{\omega\omega'} = 0, \forall m \in \mathbb{N}$ which contradicts that $\omega', \omega \in A$, or $\omega' = \omega$. By above arguments it follows that $p_{\omega\omega} = 1$ for each $\omega \in H$ such that $\{\omega\} = A$, $\forall A \in H$. **q.e.d.**

Lemma 7 *Suppose* $\omega^S \in H$. *Then* $M(\omega) \cap B(\{\omega^S\}) \neq \emptyset$, $\forall \{\omega\} = A \subseteq Z \backslash \{\omega^S\}$.

Proof Lemma. Assume $\omega^S \in H$. It is sufficient to show that $\forall q \in \Gamma$, where q is a component of an arbitrary $\omega \in H$, $q \neq q^S$, $k \in \mathbb{N}$, $1 \leq k \leq \theta n$,

$$\begin{aligned} q^S p((\theta n - k)q + kq^S + (1-\theta)nq^D) - c(q^S) > \\ qp((\theta n - k)q + kq^S + (1-\theta)nq^D) - c(q), \end{aligned} \tag{15}$$

where $q^D = b((\theta n - k)q + kq^S + ((1-\theta)n - 1)q^D)$.

By previous lemma 1 inequality 8 holds for $\pi^I(q, Q)$ in (q, Q) on $\Gamma \times \{0, \delta, 2\delta, ..., n\nu\delta\}$. Strict cardinal submodularity (inequality 8) implies strict quasisubmodularity (formulas 10 and 11) of imitator's payoff function. Define $q' := q^S$, $q'' := q$, $Q' := (\theta n - k)q + kq^S + (1-\theta)nq^D$ and $Q'' := \theta n q^S + (1-\theta)nq^{D''}$. If $q' > q''$ then $\theta n q' > (\theta n - k)q'' + kq'$. By symmetry among optimizers and condition 14, I get $q^{D''} > q^{D'}$, $\Longrightarrow Q'' > Q'$. If $q' < q''$ then $\theta n q' < (\theta n - k)q'' + kq'$. By symmetry among optimizers and condition 14, I get $q^{D''} < q^{D'} \Longrightarrow Q'' < Q'$. It follows that the left hand side of $'' \Longrightarrow''$ in formula 10 (11) is given by inequality 12 of the definition of Pseudo-Stackelberg solution (i) if $q < q^S$ (if $q > q^S$). The right hand side of "$\Longrightarrow$" in formula 10(11) yields inequality 15 of above claim if $q < q^S$ (if $q > q^S$).

Set $k = 1$ to see that one suitable mutation only is required to connect every $\omega \in H$ to $\omega^S \in H$. **q.e.d.**

By lemma 2 it follows that if $\omega^S \in H \Longrightarrow \omega^S \in S$. The opposite direction $\Longleftarrow$ is obvious. This completes the proof of the theorem. **q.e.d.**

We can not show generally that $\omega^S \in H$ is the unique long run outcome. To see this suppose we are in ω^S and consider a single mutation where one firm i sets a huge output $\hat{q}$ such that

$$qp(Q - q_i + \hat{q}) - c(q) < 0, \forall q > 0. \tag{16}$$

It is not clear that the induced dynamics lead automatically back to ω^S. Following example 2 may provide some intuition for a possible adjustment process leading to a different state than the Pseudo-Stackelberg solution.

Example 2. Consider the same numerical setting as in the example 1. Below table provides an example of a possible adjustment process. The last column indicates the actions taken in that period.

t	q_1	π_1	q_2	π_2	q_O	π_O	Actions
1	1	4.5	1.5	6.375	2.5	9.375	arbitrary starting state
2	2.5	3.125	2.5	3.125	2.5	3.125	1 and 2 mimic O
3	2.5	5.21	2.5	5.21	1.67	4.17	O adjusts best response
4	2.86	4.41	2.5	4.31	1.67	3.57	1 innovates with q^S
5	2.86	3.73	2.86	3.73	1.55	3.03	2 mimics, O adjusts
6	2.86	4.07	2.86	4.07	1.43	3.05	O adjusts, reaching ω^S
7	6	−19.72	2.86	−4.91	1.43	−1.43	mistake by 1
8	6	−2.56	1.43	2.65	0	0	2 mimics O, O adjusts
9	1.43	7.95	1.43	7.95	0.86	5.03	1 mimics 2, O adjusts
10	1.43	5.78	1.43	5.78	2.38	8.51	O adjusts
11	2.38	3.96	2.38	3.96	2.38	3.96	1 and 2 mimic O
12	2.38	5.48	2.38	5.48	1.75	4.57	O adjusts, absorb. state

O denotes the optimizer, index 1 and 2 the first and second imitator

In $t = 1$ the process starts in an arbitrary state. The optimizer happens to make the highest profit. Therefore the imitators mimic her output decision in $t = 2$. In $t = 3$ the optimizer sets her best response to the output in $t = 2$. In $t = 4$ the first imitator innovates with the Pseudo-Stackelberg output. Since she makes most of the profit the second imitator follows suit in $t = 5$ and the optimizer adjusts best response with a one period lag till in period $t = 6$ the Pseudo-Stackelberg solution is reached. Note, that when jumping to the Pseudo-Stackelberg output, an imitator can decrease her profit. However, at the same time all other firms are worse off. In $t = 7$ the first imitator suddenly mutates and sets a large output $q_1 = 6$. Profits are negative but optimizer's loss is less than the imitators' loss. That's why in $t = 8$ the second imitator, who gets the chance the adjust her output, mimics the optimizer who in turn sets best response to opponents' outputs of previous period. In round $t = 9$ the first imitator can adjust her output and mimics the second imitator. The optimizer sets best response to previous round. Since both imitators are happy they don't adjust output in round $t = 10$. However, the optimizer sets best response to previous round making higher profits than imitators. This situation leads the imitators to mimic optimizer's output decision in round $t = 11$. In round $t = 12$ the optimizer adjusts her best response. In this last move the optimizer can improve her profit but improves the profits of imitators even more. The dynamics is back in an absorbing state that is different to the Pseudo-Stackelberg solution in period $t = 6$.

5.5 A Refined Result

One may wonder whether the long run distribution can be characterized further. In particular one want to have an exact characterization of its support S. To obtain a stronger result it is convenient to introduce following assumptions.

Assumption 6 (Quasiconcavity) *Let π be quasiconcave in q_i, $\forall i \in I$. That is $\forall q, q' \in \Gamma$, $\forall \lambda \in [0, 1]$*

$$\pi(\lambda q + (1 - \lambda q'), Q) \geq \min\{\pi(q, Q), \pi(q', Q)\}, \forall Q \in \{0, \delta, 2\delta, ..., n\nu\delta\}. \quad (17)$$

The intuition for quasiconcavity is single-peakedness. The set of maxima is convex. If the inequality holds strict, then π is strict quasiconcave and the maximum is unique. Concavity and weak concavity implies quasiconcavity but not vice versa.

Definition 9 (Monopoly Solution) *A quantity* $q^\bullet \in \Gamma$ *is a monopoly solution if*

$$q^\bullet p(q^\bullet) - c(q^\bullet) \geq qp(q) - c(q), \forall q \in \Gamma. \tag{18}$$

Assumption 7 *Let* π *such that*

$$q^\circ p((2n-3)q^\circ) - c(q^\circ) \leq 0, \tag{19}$$

$$(n-1)q^\circ p((n-1)q^\circ) - c((n-1)q^\circ) \leq 0, \tag{20}$$

$$(q^\bullet + \delta)p(\theta n(q^\bullet + \delta) + (1-\theta)nq^D) - c(q^\bullet + \delta) \leq 0. \tag{21}$$

Inequality 19 means that if one firms sets zero output, another firm sets $n-1$ times the Cournot Nash equilibrium quantity and all remaining $n-2$ firms set each the Cournot Nash equilibrium output, all firms with strict positive output make nonpositive profits. A stronger but a bit more intuitive assumption would be to say that if twice the total number of firms set Cournot Nash equilibrium output then no firm setting Cournot Nash equilibrium output would make positive profits. Inequality 20 says that any firm operating with $(n-1)q^\circ$ make nonpositive profits. Inequality 21 means that if all imitators set more than monopoly output $q^\bullet$, each imitator makes nonpositive profits. These assumptions may easily satisfied if for instance n is sufficiently large. In particular all assumptions are satisfied if for example $p(Q) = 10 - Q$, $c(q_i) = q_i^2$, and $n \geq 5$.

Definition 10 *Define* $\bar{\omega} \in H$ *as the Pseudo-Stackelberg state with the largest possible (identical) output of imitators, that is*

$$\begin{aligned} &\bar{q}p(\theta n\bar{q} + (1-\theta)nq^D) - c(\bar{q}) > \\ &q^D p(\theta n\bar{q} + (1-\theta)nq^D) - c(q^D), \\ &\textit{where } q^D = b(\theta n\bar{q} + ((1-\theta)n-1)q^D) \end{aligned} \tag{22}$$

$$\begin{aligned} &(\bar{q}+\delta)p(\theta n(\bar{q}+\delta) + (1-\theta)nq^{D_\delta}) - c(\bar{q}+\delta) \leq \\ &q^{D_\delta} p(\theta n(\bar{q}+\delta) + (1-\theta)nq^{D_\delta}) - c(q^{D_\delta}), \\ &\textit{where } q^{D_\delta} = b(\theta n(\bar{q}+\delta) + ((1-\theta)n-1)q^{D_\delta}) \end{aligned} \tag{23}$$

Assumption 8 $\exists$ *unique* $\bar{\omega} \in H$.

It is easy to see that $\bar{\omega}$ is indeed in H, since by definition all imitators set identical output $\bar{q}$, all optimizer play best response q^D, and imitators make strictly higher profits than optimizers. It is also easy to see that $\bar{q}$ is the largest identical output imitators can set in the play of Pseudo-Stackelberg states since an increase by δ yields a state not in H. It is also easy to see that by assumption 7, inequality 21, we have $\bar{q} \leq q^{\bullet}$.

Assumption 6, 7, and 8 are sufficient conditions for the following clear cut result.

Theorem 4 *Suppose $\theta \in (0,1)$. By previous assumptions we have $S = H$.*

The result says that given previous assumptions the support of the long run distribution is exactly the set of Pseudo-Stackelberg states. That is in a population of heterogeneous firms the long run outcome is not unique.

The proof is divided into three lemmata. The first lemma shows that I can connect any Pseudo-Stackelberg state to the Cournot state by just one suitable mutation. The second lemma shows that I can connect also any Pseudo-Stackelberg state to the Pseudo-Stackelberg state with the largest possible output of imitators by just one suitable mutation. The third lemma shows that I can find a sequence of Pseudo-Stackelberg state such that I can move through them by single mutation. That is along the sequence of Pseudo-Stackelberg states I can jump from Pseudo-Stackelberg state to Pseudo-Stackelberg state by single suitable mutations. This implies that the set of Pseudo-Stackelberg states is the unique recurrent set. Hence it is the support of the long run distribution. The assumption of quasiconcavity in connection with the observation of quasisubmodularity ensures that those jumps are indeed possible.

Proof Theorem. We show that the recurrent set $R = Z$. Recall that by lemma 6 we know $Z = H$.

Lemma 8 *Suppose $\theta \in (0,1)$ and that assumptions 1, 2, 4, and 7 hold. Then $M(A) \cap B(\{\omega^{\circ}\}) \neq \emptyset$, $\forall A \in H \backslash \{\omega^{\circ}\}$*

Proof Lemma. Suppose in t any arbitrary state $\omega(t) \in H$. W.l.o.g. assume that by $\varepsilon > 0$ and $\rho \in (0,1)$ in $t+1$ a mutation by one firm $i \in N$ such that $q_i(t+1) = (n-1)q^{\circ}$. $Q(t+1) > (2n-3)q^{\circ}$. By assumption 7, inequality 19, $\pi_i(t+1) < 0$, $\forall j \in N$. W.l.o.g. assume that by $\rho \in (0,1)$ a firm $k \in N \backslash I$ has the opportunity to adjust output in $t+2$. Since $D_O(t+1) = 0$, we have $q_k(t+2) = 0$. $Q(t+1) \geq (2n-3)q^{\circ}$. By assumption 7, inequality 19, $\pi_j(t+1) < 0$, $\forall j \in N \backslash \{k\}$. W.l.o.g. assume that by $\rho \in (0,1)$ and $D_O(t+2) = D_I(t+2) = 0$ all $j \in N \backslash \{i\}$ adjust output in $t+3$ such that $Q(t+3) = q_i(t+3) = q_i(t+1) = (n-1)q^{\circ}$. By assumption 7, inequality 20, $\pi_i(t+3) \leq 0$. W.l.o.g. assume that by $\rho \in (0,1)$ and $D_O(t+3) = b((n-1)q^{\circ}) = q^{\circ}$ a firm $k \in N \backslash I$ has the opportunity to adjust output in $t+4$. Since

$\pi_k(t+4) > \pi_j(t+4)$, $j \in N\backslash\{k\}$ we can assume w.l.o.g. that by $\rho \in (0,1)$ and $D_I(t+4) = q^\circ$ all $j \in I$ adjust output. In the subsequent periods let all remaining optimizers adjust output such that with positive probability ω° is reached in finite time. Since we started in any arbitrary state $\omega(t) \in H$ (in particular also $\omega^S \in H$) we have shown that $M(A) \cap B(\{\omega^\circ\}) \neq \emptyset$, $\forall A \in Z\backslash\{\omega^\circ\}$. **q.e.d.**

Remark 5 *$\omega^\circ \in H$ is the state with the lowest (identical) output of imitators in the set of Pseudo-Stackelberg states H.*

Lemma 9 *Suppose $\theta \in (0,1)$, assumptions 2, 4, 7, and 8. Then $M(A) \cap B(\{\bar{\omega}\}) \neq \emptyset$, $\forall A \in H\backslash\{\bar{\omega}\}$.*

Proof Lemma. Assume for a moment that $\bar{q} = q^\bullet$. Suppose in t any arbitrary state $\omega(t) \in H$. W.l.o.g. assume that by $\varepsilon > 0$ and $\rho \in (0,1)$ in $t+1$ a mutation by imitator $i \in I$ occurs setting $\hat{q}_i$ (see inequality 16) such that $\pi_j(t+1) < 0$, $\forall j \in N$. W.l.o.g. assume that by $\rho \in (0,1)$ all optimizers in $N\backslash I$ have the opportunity to adjust output in $t+2$. Since $D_O(t+1) = 0$, we have $q^D(t+2) = 0$ with $\pi^D(t+2) = 0$. By assumption 7, inequality 21, we have $\pi^I(t+2) < \pi^D(t+2)$. W.l.o.g. assume that by $\rho \in (0,1)$ all imitators in I adjust output in $t+3$ to $D_I(t+2) = q^I(t+3) = 0$. Hence, $Q(t+3) = 0$. W.l.o.g. assume that by $\rho \in (0,1)$ in $t+4$ an optimizers k adjusts output such that $D_O(t+3) = b(0) = q^\bullet(t+4)$. W.l.o.g. assume that by $\rho \in (0,1)$ in $t+5$ all imitators in I adjusts output such that $D_I(t+4) = q^\bullet(t+5)$. Let all optimizers in $N\backslash I$ adjust output in the subsequent periods such that a state $\omega^\bullet = (q_1^\bullet, ..., q_{\theta n}^\bullet, q_{\theta n+1}^D, ..., q_n^D)$ is reached in finite time. Above we assumed for the moment $\bar{q} = q^\bullet$. Hence $\omega^\bullet = \bar{\omega}$.

If $\bar{q} < q^\bullet$ then there exists an adjustment path similar to above were at $t+s$, we have $\bar{q} = b(q_{-i})$, for some $q_{-i} \in \{0, \delta, 2\delta, ..., n\nu\delta\}$, δ arbitrary small.[26]

Since by assumption 7, inequality 21, $\bar{q} \leq q^\bullet$, we have shown that $\bar{\omega}$ can be reached by a single suitable mutation from any arbitrary ω. **q.e.d.**

Lemma 10 *Suppose $\theta \in (0,1)$ and previous assumptions. Then for any two distinct absorbing sets A_1 and A_m in H there exists a sequence of absorbing sets $A_2, ..., A_{m-1}$ in H such that $M(A_j) \cap B(A_{j+1}) \neq \emptyset$ for $j = 1, ..., m-1$.*

Proof Lemma. By lemma 6 we know each absorbing set is singleton in H.

[26] We have just shown that we can reach by a single mutation $q^\bullet$ since we could construct some sequence of adjustments in which monopoly output is once best reply. It is clear that we can make the grid arbitrary fine such that $\bar{q}$ is best reply to some output.

- Case $q^S \geq \bar{q}$: Define a sequence of absorbing states such that $\omega_1 = \omega^\circ$, ω_2 s.t. $q^I = q^\circ + \delta$, ω_3 s.t. $q^I = q^\circ + 2\delta$, ..., ω_{m-1} s.t. $q^I = \bar{q} - \delta$ and $\omega_m = \bar{\omega}$. In order to show $M(\omega_j) \cap B(\{\omega_{j+1}\}) \neq \emptyset$ for $j = 1, ..., m-1$ we have to show for $k = 1$, $\forall q \in [q^\circ, \bar{q}) \subset \Gamma$,

$$(q+\delta)p((\theta n - k)q + k(q+\delta) + (1-\theta)nq^D) - c(q+\delta) > \\ qp((\theta n - k)q + k(q+\delta) + (1-\theta)nq^D) - c(q), \qquad (24)$$

  where $q^D = b((\theta n - k)q + k(q+\delta) + ((1-\theta)n - 1)q^D)$.

  By lemma 1, π is strictly cardinal submodular (inequality 8). It implies strict quasisubmodularity (formula 10 and 11) of imitators' payoff functions. Define $q' = q + \delta$, $q'' = q$, $Q' = (\theta n - k)q + k(q+\delta) + (1-\theta)nq_D$ and $Q'' = \theta n(q+\delta) + (1-\theta)nq^{D''})$. Then the right hand side of "$\Longrightarrow$" in the upper formula 10 is equivalent to inequality 24 if $q^S \geq \bar{q}$.

  What is left to show is that the left hand side of "$\Longrightarrow$" in the upper formula 10 is implied by the assumption 6 of quasiconcavity. To see this note that for each $q \in [q^\circ, \bar{q}] \subset \Gamma$ there exists a $\lambda \in [0,1]$ s.t. $q + \delta = \lambda q + (1-\lambda)\bar{q}$. Since $q^S \geq \bar{q}$ we have $\min\{\pi(q,Q), \pi(\bar{q},Q)\} = \pi(q,Q)$, $\forall q \in [q^\circ, \bar{q}]$, $\forall Q \in \{0, \delta, 2\delta, ..., n\nu\delta\}$. Hence $\pi(q+\delta, Q'') \geq \pi(q, Q'')$, $\forall q \in [q^\circ, \bar{q}]$, which is equivalent to the left hand side of "$\Longrightarrow$" in the upper formula 10. Thus we have shown that there exists a sequence of absorbing states through which we can move from the absorbing state with the lowest output of imitators ω° to the absorbing state with the highest output of imitators $\bar{\omega}$ by a sequence of single suitable mutations. By lemma 8 we can also move from $\bar{\omega}$ to ω° by a single suitable mutation. Hence in this case $H = R$.

- Case $q^S < \bar{q}$: Partition $[q^\circ, \bar{q}] \subset \Gamma$ into $\{[q^\circ, q_S), q^S, (q^S, \bar{q}]\}$. For any $q \in [q^\circ, q^S)$ we can show inequality 24 analogous to previous case.

  Consider the interval $(q^S, \bar{q}]$. Define a sequence of absorbing states such that $\omega_1 = \bar{\omega}$, ω_2 s.t. $q^I = \bar{q} - \delta$, ω_3 s.t. $q^I = \bar{q} - 2\delta$, ..., ω_{m-1} s.t. $q^I = q^S + \delta$ and $\omega_m = \omega^S$. In order to show $M(\omega_j) \cap B(\{\omega_{j+1}\}) \neq \emptyset$ for $j = 1, ..., m-1$ we have to show for $k = 1$, $\forall q \in (q^S, \bar{q}] \subset \Gamma$,

$$(q-\delta)p((\theta n - k)q + k(q-\delta) + (1-\theta)nq^D) - c(q-\delta) > \\ qp((\theta n - k)q + k(q-\delta) + (1-\theta)nq^D) - c(q), \qquad (25)$$

  where $q^D = b((\theta n - k)q + k(q-\delta) + ((1-\theta)n - 1)q^D)$.

  By lemma 1, π is strictly cardinal submodular (inequality 8). It implies strict quasisubmodularity (formula 10 and 11) of imitators' payoff functions. Define $q' = q-\delta$, $q'' = q$, $Q' = (\theta n-k)q+k(q-\delta)+(1-\theta)nq_D$ and $Q'' = \theta n(q-\delta)+(1-\theta)nq^{D''}$. Since $q^S < \bar{q}$ the right hand side of "$\Longrightarrow$" in the lower formula 11 is equivalent to inequality 25.

What is left to show is that the left hand side of "$\Longrightarrow$" in the lower formula 11 is implied by the assumption 6 of quasiconcavity. To see this note that for each $q \in [q^S, \bar{q}] \subset \Gamma$ there exists a $\lambda \in [0,1]$ s.t. $q-\delta = \lambda q + (1-\lambda)q^S$. Since $q^S < \bar{q}$ we have $\min\{\pi(q,Q), \pi(q^S,Q)\} = \pi(q,Q)$, $\forall q \in [q^S, \bar{q}]$, $\forall Q \in \{0, \delta, 2\delta, ..., n\nu\delta\}$. Hence $\pi(q-\delta, Q'') \geq \pi(q, Q'')$, $\forall q \in [q^S, \bar{q}]$, which is equivalent to the left hand side of "$\Longrightarrow$" in the lower formula 11.

Thus we have shown that there exists a single-mutation sequence of absorbing states through which we can move from the absorbing state with the lowest output of imitators ω° to ω^S and a sequence from the absorbing state with the highest output of imitators $\bar{\omega}$ to ω^S. By lemma 8 and 9 we can also move by single suitable mutations from $\bar{\omega}$ to ω° and vice versa. Hence also in this case $H = R$. **q.e.d.**

By lemma 2 $Z = H = R$ implies that $S = H$. This completes the proof of the theorem. **q.e.d.**

Remark 6 *If $\omega^S \notin H$ then $q^S > \bar{q}$.*

To see above remark, notice that the Pseudo-Stackelberg solution ω^S satisfies conditions (i) to (iii) of the Pseudo-Stackelberg state per definition. Hence if $\omega^S \notin H$ then the condition (iv) of the definition 8 of Pseudo-Stackelberg states must be violated. It is violated if $\pi^I(q^S, q^D, n, \theta) = \pi^D(q_S, q_D, n, \theta)$ but $q^S \neq q^D$. Then by definition 10 of $\bar{\omega}$, inequality 23 we must have $q^S \geq \bar{q} + \delta$. Hence $q^S > \bar{q}$.

5.6 Discussion

In [67] Vega-Redondo's (1997) case of imitators one could see that all absorbing states were connected to the competitive solution by one mutation only. Similar I can show in theorem 3 that if the Pseudo-Stackelberg solution exists in H, it can be reached from any other absorbing state by just one suitable mutation. However, as I showed in example 2 and more generally in theorem 4 it does not imply that the Pseudo-Stackelberg solution is generally the unique long run outcome since also the Pseudo-Stackelberg solution can be destabilized by just one suitable mutation. Thus the long run distribution does not need to settle on a singleton support in H. In particular one can show under some additional mild assumptions that $S = H$, that is the support of the unique long run distribution is the entire set of Pseudo-Stackelberg states. This is in contrast to previous results on long run outcomes in homogeneous populations of either imitators or optimizers. My results imply that [67] Vega-Redondo's (1997) result is not robust against a single optimizer and the Cournot result is not robust against a single imitator in the population of firms.

We showed that by assumption of quasiconcavity there exists a sequence of Pseudo-Stackelberg state such that we can move through them by single mutation. That is along a sequence of Pseudo-Stackelberg states we can jump from Pseudo-Stackelberg state to Pseudo-Stackelberg state by single suitable mutations. This implies:

Remark 7 *For heterogeneous or homogeneous populations ($\theta = [0, 1]$) at most one suitable mutation is required to trigger convergence to the long run outcome.*

My results appear to be robust in the sense that essentially just one suitable mutation is required to induce the results. In other words, the results do not depend on a multiplicity of highly unlikely mutations. Thus the convergence to the long run outcome is rather fast.

The Pseudo-Stackelberg states being essentially the support of the stochastically stable solution is puzzling in the regard that the more rational firm, the optimizer, is at least weakly worse off than the imitator. The optimizer has to solve a strategic optimization problem taking into account the demand function, cost function and the outputs of all other firms. The imitator just needs to know the level of output corresponding to the firm(s) with the highest profit. In Stackelberg à la subgame perfect equilibrium, the leader must know more than the follower since her optimization problem requires knowledge of the followers' reaction correspondence. The usual conclusion in industrial organization is that it pays to know something about opponents. This is contrary to my model where similarly as in Stackelberg games imitators are able to slip in the role of independent firms although they appear to be less sophisticated than optimizers. This sort of thought leads to a couple of interesting questions such as when are cognitive bounded players more successful than fully rational ones, when doesn't it pay to learn, etc.

Note that above result is made possible because firms are committed to their decision rules per definition. That is I do not consider any endogenous change of decision rules. However, the result of such endogenous change of decision rule such as in a population game would be clear. If for example the survival probability of the firm or decision rule dependents monotonically on the profits, then I conjecture that optimizers would be driven out of the market in finite time, since imitators are relatively better off than optimizers. To see this note, that being an imitator weakly dominates being an optimizer. Although an optimizer in a homogeneous population makes higher profits than an imitator in a homogeneous population, she is weakly worse off in a heterogeneous population. To put it differently, an imitator makes less profits in a homogeneous population than an optimizers but is weakly better off in a heterogeneous population. Hence, [19] Friedman's (1953) arguments have to be rejected in this model. It seems that from an evolutionary view in this model imitation behavior provides strong support for competitive outcomes in the long run in markets where market power could be exercised, no matter whether initially some firms were actually able to exercise market power or not. Nevertheless, these conclusions remain to be shown explicitly in further research.

One could argue that the optimizers in this model are not very rational. Why do they only take the previous period into account when taking their output decisions? Why don't they form more or less sophisticated beliefs about the output of others? Suppose we would allow optimizers to take a longer history of output decisions into account when deciding which output level to set. Then results would not change much but convergence may be slower since their adjustment process is similar to ficticious play[27]. Suppose now that optimizers could form some sort of sophisticated beliefs about the actions of others. Assume that they can forecast the actions of others perfectly. What does it help them if imitators set some large output, which they will do in finite time by the noise assumed? All the optimizers can do is to play best reply against their beliefs such that they play a small output with smaller profits than imitators.

Finally, above discussion should make clear that "benchmarking", "best practices" and other kinds of imitating behavior in reality may not be bad decisions rules. "Not be bad" is meant in a double sense. As long as there is a heterogeneous population, imitation behavior may make firms better off than optimization. Once there are only imitators left, we converge to the best outcome in terms of welfare considerations.

6 Conclusion

In this paper I study imitators and optimizers in a symmetric n-firm Cournot oligopoly. Imitators mimic the output decision of the most successful firm(s) in the previous period. Optimizers play best reply to opponents' output

[27]Regarding ficitious play refer for example to [20] Fudenberg/Levine (1998), pp. 29

of the previous period. As expression of bounded rationality firms are allowed to make small mistakes. The dynamics and noise induces a perturbed Markov chain. I study long run distributions as the noise goes to zero. For a homogeneous population of imitators the long run outcome is the competitive solution. I provide a simpler proof of [67] Vega-Redondo's (1997) result based on submodular games and Noeldeke/Samuelson recurrent sets. An analogous result of convergence to Cournot Nash equilibrium is proved for a homogeneous population of optimizers. For a heterogeneous population of imitators and optimizers the support of the long run distribution is a set of states where imitators are weakly better off than optimizers. The intuition is that imitators may set independently higher outputs than Cournot Nash equilibrium output pushing optimizers in a position similar to the dependent firm in Stackelberg settings.

The conclusions can be summarized as follows:

- [67] Vega-Redondo's (1997) result of convergence to the competitive solution is not robust against a single optimizer in the population of firms.
- Convergence to Cournot Nash equilibrium in the Cournot tatonnement is not robust against a single imitator in the population of firms.
- In the average, imitators are better off than optimizers. It does not always pay to be more intelligent.

Based on the conclusions we can conjecture that

- Imitators may drive out optimizers in the long run. Taking such selection process into account the competitive solution may be the unique long run outcome. [19] Friedman's (1953) argument for assuming firms as "as if" profit maximizers, since otherwise they are driven out of the market, appears to be false in this model.
- "Benchmarking", "best practices" and other imitating behavior in reality may not be bad decision rules.
- The results appear to be robust even if optimizers are more sophisticated than myopic best responders.

Since both results - on imitation behavior as well as on best response - are supported by experimental findings in Cournot markets ([51] Sauermann/Selten, 1959, [25], [24] Huck/Normann/Oechssler, 1999, 2000) depending on the amount of information provided to subjects, it is only natural to test whether my results can be supported experimentally if different amounts of information is given to various firms in an oligopoly experiment. This and an explicit derivation of above conjectures is left for further research.

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