Further results on the asymptotics for nonlinear transformations of integrated time series*

Robert M. de Jong and Chien-Ho Wang
Department of Economics
217 Marshall Hall
Michigan State University
East Lansing, MI 48824

February 2, 2002

Abstract

This paper establishes various results involving functions of integrated processes that all build upon recent work by Park and Phillips. First, a result is established for the average of a locally integrable function of a rescaled integrated process, and the function under consideration is explicitly allowed to have pole. Two additional results are given that characterize asymptotic behavior in the case of non-integrable poles; the observations close to the pole take over asymptotic behavior in that case. Furthermore, two theorems - that improve similar results by Park and Phillips - are proven for averages of functions of an integrated process that has not been rescaled by the square root of sample size. Throughout, we make the assumption that the innovations of the integrated process are a linear process.

*This version was prepared for the ESEM conference, August 2002, Venice. The authors thank Peter Phillips for comments and for suggesting the result of Theorem 1.

1 Introduction

In recent work by Park and Phillips (see Park and Phillips (1999)), a range of new asymptotic results involving functions of integrated processes is proven. This paper seeks to extend those results into various directions. First, a new result involving averages of functions of scaled integrated processes is proven. Furthermore, results are proven regarding averages of non-integrable functions of integrated processes, and another result is conjectured. Finally, we prove two theorems involving averages of functions of integrated processes that have not been rescaled by the square root of sample size.

For our first result, note that the continuous mapping theorem ensures that for I(1) processes x_t,

$$n^{-1} \sum_{t=1}^{n} T(n^{-1/2} x_t) \xrightarrow{d} \int_0^1 T(\sigma W(r)) dr \tag{1}$$

for continuous functions $T(.)$, where $\xrightarrow{d}$ denotes convergence in distribution, $n^{-1/2} x_{[rn]} \Rightarrow \sigma W(r)$ where $W(.)$ denotes Brownian motion and "$\Rightarrow$" denotes weak convergence, and $\sigma^2 = \lim_{n\to\infty} n^{-1} E x_n^2$. Theorem 3.2 of Park and Phillips (1999) shows that for functions that are "regular" (as defined in that paper) and locally integrable, the result of Equation (1) still holds. Among the functions that are dealt with by that theorem is, for example, $T(x) = I(x > 0)$. Park and Phillips' Theorem 3.2 however fails to show that for all $\alpha > -1$,

$$Z_n = n^{-1} \sum_{t=1}^{n} |n^{-1/2} x_t|^\alpha \xrightarrow{d} \int_0^1 |\sigma W(r)|^\alpha dr, \tag{2}$$

and also fails to establish the result of Equation (1) for $T(x) = \log |x|$. For $\alpha \geq 0$, the above result follows from the continuous mapping theorem in a straightforward way, because $|x|^\alpha$ is continuous on $\mathbb{R}$ for $\alpha \geq 0$. But for $-1 < \alpha < 0$, the pole at 0 will make it impossible to apply the continuous mapping theorem directly, and this paper shows that under regularity conditions, the result of Equation (2) still holds. Note that since the right-hand side of Equation (2) is undefined for $\alpha \leq -1$, there is no hope of proving the result of Equation (2) for such values of α. Instead of the result of Equation (2), Park and Phillips (1999) showed that for $\alpha > -1$,

$$n^{-1} \sum_{t=1}^{n} |n^{-1/2} x_t|^\alpha I(|n^{-1/2} x_t| > \delta_n) \xrightarrow{d} \int_0^1 |\sigma W(r)|^\alpha dr \tag{3}$$

for a sequence δ_n that converges to 0 as $n \to \infty$. In this paper, we prove the result of Equation (1) for a function class that is different from that of Park and Phillips, and we show that the result of Equation (2) holds under the exact same assumptions that were needed in Park and Phillips (1999) for the proof of the result of Equation (3). Among those assumptions is one that guarantees that the increments w_t of x_t have an absolutely continuous distribution. In the main theorem, we show that as long as it is possible to "integrate over the poles" of the function $T(.)$, the result of Equation (1) can still be proven.

Of course, the above analysis leaves the interesting open question as to what will happen if the function $T(.)$ possesses a pole that cannot be integrated over. We present two results that illustrate that we should not expect a simple modification of the result of Equation (1) to hold in such a situation, but that instead, the limit behavior will be determined by the observations close to the pole of $T(.)$.

Also in Park and Phillips (1999), it is proven that for integrable functions $T(.)$ and for I(1) processes x_t,

$$n^{-1/2}\sum_{t=1}^{n} T(x_t) \xrightarrow{d} \left(\int_{-\infty}^{\infty} T(s)ds\right)L(1,0), \tag{4}$$

where $L(t,s)$ is a two-parameter stochastic process called *local time*. The remarkable thing about this result is that it establishes limit theory for a function of an I(1) process that has not been rescaled by $n^{-1/2}$. Park and Phillips establish the above result under some regularity conditions on the I(1) process x_t and the integrable function $T(.)$. In this paper, we show that Park and Phillips' regularity conditions for the above result can be relaxed.

Another result that is proven in Park and Phillips (1999) is that for functions $T(.)$ that satisfy

$$T(\lambda x) = \nu(\lambda)H(x) + R(x,\lambda) \tag{5}$$

under conditions on $R(.,.)$ that basically serve to ensure asymptotic negligibility of

$$n^{-1}\sum_{t=1}^{n} R(x_t, n^{1/2}), \tag{6}$$

we have

$$\nu(n^{1/2})^{-1}n^{-1}\sum_{t=1}^{n} T(x_t) \xrightarrow{d} \int_0^1 H(\sigma W(r))dr. \tag{7}$$

Such functions are coined *asymptotically homogeneous* by Park and Phillips. For this result also, the interesting aspect is the fact that a function of an integrated process is considered that has not been rescaled by $n^{-1/2}$. The asymptotically homogeneous condition is trivially satisfied for $T(x) = |x|^a$ for $a \geq 0$, but is general enough to also deal with functions such as $T(x) = |x|^a \log(x)$ for all $a \geq 0$. In this paper, we show the more general result that whenever for functions $H(.)$ and $\nu(.)$ we have

$$\nu(\lambda)^{-1}T(\lambda x) \to H(x) \quad \text{as } \lambda \to \infty \tag{8}$$

in L_1 sense, the result of Equation 7 follows. This criterion is then shown to yield a more general result than Park and Phillips' result for asymptotically homogeneous functions.

2 Assumptions and lemma

In this paper, as in Park and Phillips (1999), it is assumed that

$$x_t = x_{t-1} + w_t, \tag{9}$$

where w_t is generated according to

$$w_t = \sum_{k=0}^{\infty} \phi_k \varepsilon_{t-k} \tag{10}$$

where ε_t is assumed to be a sequence of i.i.d. random variables with mean zero, and where it is assumed that $\sum_{k=0}^{\infty} \phi_k \neq 0$. In addition, we will assume that x_0 is an arbitrary random variable that is independent of all w_t.
The main assumptions used in this paper are Assumption 2.1 and 2.2 from Park and Phillips (1999):

Assumption 1 $\sum_{k=0}^{\infty} k^{1/2}\phi(k) < \infty$ *and* $E\varepsilon_t^2 < \infty$.

Assumption 2

(a) $\sum_{k=0}^{\infty} k|\phi_k| < \infty$ *and* $E|\varepsilon_t|^p < \infty$ *for some* $p > 2$.

(b) *The distribution of* ε_t *is absolutely continuous with respect to the Lebesgue measure and has characteristic function* $\psi(s)$ *for which* $\lim_{s\to\infty} s^\eta \psi(s) = 0$ *for some* $\eta > 0$.

Assumption 1 guarantees that $n^{-1/2}x_{[rn]} \Rightarrow \sigma W(r)$, while Assumption 2 in addition also guarantees a convergence rate for a Skorokhod representation of $n^{-1/2}x_{[rn]}$. Several of the manipulations in the proofs of the results in this paper require the use of local time $L(.,.)$. Local time is a random function satisfying

$$L(t, s) = \lim_{\varepsilon\to 0}(2\varepsilon)^{-1} \int_0^t I(|W(r) - s| < \varepsilon)dr. \tag{11}$$

See Park and Phillips (1999, p. 271-272) and Chung and Williams (1990, ch. 7) for more details regarding local time. For the proofs, we will also need to define, analogously to Park and Phillips (1999),

$$N_n(\nu_n; a, b) = \int_0^1 I(a \leq \nu_n W_n(r) \leq b) dr$$

and

$$N(\nu_n; a, b) = \int_0^1 I(a \leq \nu_n W(r) \leq b) dr,$$

where $W(.)$ denotes Brownian motion and $W_n(r) = \sigma^{-1} n^{-1/2} x_{[nr]}$.
In order to prove the two of the three main results of this paper, we needed the following key lemma:

Lemma 1 *Under Assumption 2, for all $y \in \mathbb{R}$, $\delta > 0$, and $n \geq N$ for some value of N,*

$$P(y \leq n^{-1/2} x_n \leq y + \delta) \leq C\delta, \tag{12}$$

where C and N do not depend on y, δ, or n.

Obviously, a result such as Lemma 1 will fail to hold for the discrete-valued x_t example at the end of the Introduction.

3 Functions of integrated processes with integrable poles

Typically in the econometrics literature, it is argued that for continuous $T(.)$, we will have the result from Equation (1) from an appeal to the continuous mapping theorem. In the theorem below, we show that local integrability of $T(.)$ is sufficient as a key condition:

Theorem 1 *Assume that $\int_{-K}^{K} |T(x)| dx < \infty$ for all $K > 0$. In addition, assume that for some $q \geq 1$ there exists a grid $\{a_1, \ldots, a_q\}$, where $a_j < a_{j+1}$ for all $j = 1, \ldots, q-1$, such that $T(.)$ is continuous at any $x \in \mathbb{R} \backslash \{a_1, \ldots, a_q\}$, and monotone on (a_{j-1}, a_j) for $j = 1, \ldots, q+1$ (defining $a_0 = -\infty$ and $a_{q+1} = \infty$). Then under Assumption 2,*

$$n^{-1} \sum_{t=1}^{n} T(n^{-1/2} x_t) \xrightarrow{d} \int_0^1 T(\sigma W(r)) dr. \tag{13}$$

A monotone function is defined as a function that is either nondecreasing or nonincreasing. A corollary of Theorem 1 is the following result, which renders the improvement of the result of Park and Phillips (1999) mentioned in the Introduction:

Corollary 1 *Under Assumption 2,*

$$n^{-1}\sum_{t=1}^{n}|n^{-1/2}x_t|^{\alpha} \xrightarrow{d} \int_0^1 |\sigma W(r)|^{\alpha} dr \tag{14}$$

for all $\alpha > -1$.

Corollary 1 follows immediately from Theorem 1 by choosing $q = 1$ and $a_1 = 0$, and noting that $|x|^{\alpha}$ is monotone on $(-\infty, 0)$ and $(0, \infty)$ and that $\int_{-K}^{K} |x|^{-\alpha}dx < \infty$ for all $K > 0$ for $\alpha > -1$. Similarly, from Theorem 1 the following result follows:

Corollary 2 *Under Assumption 2,*

$$n^{-1}\sum_{t=1}^{n}\log|n^{-1/2}x_t| \xrightarrow{d} \int_0^1 \log|\sigma W(r)|dr. \tag{15}$$

As far as this author is aware, the result of Corollary 2 has not been established elsewhere. Corollary 2 may be useful for the analysis of inference procedures when a logarithm transformation has been incorrectly applied to a time series process.
Theorem 1 suggests the theoretically interesting question as to what will be "minimal" conditions, in addition to the integrability condition on $T(.)$, for the result of Equation (1) to hold. The proof as presented in the Appendix seems to rely on the assumed monotonicity, but it is unclear whether some additional relaxation of the conditions of Theorem 1 is possible.

4 Functions of integrated processes with nonintegrable poles

The previous section raises the question as to what will happen if a non-integrable function of an integrated process is used for $T(.)$ in the statistic of Equation (1). This issue appears to have never been tackled before in either the statistics or the econometrics literature, and is far from easy. This section explores this issue, but is far from definitive in the authors' minds. This is because the only results that we were able to establish exclude observations "close to" the pole, but admit closer and closer observations as sample size increases. Also, at this time we can only deal with functions of the type

$$T(x) = |x|^{-m} I(x > 0) \tag{16}$$

and

$$T(x) = |x|^{-m}, \tag{17}$$

for $m > 1$, although from our proof it appears that quite some level of generalization should be possible. We will need a "clipping device" and we construct statistics similar to those of Equation (3).

Theorem 2 *Let $c_n = n^{-(2p+1)/3p+\eta}$ for some $\eta > 0$ such that $-(2p+1)/3p + \eta < 0$. In addition, assume that*

$$T(x) = |x|^{-m} I(x > 0). \tag{18}$$

Let $d_n = \int_{c_n}^{1} T(x)dx$. Then under Assumption 2,

$$d_n^{-1} n^{-1} \sum_{t=1}^{n} T(\sigma^{-1} n^{-1/2} x_t) I(\sigma^{-1} n^{-1/2} x_t > c_n) \xrightarrow{d} L(1, 0). \tag{19}$$

Clearly, in the above theorem $d_n = (m-1)^{-1}(c_n^{m-1} - 1)$, but we choose the above formulation to bring out better where our rescaling factor d_n originates from.
The proof of the following "two-sided" version of the above theorem is analogous and therefore omitted:

Theorem 3 *Let $c_n = n^{-(2p+1)/3p+\eta}$ for some $\eta > 0$ such that $-(2p+1)/3p + \eta < 0$. Assume that*

$$T(x) = |x|^{-m} \tag{20}$$

Let $d_n = 2\int_{c_n}^{1} T(x)dx$. Then under Assumption 2,

$$d_n^{-1} n^{-1} \sum_{t=1}^{n} T(\sigma^{-1} n^{-1/2} x_t) I(|\sigma^{-1} n^{-1/2} x_t| > c_n) \xrightarrow{d} L(1, 0). \tag{21}$$

The above theorems leave the issue wide open to what function class the above theorem can be extended. Clearly, the line of proof employed in the Appendix allows for some generalization, but it is not obvious what the outer limits are for which a result as the above might hold. Furthermore, the clipping device is intriguing, and one could conjecture that for the above definitions the theorem will remain true if c_n in the theorem and in the definition of d_n were to be replaced by $\min_{1 \le t \le n} n^{-1/2} x_t I(x_t > 0)$ and $\min_{1 \le t \le n} n^{-1/2} |x_t|$ respectively.

5 Integrable functions

Park and Phillips (1999) establish the following result for integrable functions of integrated random variables:

Theorem 4 *Suppose that $T(.)$ is integrable and Assumption 2 holds with $p > 4$. If $T(.)$ is square integrable and satisfies the Lipschitz condition*

$$|T(x) - T(y)| \leq c|x - y|^l \tag{22}$$

over its support for some constants c and $l > 6/(p - 2)$, then

$$n^{-1/2} \sum_{t=1}^{n} T(x_t) \xrightarrow{d} (\int_{-\infty}^{\infty} T(s)ds)L(1, 0). \tag{23}$$

For differentiable functions $T(.)$, we need to set $l = 1$, implying that we need $p > 8$ in order for the theorem to work. In order to improve the above result, Lemma 1 turned out to be a key tool. Using this lemma, we were able to improve Park and Phillips' result and show the following quite general result:

Theorem 5 *Assume that $|T(x)| \leq R(x)$, and assume that $R(.)$ is integrable, continuous on $\mathbb{R}$, and monotone on $(0, \infty)$ and $(-\infty, 0)$. If $T(.)$ is continuous, then under Assumption 2,*

$$n^{-1/2} \sum_{t=1}^{n} T(x_t) \xrightarrow{d} (\int_{-\infty}^{\infty} T(s)ds)L(1, 0). \tag{24}$$

Compared to Park and Phillips' theorem, we have completely removed their Lipschitz-continuity condition and weakened it to continuity, and in addition, their requirement on p has been removed. While no $R(.)$ function such as present in Theorem 5 is explicitly used in their Theorem 4, from Park and Phillips' proof it is clear that existence of such a function is implied. Therefore, Theorem 5 is a "clean" improvement to Park and Phillips' Theorem 4.

6 Asymptotically homogeneous functions

In this section, we improve Park and Phillips' (1999) result for asymptotically homogeneous functions. Park and Phillips assume that

$$T(\lambda x) = \nu(\lambda)H(x) + R(x, \lambda) \tag{25}$$

and they show that

$$\nu(n^{1/2})^{-1}n^{-1}\sum_{t=1}^{n}T(x_t)\xrightarrow{d}\int_0^1 H(W(r))dr \tag{26}$$

if either

a. $|R(x,\lambda)|\leq a(\lambda)P(x)$, where $\limsup_{\lambda\to\infty}a(\lambda)/\nu(\lambda)=0$ and P is locally integrable, or

b. $|R(x,\lambda)|\leq b(\lambda)Q(\lambda x)$, where $\limsup_{\lambda\to\infty}b(\lambda)/\nu(\lambda)<\infty$ and Q is locally integrable and vanishes at infinity, i.e. $Q(x)\to 0$ as $|x|\to\infty$.

In this paper, we redefine their notion of an asymptotically homogeneous function, as follows:

Definition 1 *A function $T(.)$ is called* asymptotically homogeneous *if for all $K>0$ and some function $H(.)$,*

$$\lim_{\lambda\to\infty}\int_{-K}^{K}|\nu(\lambda)^{-1}T(\lambda x)-H(x)|dx=0. \tag{27}$$

Obviously from the dominated convergence theorem it follows that if for some $\nu(.)$ and $H(.)$, pointwise in x,

$$\nu(\lambda)^{-1}T(\lambda x)\to H(x)\quad\text{as }\lambda\to\infty \tag{28}$$

and $|\nu(\lambda)^{-1}T(\lambda x)|\leq G(x)$ for a locally integrable function $G(.)$, then $T(.)$ is asymptotically homogeneous. Below, we will call a function *monotone regular* if for some $\{a_1,\ldots,a_q\}$, $T(.)$ is monotone on (a_j,a_{j+1}) for $j=0,\ldots,q$ (setting $a_0=-\infty$ and $a_{q+1}=\infty$).
The main result of this section is the following:

Theorem 6 *Suppose Assumption 1 holds. Also assume that $T(.)$ is asymptotically homogeneous. In addition, assume that $H(.)$ is continuous and $T(.)$ is monotone regular. Then*

$$\nu(n^{1/2})^{-1}n^{-1}\sum_{t=1}^{n}T(x_t)\xrightarrow{d}\int_0^1 H(W(r))dr=\int_{-\infty}^{\infty}H(s)L(1,s)ds. \tag{29}$$

It is also possible to show that our definition of an asymptotically homogeneous function is more general than Park and Phillips'. Under Assumption a. above,

$$\int_{-K}^{K}|\nu(\lambda)^{-1}T(\lambda x)-H(x)|dx=\nu(\lambda)^{-1}\int_{-K}^{K}|R(x,\lambda)|dx$$

$$\leq a(\lambda)\nu(\lambda)^{-1}\int_{-K}^{K} P(x)dx \to 0 \tag{30}$$

as $\lambda \to \infty$ if $P(.)$ is locally integrable. Under Assumption b. above,

$$\int_{-K}^{K} |\nu(\lambda)^{-1}T(\lambda x) - H(x)|dx = \nu(\lambda)^{-1}\int_{-K}^{K} |R(x,\lambda)|dx$$

$$\leq b(\lambda)\nu(\lambda)^{-1}\int_{-K}^{K} Q(\lambda x)dx \to 0 \tag{31}$$

as $\lambda \to \infty$, because $\limsup_{\lambda\to\infty} b(\lambda)\nu(\lambda)^{-1} < \infty$ and $\lim_{\lambda\to\infty}\int_{-K}^{K} Q(\lambda x)dx = 0$ by boundedness of $Q(.)$ (which is also assumed in Park and Phillips (1999)). Therefore, obviously the set of functions that is "asymptotically homogeneous" in this paper is wider than in Park and Phillips (1999). But clearly, most functions that one may expect to be useful for applications should be expected to already be in Park and Phillips' class of asymptotically homogeneous functions, and the main achievement of our analysis is the redefinition of the class of asymptotically homogeneous functions to as large as possible a collection of functions. The proof of the theorem below appears to us to be close to the limits of what should be possible, and for the authors of this paper, it is hard to see how the above definition of the class of asymptotically homogeneous functions can be relaxed further to yield an even larger function class that generates similar behavior.

References

Akonom, J. (1993), Comportement asymptotique du temps d'occupation du processus des sommes partielles, Annales de l'Institut Henri Poncaré 29, 57-81.

Chung, K.L. (1974), A course in probability theory. New York: Academic Press.

Davidson, J. (1994), *Stochastic limit theory.* Oxford: Oxford University Press.

Park, J.Y. and P.C.B. Phillips (1999), Asymptotics for nonlinear transformations of integrated time series, Econometric Theory 15, 269-298.

Proofs

Proof of Lemma 1:

Note that by Theorem 6.2.1 of Chung (1974)

$$P(y \le n^{-1/2}x_n \le y+\delta) = \lim_{T\to\infty}(2\pi)^{-1}\int_{-T}^{T}(is)^{-1}(\exp(-isy)-\exp(-is(y+\delta)))f_n(s)ds \quad (32)$$

where $f_n(.)$ is the characteristic function of $n^{-1/2}x_n$. Now, note that

$$x_n = x_0 + \sum_{t=1}^{n}\sum_{k=0}^{\infty}\phi_k\varepsilon_{t-k} = x_0 + \sum_{t=1}^{n}\sum_{j=-\infty}^{t}\phi_{t-j}\varepsilon_j = x_0 + \sum_{j=-\infty}^{n}\varepsilon_j\sum_{t=1}^{n}\phi_{t-j}I(j\le t), \quad (33)$$

and therefore

$$f_n(s) = E\exp(isn^{-1/2}x_n) = \prod_{j=-\infty}^{n}E\exp(is(n^{-1/2}x_0 + n^{-1/2}\varepsilon_j\sum_{t=1}^{n}\phi_{t-j}I(j\le t))). \quad (34)$$

This in turn implies that

$$|f_n(s)| \le \prod_{j=1}^{n}|\psi(n^{-1/2}s\sum_{t=1}^{n}\phi_{t-j}I(j\le t))| = \prod_{j=1}^{n}|\psi(n^{-1/2}s\sum_{l=0}^{n-j}\phi_l)|. \quad (35)$$

Therefore, noting that $s^\eta\psi(s)\to 0$ as $|s|\to\infty$ for some $\eta>0$, for λ such that $\lambda^{p-2} = (1/6)\sigma^2/E|\varepsilon_t|^p$ we have

$$P(y \le n^{-1/2}x_n \le y+\delta)$$

$$\le (2\pi)^{-1}\int_{-\infty}^{\infty}|s|^{-1}|\exp(-isy)-\exp(-is(y+\delta))|\prod_{j=1}^{n}|\psi(n^{-1/2}s\sum_{l=0}^{n-j}\phi_l)|ds$$

$$\le (2\pi)^{-1}\int_{|s|\le\lambda n^{1/2}/|2\sum_{l=0}^{\infty}\phi_l|}|s|^{-1}|\exp(-isy)-\exp(-is(y+\delta))|\prod_{j=1}^{n}|\psi(n^{-1/2}s\sum_{l=0}^{n-j}\phi_l)|ds$$

$$+(2\pi)^{-1}\int_{|s|\ge\lambda n^{1/2}/|2\sum_{l=0}^{\infty}\phi_l|}|s|^{-1}|\exp(-isy)-\exp(-is(y+\delta))|\prod_{j=1}^{n}|\psi(n^{-1/2}s\sum_{l=0}^{n-j}\phi_l)|ds$$

$$\le (2\pi)^{-1}\delta\int_{|s|\le\lambda n^{1/2}/|2\sum_{l=0}^{\infty}\phi_l|}\prod_{j=1}^{n}|\psi(n^{-1/2}s\sum_{l=0}^{n-j}\phi_l)|ds$$

$$+(2\pi)^{-1}\delta \int_{|\xi|\geq\lambda/|2\sum_{l=0}^{\infty}\phi_l|} n^{1/2} \prod_{j=1}^{n} |\psi(\xi \sum_{l=0}^{n-j} \phi_l)| d\xi$$

$$= T_1 + T_2, \tag{36}$$

say, where in the last inequality we set $\xi = sn^{-1/2}$ and used the inequality $|\exp(ia) - \exp(ib)| \leq |a-b|$. To deal with T_2, note that because

$$\sum_{l=0}^{\infty} \phi_l \neq 0, \tag{37}$$

we have

$$|\sum_{l=0}^{k} \phi_l - \sum_{l=0}^{\infty} \phi_l| \leq (1/2)|\sum_{l=0}^{\infty} \phi_l| \tag{38}$$

for $k \geq M$ for some M. Therefore for $n \geq N = M + [2/\eta + 1]$, under Assumption 1,

$$T_2 \leq (2\pi)^{-1}\delta \int_{|\xi|\geq\lambda/|2\sum_{l=0}^{\infty}\phi_l|}^{\infty} n^{1/2} \prod_{j=1}^{n-N} |\psi(\xi \sum_{l=0}^{n-j} \phi_l)| d\xi$$

$$\leq C'\delta n^{1/2} (\sup_{|\xi|\geq\lambda/4} |\psi(\xi)|)^{n-N-[2/\eta+1]} \int_{|\xi|\geq\lambda/|2\sum_{l=0}^{\infty}\phi_l|}^{\infty} \prod_{j=n-N-[2/\eta+1]+1}^{n-N} |\xi \sum_{l=0}^{n-j} \phi_l|^{-\eta} d\xi$$

$$\leq C'\delta n^{1/2} (\sup_{|\xi|\geq\lambda/4} |\psi(\xi)|)^{n-N-[2/\eta+1]} \int_{|\xi|\geq\lambda/|2\sum_{l=0}^{\infty}\phi_l|}^{\infty} |\xi|^{-2} |(1/2) \sum_{l=0}^{\infty} \phi_l|^{-2} d\xi$$

$$\leq c_n \delta, \tag{39}$$

say, where $c_n \to 0$ as $n \to \infty$ because

$$\sup_{|\xi|\geq\lambda/4} |\psi(\xi)| < 1, \tag{40}$$

C' denotes some positive constant not depending on y, δ or n, and the second inequality follows from the assumptions made on $\psi(.)$ in Assumption 1. The result of Equation (40) follows because by Theorem 6.4.7 of Chung (1974), characteristic functions of absolutely continuous random variables will not equal 1 at any point other than $\xi = 0$ if the underlying distribution is absolutely continuous.

In order to deal with T_1, note that by Theorem 11.6 of Davidson (1994), we have for $|s|n^{-1/2} \leq \lambda$, and because $\lambda^{p-2} = (1/6)\sigma^2/E|\varepsilon_t|^p$,

$$
\begin{aligned}
&|\psi(n^{-1/2}s) - (1 - (1/2)n^{-1}\sigma^2 s^2)| \\
&\leq E \min(n^{-1}s^2\varepsilon_t^2, (1/6)n^{-3/2}|s|^3|\varepsilon_t|^3) \\
&= E \min(n^{-1}s^2\varepsilon_t^2, (1/6)n^{-3/2}|s|^3|\varepsilon_t|^3)(I(|s\varepsilon_t| \leq n^{1/2}) + I(|s\varepsilon_t| > n^{1/2})) \\
&\leq n^{-1}s^2 E\varepsilon_t^2 I(|\varepsilon_t| > |s|^{-1}n^{1/2}) + (1/6)n^{-1}\sigma^2 s^2 \\
&\leq n^{-1}s^2(\lambda^{-1})^{2-p}E|\varepsilon_t|^p + (1/6)n^{-1}\sigma^2 s^2 \\
&= n^{-1}s^2(\lambda^{p-2}E|\varepsilon_t|^p + (1/6)\sigma^2) \\
&= (1/3)n^{-1}s^2\sigma^2. \qquad (41)
\end{aligned}
$$

The first inequality here was Theorem 11.6 of Davidson (1994), and the third uses $E|X|^2 I(|X| > K) \leq E|X|^p K^{2-p}$. Therefore, $|\psi(n^{-1/2}s)| \leq 1 - (1/6)n^{-1}s^2\sigma^2$ for $|s|n^{-1/2} \leq \lambda$. Next, again assume that $n \geq N$, where N is defined after Equation (38). Therefore, for $|s|n^{-1/2} \leq \lambda/|2\sum_{l=0}^{\infty}\phi_l|$,

$$
\begin{aligned}
&\prod_{j=1}^{n} |\psi(n^{-1/2}s\sum_{l=0}^{n-j}\phi_l)| \leq \prod_{j=1}^{n-N} |\psi(n^{-1/2}s\sum_{l=0}^{n-j}\phi_l)| \\
&\leq \exp(\sum_{j=1}^{n-N}\log(1 - (1/6)n^{-1}s^2\sigma^2(\sum_{l=0}^{n-j}\phi_l)^2)) \\
&\leq \exp(-(1/6)s^2\sigma^2 n^{-1}\sum_{j=1}^{n-N}(\sum_{l=0}^{n-j}\phi_l)^2), \qquad (42)
\end{aligned}
$$

where the last inequality uses that $\log(1+x) \leq x$ for $x > -1$, and it is easily verified that for $|s|n^{-1/2} \leq \lambda/|2\sum_{l=0}^{\infty}\phi_l|$ and $j \leq n - N$,

$$
(1/6)n^{-1}s^2\sigma^2(\sum_{l=0}^{n-j}\phi_l)^2 \leq (3/32)\lambda^2\sigma^2 \leq 3/32.
$$

This implies that

$$T_1 = (2\pi)^{-1}\delta \int_{|s|\le \lambda n^{1/2}/|2\sum_{l=0}^{\infty}\phi_l|} \prod_{j=1}^{n} |\psi(n^{-1/2} s \sum_{l=0}^{n-j} \phi_l)| ds$$

$$\le (2\pi)^{-1}\delta \int_{-\infty}^{\infty} \exp(-(1/6)s^2\sigma^2 n^{-1} \sum_{j=1}^{n-N} (\sum_{l=0}^{n-j} \phi_l)^2) ds, \tag{43}$$

and because by the definition of N,

$$n^{-1} \sum_{j=1}^{n-N} (\sum_{l=0}^{n-j} \phi_l)^2 \ge (1/4)(\sum_{l=0}^{\infty} \phi_l)^2 > 0, \tag{44}$$

it now follows that for n large enough,

$$T_1 \le C_1 \delta. \tag{45}$$

In conclusion, we have now shown that for large enough n,

$$T_1 + T_2 \le C\delta \tag{46}$$

for some $C > 0$ not depending on y, δ, or n, which establishes the result. □

Proof of Theorem 1:

First note that $Y_n \xrightarrow{d} Y$ if for all $\varepsilon > 0$ and some $Y_{n\varepsilon}$, $Y_{n\varepsilon} \xrightarrow{d} Y_\varepsilon$ as $n \to \infty$, $Y_\varepsilon \xrightarrow{d} Y$ as $\varepsilon \to 0$, and

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} E|Y_{n\varepsilon} - Y_n| = 0. \tag{47}$$

This is because

$$\lim_{n\to\infty} |E\exp(i\xi Y_n) - E\exp(i\xi Y)|$$

$$\le \lim_{\varepsilon\to 0} \lim_{n\to\infty} |E\exp(i\xi Y_n) - E\exp(i\xi Y_{n\varepsilon})|$$

$$+ \lim_{\varepsilon\to 0} \lim_{n\to\infty} |E\exp(i\xi Y_{n\varepsilon}) - E\exp(i\xi Y_\varepsilon)|$$

$$+\lim_{\varepsilon\to 0}|E\exp(i\xi Y_\varepsilon)-E\exp(i\xi Y)|. \tag{48}$$

Next, note that the second term and third term in the last equation converge to 0 by assumption, and the first term can be bounded by

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}|\xi|E|Y_{n\varepsilon}-Y_\varepsilon|, \tag{49}$$

which equals 0 by assumption.
Next, we will argue that for any $j=0,\ldots,q$,

$$Y_n = n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t)I(a_j\le n^{-1/2}x_t\le a_{j+1})$$

$$\xrightarrow{d}\int_0^1 T(\sigma W(r))I(a_j\le W(r)\le a_{j+1})dr = Y. \tag{50}$$

By the Skorokhod representation theorem, the joint convergence in distribution of the $q+1$ terms follows, which then implies the result of the theorem.
Now define, for small enough $\varepsilon>0$ and for $j=1,\ldots,q-1$,

$$T_\varepsilon(x)=T(x)I(a_j+\varepsilon\le x\le a_{j+1}-\varepsilon),$$

$$Y_{n\varepsilon}=n^{-1}\sum_{t=1}^{n}T_\varepsilon(n^{-1/2}x_t), \tag{51}$$

and

$$Y_\varepsilon=\int_0^1 T_\varepsilon(\sigma W(r))dr. \tag{52}$$

Note that the argument for $j=0$ and $j=q$ is analogous. Then by Park and Phillips' (1999) Theorem 3.2, it follows that for all $\varepsilon>0$, as $n\to\infty$,

$$Y_{n\varepsilon}=n^{-1}\sum_{t=1}^{n}T_\varepsilon(n^{-1/2}x_t)\xrightarrow{d}\int_0^1 T_\varepsilon(\sigma W(r))dr = Y_\varepsilon \tag{53}$$

because $T_\varepsilon(x)$ is regular and locally integrable. Furthermore, as $\varepsilon\to 0$, by the occupation times formula,

$$|Y_\varepsilon - Y| = |\int_0^1 T_\varepsilon(\sigma W(r))dr-\int_0^1 T(\sigma W(r))I(a_j\le W(r)\le a_{j+1})dr|$$

$$\leq |\int_{a_j}^{a_j+\varepsilon} L(1,s)T(\sigma s)ds| + |\int_{a_{j+1}-\varepsilon}^{a_{j+1}} L(1,s)T(\sigma s)ds|$$

$$\leq \sup_s |L(1,s)| \int_{a_j}^{a_j+\varepsilon} |T(\sigma s)|ds + \sup_s |L(1,s)| \int_{a_{j+1}-\varepsilon}^{a_{j+1}} |T(\sigma s)|ds \to 0 \tag{54}$$

(where $L(t,s)$ denotes local time; see Park and Phillips (1999)) because $\int_{a_j}^{a_{j+1}} |T(\sigma s)|ds < \infty$ by assumption. Finally, note that because of absolute continuity of the distributions of x_t,

$$|Y_n - Y_{n\varepsilon}| \leq O_P(1) + |n^{-1} \sum_{t=N+1}^{n} T(n^{-1/2}x_t)I(a_j \leq n^{-1/2}x_t \leq a_j + \varepsilon)|$$

$$+|n^{-1} \sum_{t=N+1}^{n} T(n^{-1/2}x_t)I(a_{j+1} - \varepsilon \leq n^{-1/2}x_t \leq a_{j+1})|. \tag{55}$$

I will only deal with the second term, since the third term can be dealt with analogously. Defining $\varepsilon_k = \varepsilon 2^{-k}$, the expectation of that term can be bounded by

$$\sum_{k=0}^{\infty} E|n^{-1} \sum_{t=N+1}^{n} T(n^{-1/2}x_t)I(a_j + \varepsilon_{k+1} \leq n^{-1/2}x_t \leq a_j + \varepsilon_k)|$$

$$\leq \sum_{k=0}^{\infty}(|T(a_j + \varepsilon_{k+1})| + |T(a_j + \varepsilon_k)|)n^{-1} \sum_{t=N+1}^{n} EI(a_j + \varepsilon_{k+1} \leq n^{-1/2}x_t \leq a_j + \varepsilon_k)$$

$$\leq \sum_{k=0}^{\infty}(|T(a_j + \varepsilon_{k+1})| + |T(a_j + \varepsilon_k)|)n^{-1} \sum_{t=N+1}^{n} (\varepsilon_k - \varepsilon_{k+1})n^{1/2}t^{-1/2}$$

$$\leq (\sup_{n\geq 1} n^{-1/2} \sum_{t=N+1}^{n} t^{-1/2}) \sum_{k=0}^{\infty} |T(a_j + \varepsilon_{k+1})| + |T(a_j + \varepsilon_k)|)\varepsilon_{k+1}$$

$$\leq (\sup_{n\geq 1} n^{-1/2} \sum_{t=1}^{n} t^{-1/2})(3/2) \sum_{k=0}^{\infty} |T(a_j + \varepsilon_k)|\varepsilon_k$$

$$= (\sup_{n\geq 1} n^{-1/2} \sum_{t=1}^{n} t^{-1/2})(3/2) \int_0^{\infty} |T(a_j + \varepsilon_{[k]})|\varepsilon_{[k]}dk$$

$$\leq C \int_{a_j}^{a_j+\varepsilon} |T(x)| dx, \tag{56}$$

for some constant $C > 0$, where the first and fifth inequality follow from monotonicity of $T(.)$ on (a_{j-1}, a_j), the second uses Lemma 1, and the third uses the definition of ε_k. The last expression obviously converges to 0 as $\varepsilon \to 0$ under the assumption of the theorem, which completes the proof of this theorem. □

The following two lemmas were needed in order to prove the results of Section 4.

Lemma 2 *Under the assumptions of Theorem 2, as $n \to \infty$,*

$$E(N_n(\nu_n; 0, \delta) - N_n(\nu_n; k\delta, (k+1)\delta))^2 \leq c(\delta/n\nu_2)(1 + k\delta^2 n \log(n)/\nu_n).$$

Proof:

See Park and Phillips (1999). □

Lemma 3 *For all integers k_1, k_2 and k_3, and any sequence b_n such that $b_n = o(c_n)$, under the assumptions of Theorem 2,*

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \sum_{j \geq \max(1-k_1, 1-k_2, 1-k_3)}^{\infty} T((j+k_1)\varepsilon c_n) I((j+k_2)\varepsilon c_n > c_n) I((j+k_3)\varepsilon c_n \leq b_n) d_n^{-1} \varepsilon c_n = 1.$$

Proof of Lemma 3:

$$\sum_{j \geq \max(1-k_1, 1-k_2, 1-k_3)}^{\infty} T((j+k_1)\varepsilon c_n) I((j+k_2)\varepsilon c_n > c_n) I((j+k_3)\varepsilon c_n \leq b_n) d_n^{-1} \varepsilon c_n$$

$$= \int_{j \geq \max(1-k_1, 1-k_2, 1-k_3)}^{\infty} T(([j]+k_1)\varepsilon c_n) I(([j]+k_2)\varepsilon c_n > c_n) I(([j]+k_3)\varepsilon c_n \leq b_n) d_n^{-1} \varepsilon c_n dj$$

$$\leq \int_{j \geq \max(1-k_1, 1-k_2, 1-k_3)}^{\infty} T((j+k_1)\varepsilon c_n) I((j+k_2-1)\varepsilon c_n > c_n) I((j+k_3)\varepsilon c_n \leq b_n) d_n^{-1} \varepsilon c_n dj$$

$$= \int_{x=0}^{\infty} T(x) I(x + (k_2 - 1 - k_1)\varepsilon c_n > c_n) I(x + (k_3 - k_1)\varepsilon c_n \le b_n) d_n^{-1} dx$$

$$= (\int_{x=0}^{1} T(x) I(x > c_n) dx)^{-1} \int_{x=0}^{\infty} T(x) I(x + (k_2 - 1 - k_1)\varepsilon c_n > c_n) I(x + (k_3 - k_1)\varepsilon c_n \le b_n) dx.$$

Now because $T(x) = |x|^{-m} I(x > 0)$, it follows that the last expression converges to 1 as $n \to \infty$ under the conditions of the lemma. The same argument will hold for a lower bound, which then completes the proof of the lemma. □

Proof of Theorem 2:

For this proof, I will assume without loss of generality that $\sigma^2 = 1$. Note that, for $b_n = c_n^{1-1/2-\alpha}$ for some $\alpha > 0$ small enough that $d_n^{-1} T(b_n) \to 0$,

$$d_n^{-1} n^{-1} \sum_{t=1}^{n} T(n^{-1/2} x_t) I(n^{-1/2} x_t > c_n)$$

$$= d_n^{-1} n^{-1} \sum_{t=1}^{n} T(n^{-1/2} x_t) I(n^{-1/2} x_t > c_n) I(n^{-1/2} x_t \le b_n)$$

$$+ d_n^{-1} n^{-1} \sum_{t=1}^{n} T(n^{-1/2} x_t) I(n^{-1/2} x_t > b_n), \tag{57}$$

and the second term is $o_P(1)$ because

$$d_n^{-1} n^{-1} \sum_{t=1}^{n} T(n^{-1/2} x_t) I(n^{-1/2} x_t > b_n)$$

$$\le d_n^{-1} T(b_n) \to 0 \tag{58}$$

by assumption. Now note that trivially, for all $\varepsilon > 0$,

$$d_n^{-1} n^{-1} \sum_{t=1}^{n} T(n^{-1/2} x_t) I(n^{-1/2} x_t > c_n) I(n^{-1/2} x_t \le b_n)$$

$$= \sum_{j=0}^{\infty} d_n^{-1} \int_0^1 T(n^{-1/2}W_n(r))I(W_n(r) > c_n)I(W_n(r) \le b_n)I(j\varepsilon c_n \le W_n(r) < (j+1)\varepsilon c_n)dr. \quad (59)$$

An upper bound for the last term is

$$\sum_{j=0}^{\infty} T(j\varepsilon c_n) d_n^{-1} \int_0^1 I(W_n(r) > c_n)I(W_n(r) \le b_n)I(j\varepsilon c_n \le W_n(r) < (j+1)\varepsilon c_n)dr$$

$$\le \sum_{j=0}^{\infty} T(j\varepsilon c_n)I((j+1)\varepsilon c_n > c_n)I(j\varepsilon c_n \le b_n)d_n^{-1} \int_0^1 I(j\varepsilon c_n \le W_n(r) < (j+1)\varepsilon c_n)dr$$

$$= \sum_{j=0}^{\infty} T(j\varepsilon c_n)I((j+1)\varepsilon c_n > c_n)I(j\varepsilon c_n \le b_n)d_n^{-1}N_n(1; j\varepsilon c_n, (j+1)\varepsilon c_n). \quad (60)$$

Similarly, a lower bound is

$$\sum_{j=0}^{\infty} T((j+1)\varepsilon c_n)I(j\varepsilon c_n > c_n)I((j+1)\varepsilon c_n \le b_n)d_n^{-1}N_n(1; j\varepsilon c_n, (j+1)\varepsilon c_n). \quad (61)$$

I will only consider the lower bound and determine its limit, but the argument for the upper bound is identical and renders the same limit. By Lemma 2,

$$E\sum_{j=0}^{\infty} T(j\varepsilon c_n)I((j+1)\varepsilon c_n > c_n)I(j\varepsilon c_n \le b_n)d_n^{-1}|N_n(1; j\varepsilon c_n, (j+1)\varepsilon c_n) - N_n(1; 0, \varepsilon c_n)|$$

$$\le \sum_{j=0}^{\infty} T(j\varepsilon c_n)I((j+1)\varepsilon c_n > c_n)I(j\varepsilon c_n \le b_n)d_n^{-1}(c(\varepsilon c_n/n)(1 + (j(\varepsilon c_n)^2 n\log(n))))^{1/2}$$

$$\le (d_n^{-1}\varepsilon c_n \sum_{j=0}^{\infty} T(j\varepsilon c_n)I((j+1)\varepsilon c_n > c_n))(c(\varepsilon c_n^{-1}/n)(1+((b_n/(\varepsilon c_n))(\varepsilon c_n)^2 n\log(n))))^{1/2}. \quad (62)$$

Now, by Lemma 3,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}(d_n^{-1}\varepsilon c_n \sum_{j=0}^{\infty} T(j\varepsilon c_n)I((j+1)\varepsilon c_n > c_n)) = 1, \quad (63)$$

and therefore the expression converges to zero in probability if

$$(\varepsilon c_n^{-1}/n)(1 + ((b_n/(\varepsilon c_n))(\varepsilon c_n)^2 n \log(n))) \to 0. \tag{64}$$

First, note that by assumption $c_n^{-1}n^{-1} \to 0$, and that

$$\begin{aligned}&(\varepsilon c_n^{-1}/n)((b_n/(\varepsilon c_n))(\varepsilon c_n)^2 n \log(n))) \\ &= \varepsilon^2 b_n \log(n) \to 0 \end{aligned} \tag{65}$$

by assumption. Therefore, it suffices to consider

$$\sum_{j=0}^{\infty} T((j+1)\varepsilon c_n) I(j\varepsilon c_n > c_n) I((j+1)\varepsilon c_n \le b_n) d_n^{-1} \varepsilon c_n (N_n(1; 0, j\varepsilon c_n)/(\varepsilon c_n)). \tag{66}$$

Now by the comment following Lemma 2.5 in Park and Phillips (1999),

$$N_n(1; 0, j\varepsilon c_n)/(\varepsilon c_n) = L(1, 0) + o_P(1) \tag{67}$$

if $\varepsilon c_n \ge n^{-(2p-1)/3p+\eta}$ for some $\eta > 0$, which is the case by assumption. Therefore, we only need consider

$$L(1, 0) \sum_{j=0}^{\infty} T((j+1)\varepsilon c_n) I(j\varepsilon c_n > c_n) I((j+1)\varepsilon c_n \le b_n) d_n^{-1} \varepsilon c_n. \tag{68}$$

Now by Lemma 3, it follows that by choosing ε arbitrarily small, the limit distribution will be arbitrarily close to $L(1, 0)$; and noting that the same argument will work for the upper bound, this suffices to prove the result. □

Proof of Theorem 5:

Define $T_K(x) = T(x)I(|x| \le K)$, $T'_K(x) = T(x)I(x > K)$, and $T''_K(x) = T(x)I(x < -K)$. We will show that

$$\lim_{K\to\infty} \limsup_{n\to\infty} E|n^{-1/2} \sum_{t=1}^{n} T'_K(x_t)| = 0 \tag{69}$$

and the same argument, mutatis mutandis, will hold for $n^{-1/2} \sum_{t=1}^{n} T''_K(x_t)$. Then, we will show that for all $K > 0$,

$$n^{-1/2} \sum_{t=1}^{n} T_K(x_t) \xrightarrow{d} (\int_{-K}^{K} T(s) ds) L(1, 0), \tag{70}$$

and the result then follows. To show the result of Equation (69), note that

$$n^{-1/2}\sum_{t=1}^{M}T(x_t)I(x_t>K)\le Mn^{-1/2}R(K)\to 0 \tag{71}$$

as $n\to\infty$, and

$$\begin{aligned}
&E|n^{-1/2}\sum_{t=M+1}^{n}T(x_t)I(x_t>K)|\\
&=E|\sum_{j=1}^{\infty}n^{-1/2}\sum_{t=M+1}^{n}T(x_t)I(Kj<x_t\le K(j+1))|\\
&\le E\sum_{j=1}^{\infty}n^{-1/2}\sum_{t=M+1}^{n}R(Kj)I(Kjt^{-1/2}<t^{-1/2}x_t\le K(j+1)t^{-1/2})\\
&\le \sum_{j=1}^{\infty}n^{-1/2}\sum_{t=1}^{n}R(Kj)CKt^{-1/2}\\
&\le C(\sup_{n\ge1}n^{-1/2}\sum_{t=1}^{n}t^{-1/2})K\sum_{j=1}^{\infty}R(Kj)\\
&=C'\int_1^{\infty}R(K[j])d(Kj)\\
&=C'\int_K^{\infty}R(K[x/K])dx=C'\int_K^{2K}R(K[x/K])dx+C'\int_{2K}^{\infty}R(K[x/K])dx\\
&\le C'(KR(K)+\int_K^{\infty}R(x)dx)\to 0
\end{aligned} \tag{72}$$

as $K\to\infty$, where $C'=C\sup_{n\ge1}n^{-1/2}\sum_{t=1}^{n}t^{-1/2}$, and $KR(K)\to 0$ under the assumptions of the theorem because

$$R(2K)K\le\int_K^{2K}R(x)dx\le\int_K^{\infty}R(x)dx\to 0 \tag{73}$$

as $K \to \infty$. The first inequality follows from the assumed boundedness of $|T(.)|$ by $R(.)$ and the assumed monotonicity of $R(.)$, and the second is an application of Lemma 1. This completes the proof of the result of Equation (69). The remainder of the proof follows the line of proof of Park and Phillips (1999, proof of Theorem 5.1), but some modifications will be made. In order to show the result of Equation (70) and thereby make the proof of Theorem 5 complete, define for $\delta > 0$

$$T_\delta(x) = \sum_{k=-K/\delta}^{K/\delta} T(k\delta)I(k\delta \le x_t \le (k+1)\delta) \tag{74}$$

where the notation $\sum_{t=a}^{b} c_t$ is defined as a summation over all integer-valued elements in $[a, b]$, and note that for all $K > 0$,

$$\begin{aligned}
&E|\sum_{k=-K/\delta}^{K/\delta} n^{-1/2} \sum_{t=1}^{n} (T_K(x_t) - T_\delta(x_t))| \\
&= E|\sum_{k=-K/\delta}^{K/\delta} n^{-1/2} \sum_{t=1}^{n} (T(k\delta) - T(x_t))I(k\delta \le x_t \le (k+1)\delta)| \\
&\le \sup_{x\in[-K,K]} \sup_{x'\in[-K,K]:|x-x'|\le\delta} |T(x) - T(x')| E \sum_{k=-K/\delta}^{K/\delta} n^{-1/2} \sum_{t=1}^{n} I(k\delta \le x_t \le (k+1)\delta) \\
&= \sup_{x\in[-K,K]} \sup_{x'\in[-K,K]:|x-x'|\le\delta} |T(x) - T(x')| n^{-1/2} \sum_{t=1}^{n} P(-[K/\delta]\delta \le x_t \le [K/\delta]\delta) \\
&\le \sup_{x\in[-K,K]} \sup_{x'\in[-K,K]:|x-x'|\le\delta} |T(x) - T(x')| n^{-1/2} \sum_{t=1}^{n} 2CKt^{-1/2} \\
&\le 2C'K \sup_{x\in[-K,K]} \sup_{x'\in[-K,K]:|x-x'|\le\delta} |T(x) - T(x')| \to 0
\end{aligned} \tag{75}$$

as $\delta \to 0$ by continuity of $T(.)$, where the second inequality is Lemma 1. Therefore, we can consider $n^{-1/2} \sum_{t=1}^{n} T_\delta(x_t)$ instead of $n^{-1/2} \sum_{t=1}^{n} T_K(x_t)$. Now

$$n^{-1/2} \sum_{t=1}^{n} T_\delta(x_t)$$

$$= \sum_{k=-K/\delta}^{K/\delta} T(k\delta) n^{-1/2} \sum_{t=1}^{n} I(k\delta \le x_t \le (k+1)\delta)$$

$$= \sum_{k=-K/\delta}^{K/\delta} T(k\delta) n^{1/2} N_n(n^{1/2}; k\delta, (k+1)\delta), \tag{76}$$

and

$$| \sum_{k=-K/\delta}^{K/\delta} T(k\delta) n^{1/2} N_n(n^{1/2}; k\delta, (k+1)\delta) - \sum_{k=-K/\delta}^{K/\delta} T(k\delta) n^{1/2} N_n(n^{1/2}; 0, \delta)| = o_P(1) \tag{77}$$

because by the Cauchy-Schwartz inequality,

$$E(\sum_{k=-K/\delta}^{K/\delta} T(k\delta) n^{1/2} N_n(n^{1/2}; k\delta, (k+1)\delta) - \sum_{k=-K/\delta}^{K/\delta} T(k\delta) n^{1/2} N_n(n^{1/2}; 0, \delta))^2$$

$$\le n \sum_{k=-K/\delta}^{K/\delta} R(k\delta)^2 \sum_{k=-K/\delta}^{K/\delta} E(N_n(n^{1/2}; k\delta, (k+1)\delta) - N_n(n^{1/2}; 0, \delta))^2$$

$$\le n \int_{-K}^{K} R(k\delta)^2 dk \sum_{k=-K/\delta}^{K/\delta} c(\delta n^{-3/2})(1 + (k\delta^2 n \log(n)/n))$$

$$\le n^{-1/2}(1/\delta)(\int_{-K}^{K} R(s)^2 ds) c(1 + \delta \log(n)) = o(1), \tag{78}$$

where the second inequality is Lemma 2.5 (a) of Park and Phillips (1999). Therefore, it suffices to consider $\sum_{k=-K/\delta}^{K/\delta} T(k\delta) n^{1/2} N_n(n^{1/2}; 0, \delta)$. Now note that

$$\sum_{k=-K/\delta}^{K/\delta} T(k\delta) n^{1/2} N_n(n^{1/2}; 0, \delta) = (\int_{-K/\delta}^{K/\delta} T([k]\delta) dk) n^{1/2} N_n(n^{1/2}; 0, \delta)$$

$$= (\int_{-K}^{K} T([s/\delta]\delta) ds)(1/\delta) n^{1/2} N_n(n^{1/2}; 0, \delta), \tag{79}$$

and

$$|n^{1/2}N_n(n^{1/2};0,\delta) - n^{1/2}N(n^{1/2};0,\delta)| = o_P(n^{1/2}n^{-(2p-1)/3p}) = o_P(n^{(1-p/2)/(3p)}) = o_P(1) \quad (80)$$

by Equation (b) of Lemma 2.5 of Park and Phillips (1999). Therefore,

$$|\sum_{k=-K/\delta}^{K/\delta} T(k\delta)n^{1/2}N_n(n^{1/2};0,\delta) - \sum_{k=-K/\delta}^{K/\delta} T(k\delta)n^{1/2}N(n^{1/2};0,\delta)| = o_P(1), \quad (81)$$

implying that it suffices to analyze

$$(\delta \sum_{k=-K/\delta}^{K/\delta} T(k\delta))(\delta^{-1}n^{1/2}N(n^{1/2};0,\delta)). \quad (82)$$

As $n \to \infty$,

$$\delta^{-1}n^{1/2}N(n^{1/2};0,\delta) \to L(1,0) \quad \text{almost surely}, \quad (83)$$

as explained in the text following Lemma 2.5 of Park and Phillips (1999). In addition, as $\delta \to 0$, by continuity of $T(.)$,

$$\int_{-K}^{K} T([s/\delta]\delta)ds \to \int_{-K}^{K} T(s)ds. \quad (84)$$

Therefore,

$$n^{-1/2}\sum_{t=1}^{n} T_K(x_t) \xrightarrow{d} (\int_{-K}^{K} T(s)ds)L(1,0), \quad (85)$$

implying that the condition of Equation (70) is now verified. This completes the proof. □

For the proof of Theorem 6, we need the following lemma:

Lemma 4 *Under Assumption 1, for any $K > 0$,*

$$n^{-1}\sum_{t=1}^{n} I(n^{-1/2}x_t \leq x) \Rightarrow \int_0^1 I(W(r) \leq x)dr, \quad (86)$$

where "$\Rightarrow$" denotes weak convergence on $D[-K,K]$.

Proof of Lemma 4:

Pointwise in x, the result follows from Theorem 3.2 of Park and Phillips (1999), and therefore it suffices to show stochastic equicontinuity of $n^{-1}\sum_{t=1}^{n} I(n^{-1/2}x_t \le x)$. By the Skorokhod representation, we can assume that $\sup_{r\in[0,1]}|n^{-1/2}x_{[rn]} - W(r)| \xrightarrow{as} 0$. Then for n large enough, $\sup_{r\in[0,1]}|n^{-1/2}x_{[rn]} - W(r)| \le \delta$ almost surely, implying that for n large enough

$$\sup_{|x|\le K}\sup_{x':x<x'<x+\delta}|n^{-1}\sum_{t=1}^{n}(I(n^{-1/2}x_t \le x) - I(n^{-1/2}x_t \le x'))|$$

$$\le \sup_{|x|\le K} n^{-1}\sum_{t=1}^{n} I(x \le n^{-1/2}x_t \le x+\delta)$$

$$\le \sup_{|x|\le K}\int_0^1 I(x-\delta \le W(r) \le x+2\delta)dr$$

$$= \sup_{|x|\le K}\int_{x-\delta}^{x+2\delta} L(1,s)ds \le 3\delta \sup_{|s|\le K}|L(1,s)| \tag{87}$$

where the equality follows from the occupation times formula (see Park and Phillips (1999, Lemma 2.4)) and because $\sup_{|s|\le K}|L(1,s)|$ is a well-defined random variable. The above set chain of inequality establishes stochastic equicontinuity of $n^{-1}\sum_{t=1}^{n} I(n^{-1/2}x_t \le x)$, which completes the proof. □

Proof of Theorem 6:

Because $\sup_{1\le t\le n} n^{1/2}|x_t| = O_P(1)$, it now suffices to show that for any $K > 0$,

$$\nu(n^{1/2})^{-1}n^{-1}\sum_{t=1}^{n} T(x_t)I(|n^{-1/2}x_t| \le K) \xrightarrow{as} \int_0^1 H(W(r))I(|W(r)| \le K)dr$$

$$= \int_{-K}^{K} H(s)L(1,s)ds. \tag{88}$$

Now, by Lemma 4, $n^{-1}\sum_{t=1}^{n} I(n^{1/2}x_t \leq x) \Rightarrow \int_0^1 I(W(r) \leq x)dr$. By the Skorokhod Representation Theorem, we can assume without loss of generality that $|n^{-1}\sum_{t=1}^{n} I(n^{1/2}x_t \leq x) - \int_0^1 I(W(r) \leq x)dr| = c_n \xrightarrow{as} 0$. Now for all $\varepsilon > 0$, let

$$T_{1n\varepsilon} = T_{1n} = \nu(n^{1/2})^{-1}n^{-1}\sum_{t=1}^{n} T(x_t)$$

$$= \nu(n^{1/2})^{-1}\int_{-K/\varepsilon}^{K/\varepsilon} n^{-1}\sum_{t=1}^{n} T(x_t)I(j\varepsilon \leq n^{-1/2}x_t \leq (j+1)\varepsilon)dj, \tag{89}$$

$$T_{2n\varepsilon} = \nu(n^{1/2})^{-1}\int_{-K/\varepsilon}^{K/\varepsilon} T(n^{1/2}j\varepsilon)n^{-1}\sum_{t=1}^{n} I(j\varepsilon \leq n^{-1/2}x_t \leq (j+1)\varepsilon)dj, \tag{90}$$

$$T_{3n\varepsilon} = \nu(n^{1/2})^{-1}\int_{-K/\varepsilon}^{K/\varepsilon} T(n^{1/2}j\varepsilon)\int_0^1 I(j\varepsilon \leq W(r) \leq (j+1)\varepsilon)dr\,dj \tag{91}$$

$$T_{4n\varepsilon} = \nu(n^{1/2})^{-1}\int_{-K/\varepsilon}^{K/\varepsilon} T(n^{1/2}j\varepsilon)\varepsilon L(1, j\varepsilon)dj = \nu(n^{1/2})^{-1}\int_{-K}^{K} T(n^{1/2}s)L(1,s)ds \tag{92}$$

$$T_{5n\varepsilon} = T_5 = \int_{-K}^{K} H(s)L(1,s)ds = \int_0^1 H(W(r))dr. \tag{93}$$

We will show that $\lim_{\varepsilon\to 0}\limsup_{n\to\infty}|T_{jn\varepsilon} - T_{j+1,n\varepsilon}| = 0$ for $j = 1, \ldots, 4$. By the monotone regular condition, we can act as if $T(.)$ is monotone without loss of generality. For $|T_1 - T_{2n\varepsilon}|$ we then have

$$\limsup_{n\to\infty}|T_1 - T_{2n\varepsilon}|$$

$$\leq \limsup_{n\to\infty}\nu(n^{1/2})^{-1}\int_{-K/\varepsilon}^{K/\varepsilon} n^{-1}\sum_{t=1}^{n}|T(x_t) - T(n^{1/2}j\varepsilon)|I(j\varepsilon \leq n^{-1/2}x_t \leq (j+1)\varepsilon)dj$$

$$\leq \limsup_{n\to\infty}\nu(n^{1/2})^{-1}\int_{-K/\varepsilon}^{K/\varepsilon} n^{-1}\sum_{t=1}^{n}|T(n^{1/2}(j+1)\varepsilon) - T(n^{1/2}j\varepsilon)|I(j\varepsilon \leq n^{-1/2}x_t \leq (j+1)\varepsilon)dj$$

$$\leq \limsup_{n\to\infty}\int_{-K/\varepsilon}^{K/\varepsilon}|\nu(n^{1/2})^{-1}T(n^{1/2}(j+1)\varepsilon) - \nu(n^{1/2})^{-1}T(n^{1/2}j\varepsilon) - H((j+1)\varepsilon) + H(j\varepsilon)|dj$$

$$+\int_{-K/\varepsilon}^{K/\varepsilon}|H((j+1)\varepsilon)-H(j\varepsilon)|dj=\int_{-K}^{K}|H(x+\varepsilon)-H(x)|dx, \tag{94}$$

and as $\varepsilon \to 0$, the last term disappears because of continuity of $H(.)$, and the second inequality follows from monotonicity of $T(.)$. To show that $\lim_{\varepsilon\to 0}\limsup_{n\to\infty}|T_{2n\varepsilon}-T_{3n\varepsilon}| = 0$ almost surely, note that

$$|\nu(n^{1/2})^{-1}\int_{-K/\varepsilon}^{K/\varepsilon}T(n^{1/2}j\varepsilon)(n^{-1}\sum_{t=1}^{n}I(j\varepsilon\leq n^{-1/2}x_t\leq(j+1)\varepsilon)$$

$$-\int_0^1 I(j\varepsilon\leq W(r)\leq(j+1)\varepsilon)dr)dj|$$

$$\leq 2c_n\nu(n^{1/2})^{-1}\int_{-K/\varepsilon}^{K/\varepsilon}T(n^{1/2}j\varepsilon)dj$$

$$\leq 2c_n\varepsilon^{-1}\int_{-K}^{K}|\nu(n^{1/2})^{-1}T(n^{1/2}x)-H(x)|dx+2c_n\varepsilon^{-1}\int_{-K}^{K}|H(x)|dx=o(1) \tag{95}$$

almost surely under our assumptions and by the definition of c_n. For $|T_{3n\varepsilon}-T_{4n\varepsilon}|$ we have

$$|T_{3n\varepsilon}-T_{4n\varepsilon}|$$

$$\leq \nu(n^{1/2})^{-1}\int_{-K/\varepsilon}^{K/\varepsilon}\varepsilon T(n^{1/2}j\varepsilon)(\varepsilon^{-1}\int_0^1 I(j\varepsilon\leq W(r)\leq(j+1)\varepsilon)dr-L(1,j\varepsilon))dj$$

$$\leq \nu(n^{1/2})^{-1}\int_{-K/\varepsilon}^{K/\varepsilon}\varepsilon T(n^{1/2}j\varepsilon)dj\sup_{|x|\leq K}|\varepsilon^{-1}\int_0^1 I(x\leq W(r)\leq x+\varepsilon)dr-L(1,x)|. \tag{96}$$

By the earlier argument,

$$\sup_{n\geq 1}\sup_{\varepsilon>0}\nu(n^{1/2})^{-1}\int_{-K/\varepsilon}^{K/\varepsilon}\varepsilon T(n^{1/2}j\varepsilon)dj<\infty, \tag{97}$$

and therefore it suffices to show that as $\varepsilon \to 0$,

$$\sup_{|x|\leq K}|\varepsilon^{-1}\int_0^1 I(x\leq W(r)\leq x+\varepsilon)dr-L(1,x)|\to 0. \tag{98}$$

By the occupation times formula, the above expression satisfies

$$\sup_{|x|\le K} |\varepsilon^{-1} \int_x^{x+\varepsilon} L(1,s)ds - L(1,x)| = \sup_{|x|\le K} |\varepsilon^{-1} \int_x^{x+\varepsilon} (L(1,s) - L(1,x))ds|$$

$$\le \sup_{|x|\le K} \sup_{s\in[x,x+\varepsilon]} |L(1,s) - L(1,x)| \to 0 \quad \text{as } \varepsilon \to 0 \tag{99}$$

by uniform continuity of $L(1,.)$ on $[-K,K]$. Finally, for $|T_{4n\varepsilon} - T_5|$, we have

$$\lim_{n\to\infty} | \int_{-K}^{K} (\nu(n^{1/2})^{-1} T(n^{1/2}s) - H(s))L(1,s)ds|$$

$$\le \sup_{|s|\le K} |L(1,s)| \lim_{n\to\infty} \int_{-K}^{K} |\nu(n^{1/2})^{-1} T(n^{1/2}s) - H(s)|ds = 0 \tag{100}$$

by the definition of an asymptotically homogeneous function, which completes the proof. □