Small Income Effects Destroy the Constrained Efficiency of All Equilibria in Finance Economies with Production. *

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Abstract

We consider economies with incomplete markets, one good per state, private ownership of initial endowments, a single firm, and no assets other than shares in this firm. In this simple framework, arbitrarily small income effects can render every market equilibrium resulting from some production decision constrained inefficient. Thus, even if all utility functions are approximately quasilinear, the stock market can be unable to achieve a constrained efficient allocation given the agents' characteristics. Moreover, the phenomenon persists when the efficiency requirements are substantially weakened.

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1 Introduction

We consider finance economies with production. More precisely, we consider economies with incomplete markets, one good per state, private ownership of initial endowments, production, and two time periods. Due to the incompleteness of markets, shareholders typically disagree about which production decision their firm should take. Drèze (1974) presents a way of resolving the conflict among shareholders by introducing an equilibrium concept that is based on Pareto comparisons with the aim of achieving constrained efficiency. We restrict ourselves to economies with one good per state in order to rule out price effects, which are a well-known cause of constrained inefficiency [cf. Geanakoplos et al. (1990)].

In this paper we show that, unless restrictive assumptions are made, the market in these finance economies may not be able to achieve a constrained efficient Drèze equilibrium or any other constrained efficient allocation. If quasilinearity of all utility functions is assumed, a constrained efficient Drèze equilibrium exists. However, arbitrarily small income effects are sufficient to render all Drèze equilibria constrained inefficient. To demonstrate this we consider economies with only one firm.

The firm has constant returns to scale and makes zero profit. Its state dependent output at $t = 1$ is sold on the asset market in exchange for the corresponding input. When the firm proposes a production ray, consumers choose their optimal investments and this determines their consumption in all states. The firm adjusts its production level to the market clearing scale. The resulting allocation is called a market equilibrium. The set of all allocations the market can achieve consists of all market equilibria corresponding to some production decision of the firm.

A Drèze equilibrium is a market equilibrium with the following property: The (new) shareholders of the firm meet at $t = 0$ after they have chosen their shares optimally. If these shares are held fixed, there is no other production plan such that the shareholders of the firm can achieve a Pareto improvement by adopting that production plan and by making sidepayments at time $t = 0$ to reach unanimity.[1]

Constrained efficiency means that a hypothetical planner cannot find a Pareto improvement by simultaneously choosing the production plan, the shares, and each individual's consumption at $t = 0$. Note that a constrained efficient market equilibrium is a Drèze equilibrium.

An example of an economy with a unique, but constrained inefficient Drèze equilibrium is presented in Dierker, Dierker, and Grodal (2001). This example is driven by the existence of a consumer whose preferences exhibit strong income effects. If there are no income effects, that is to say, if all consumers have

[1]For an extensive treatment of Drèze equilibria in a setting with private ownership of initial endowments, the reader is referred to Magill and Quinzii (1996), chapter 6.

quasilinear utility functions, then at least one constrained efficient Drèze equilibrium exists, since the social surplus is well defined and is maximized at a Drèze equilibrium.

We show that arbitrarily small perturbations of quasilinear utility functions can destroy the constrained efficiency of all Drèze equilibria, even if all utility functions remain additively separable. In our example, we start with a quasilinear economy with three Drèze equilibria. Two of them are surplus maxima and the third is a surplus minimum. Then we perturb the quasilinear utility functions of the example by adding a small term to the utility at $t = 0$. The perturbation does not affect the way in which future consumption streams are ranked, i.e. utility at $t = 1$ is left unchanged. These small perturbations leave the set of Drèze equilibria invariant. However, for arbitrarily small perturbations, all Drèze equilibria, and hence all market equilibria, become constrained inefficient.

Since the planner, who can implement constrained efficient allocations, is more powerful than the market, we reduce the planner's power substantially and explore whether the planner can still outperform the market. We introduce the following very weak version of constrained efficiency, in which tomorrow's consumption can only be affected by the planner through the choice of the production plan. After the planner has chosen a normalized production plan (where the input is normalized to -1), consumers choose their optimal investments subject to their budget constraints. The firm adjusts production to the market clearing scale. The planner, who is no longer allowed to alter individual consumption at $t = 1$, can only distribute the resources remaining at $t = 0$ after subtracting the input. An allocation is called minimally constrained efficient, if the planner, who is subject to these constraints, cannot find a Pareto improvement. It turns out that no market equilibrium in the perturbed quasilinear example is even minimally constrained efficient.

The notion of minimal constrained efficiency cannot be weakened further, since the planner should at least retain the possibility of changing the production plan and redistributing total consumption at $t = 0$. We conclude that, even in economies with one good per state, arbitrarily small income effects can make it impossible to select a production plan that achieves a market equilibrium satisfying at least some weak version of constrained efficiency. The question of how to choose a market equilibrium remains open and is briefly discussed at the end of the paper.

The remainder of the paper is organized as follows. In Section 2 the quasilinear and the perturbed quasilinear examples are given. In Section 3 the notion of minimal constrained efficiency is presented and discussed. It is shown that minimally constrained efficient market equilibria need not exist. Section 4 contains concluding remarks.

2 Nonexistence of Constrained Efficient Equilibria when Income Effects are Small.

An example of a finance economy with a unique, but constrained inefficient Drèze equilibrium is presented in Dierker, Dierker, and Grodal (2001). In the example, one consumer has preferences exhibiting very strong income effects.

Clearly an example without any constrained efficient market equilibrium cannot be constructed if all consumers have quasilinear preferences. In this case, consumers' surplus is well defined and can be used as a welfare measure. A constrained efficient market equilibrium is obtained by maximizing consumers' surplus.

It is natural to ask whether the existence of a constrained efficient market equilibrium in the quasilinear setting is a robust phenomenon. In order to answer this question we first analyze the efficiency properties of equilibria in a quasilinear example. We use the following framework.

We consider two periods, $t = 0, 1$, and two possible states of nature at $t = 1$ denoted $s = 1$ and $s = 2$. The unique state at $t = 0$ is included as the state $s = 0$. There is a single good in each state. There is just one firm. It transforms input at $t = 0$ into state dependent outputs at $t = 1$. We assume that there are no other assets. The firm has constant returns to scale and makes zero profits. Its technology is given by a family of normalized production plans $(-1, \lambda, 1 - \lambda)$. Denote the production set by

$$Y = \{\alpha(-1, \lambda, 1 - \lambda) \in \mathbb{R}^3 \mid \alpha \geq 0,\ \lambda \in [0.1,\ 0.9]\}.$$

The ray λ is assumed to stay in the interval $[0.1, 0.9]$ to ensure that the group of shareholders always coincides with the set of all consumers.

There are two types of consumers. Ideally, each type would be represented in the economy by a continuum of mass 1. For convenience, we refer to each continuum of identical consumers as a single consumer denoted $i = 1, 2$. The consumers have initial endowments $e^1 = e^2 = (2,\ 0,\ 0)$, consumption sets $\mathbb{R}^3_+$, and utility functions U^1, U^2, respectively.

If the firm selects the normalized production plan $(-1, \lambda, 1-\lambda)$ and consumer i chooses the investment $\alpha^i \geq 0$ in the firm, the resulting consumption bundle is $e^i + \alpha^i(-1, \lambda, 1 - \lambda)$. The consumer selects α^i so as to maximize utility in the budget set

$$B^i(\lambda) = \{e^i + \alpha^i(-1, \lambda, 1 - \lambda) \in \mathbb{R}^3_+ \mid \alpha^i \geq 0\}.$$

Let $\alpha^i(\lambda)$ denote i's optimal investment given the normalized production plan $(-1, \lambda, 1 - \lambda)$. Thus, agent i consumes $x^i(\lambda) = e^i + \alpha^i(\lambda)(-1, \lambda, 1 - \lambda)$, holds shares equal to $\vartheta^i = \alpha^i(\lambda)/(\alpha^1(\lambda) + \alpha^2(\lambda))$, and the firm produces $y(\lambda) = [\alpha^1(\lambda)+\alpha^2(\lambda)](-1, \lambda, 1-\lambda)$. For any $\lambda \in [0.1, 0.9]$, the system $(y(\lambda), x^1(\lambda), x^2(\lambda))$ is called a *market equilibrium* with respect to λ. The market equilibria are the only allocations that the market can achieve. In general, these allocations cannot be Pareto compared and the shareholders face a social choice problem. In order to resolve the problem, Drèze (1974) suggested letting shareholders use sidepayments among themselves at $t = 0$ in order to reach unanimity.

A Drèze equilibrium is a market equilibrium in which the production plan of the firm passes the following test: It is impossible for the shareholders to find another production plan and sidepayments at $t = 0$ such that all shareholders are better off, if they use their original investment levels and get the sidepayments.[2] More precisely, consider a market equilibrium $(y(\tilde{\lambda}), x^1(\tilde{\lambda}), x^2(\tilde{\lambda}))$ with respect to $\tilde{\lambda}$ and let $\mathcal{I} = \{i \mid \alpha^i(\tilde{\lambda}) > 0\}$. The market equilibrium is a *Drèze equilibrium* if it is impossible to find a normalized production plan $(-1, \lambda, 1-\lambda)$ and a system of sidepayments $(\tau^i)_{i\in\mathcal{I}}$ at $t = 0$ with $\sum_{i\in\mathcal{I}} \tau^i = 0$ such that

$$U^i(e^i + \tau^i(1, 0, 0) + \alpha^i(\tilde{\lambda})(-1, \lambda, 1-\lambda)) > U^i(x^i(\tilde{\lambda}))$$

for every $i \in \mathcal{I}$. Note that the production plan $(-1, \lambda, 1-\lambda)$ on the left hand side of the above inequality is multiplied by the investment level $\alpha^i(\tilde{\lambda})$ that is optimal at the normalized production plan $(-1, \tilde{\lambda}, 1-\tilde{\lambda})$.

We recall the definitions of feasibility constrained by market incompleteness and constrained efficiency [cf. Magill and Quinzii (1996)]. For a vector $x \in \mathbb{R}^3$ we write $x = (x_0, x_{\mathbf{1}})$, where $x_0 \in \mathbb{R}$ corresponds to $t = 0$ and $x_{\mathbf{1}} \in \mathbb{R}^2$ corresponds to $t = 1$. An allocation (y, x^1, x^2) is constrained feasible if it can be implemented by a planner who simultaneously determines the production plan $y = (y_0, y_{\mathbf{1}}) \in Y$, the shares ϑ^i of all consumers and who, moreover, freely redistributes the good x_0 at $t = 0$. More precisely, the allocation $((y_0, y_{\mathbf{1}}), (x_0^1, x_{\mathbf{1}}^1), (x_0^2, x_{\mathbf{1}}^2)) \in Y \times \mathbb{R}^3_+ \times \mathbb{R}^3_+$ is *constrained feasible* if $x_0^1 + x_0^2 = e_0^1 + e_0^2 + y_0$ and there exist shares $\vartheta^i \geq 0$ such that $x_{\mathbf{1}}^i = \vartheta^i y_{\mathbf{1}}$ for all i and $\sum_i \vartheta^i = 1$. Note that the set of constrained feasible allocations only depends on the aggregate endowments at $t = 0$ and that it is, in general, larger than the set of market equilibria. A constrained feasible allocation is called *constrained efficient* if there does not exist a Pareto superior constrained feasible allocation.

In searching for constrained efficient market equilibria we can restrict attention to the set of Drèze equilibria since a constrained efficient market equilibrium is a Drèze equilibrium.

We now analyze the efficiency properties of Drèze equilibria in an example in which all consumers have quasilinear preferences. Let the consumers have

[2] In the usual definition of a Drèze equilibrium, shares ϑ^i, and not the investment levels α^i, are taken as fixed when a production plan is evaluated. The two definitions are equivalent.

quasilinear utilities given by

$$U^1(x_0, x_1, x_2) = x_0 + x_1^{0.6},$$
$$U^2(x_0, x_1, x_2) = x_0 + x_2^{0.6},$$

respectively.

It turns out that the economy under consideration has three Drèze equilibria, A, B and C, corresponding to $\lambda_A = 0.1, \lambda_B = 1/2$, and $\lambda_C = 0.9$, respectively. In the definition of a Drèze equilibrium, shares are kept fixed when shareholders evaluate alternative production plans. In order to gain insight into the consequences of this assumption it is useful to investigate the interior equilibrium B. To do this we first consider the indirect utility $u^1(2, \lambda)$ that consumer 1 with endowment $e_0^1 = 2$ at $t = 0$ obtains, if the firm chooses the ray λ and if consumer 1 makes the optimal investment $\alpha^1(\lambda) = 0.6(0.6\lambda)^{1.5}$. Since this utility equals $u^1(2, \lambda) = 2 + 0.4(0.6\lambda)^{1.5}$, the function $u^1(2, \cdot)$ is convex. Similarly, the utility level of consumer 2 at λ equals $u^2(2, \lambda) = u^1(2, 1 - \lambda)$ and is convex in λ. As a consequence, shareholders' social surplus associated with the ray λ, $u^1(2, \lambda) + u^2(2, \lambda)$, is convex in λ. Due to the symmetry between $u^1(2, \lambda)$ and $u^2(2, \lambda)$, the social surplus has a critical point at $\lambda_B = 1/2$ which must be a global minimum [see Figure 1].

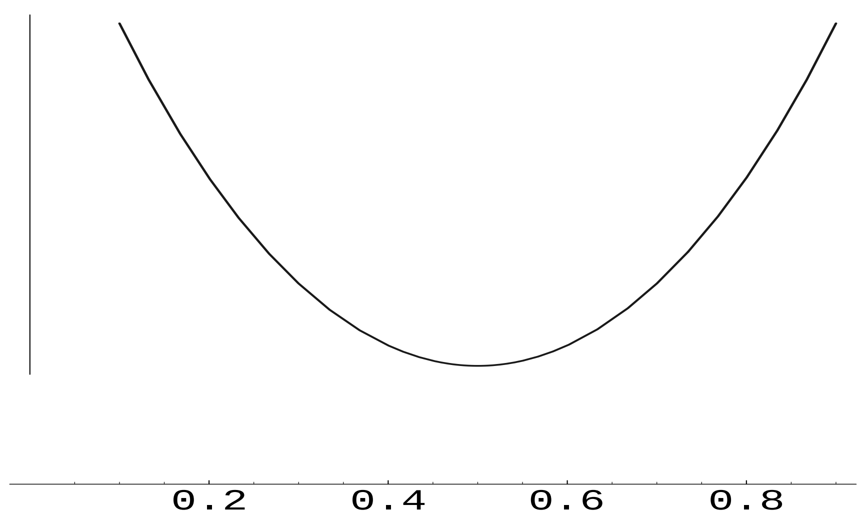

Figure 1: Surplus minimum at the Drèze equilibrium $\lambda_B = 1/2$

Observe that the situation changes drastically if the shareholders are deprived of the possibility of adjusting their shares, or, equivalently, their investment levels, when λ_B is tested against some alternative λ. Consider consumer 1 who wants to choose the investment level $\alpha^1(\lambda)$ in proportion to $\lambda^{1.5}$. If α^1 is now taken as fixed at its value at $\lambda_B = 1/2$, then the utility reached at ray λ equals $\tilde{u}^1(2, \lambda) =$

$c_0 + c_1\lambda^{0.6}$ with $c_1 > 0$, whereas the indirect utility with share adjustment is a function of the type $u^1(2,\lambda) = c_0' + c_1'\lambda^{1.5}$ with $c_1' > 0$. Thus, by disregarding how consumer 1's individual investment level $\alpha^1(\lambda)$ varies with λ, the originally convex function $u^1(2,\cdot)$ is turned into a concave function $\tilde{u}^1(2,\cdot)$. As a consequence, $\tilde{u}^1(2,\cdot) + \tilde{u}^2(2,\cdot)$ is a concave function and the critical point $\lambda = 1/2$ becomes a maximum. For this reason, λ_B yields a Drèze equilibrium. Clearly, the utility sum $\tilde{u}^1(2,\cdot) + \tilde{u}^2(2,\cdot)$ constructed by fixing the shares does not represent owners' welfare at alternative production rays correctly.

At the Drèze equilibria A and C, consumers' social surplus is maximized. Hence, A and C are constrained efficient.

Now we perturb the quasilinear example by altering the utility derived from consumption at $t = 0$ without changing the utility obtained from consumption at $t = 1$. In particular, the utility functions U^i stay additively separable after perturbation.

$$U_a^1(x_0, x_1, x_2) = x_0 + ax_0^2 + x_1^{0.6} \quad \text{and} \quad U_a^2(x_0, x_1, x_2) = x_0 + ax_0^2 + x_2^{0.6}\,, \quad (1)$$

where $0 < a \leq 0.1$. It is easy to show that i's utility function is quasiconcave in the relevant range.

As in the unperturbed example, the production ray varies in the interval $[0.1, 0.9]$ and there are three Drèze equilibria corresponding to $\lambda_A = 0.1$, $\lambda_B = 0.9$, and $\lambda_C = 0.5$, respectively. However, the boundary equilibria are no longer constrained efficient for any $a > 0$.

Let, for example, $a = 0.1$ and consider the ray corresponding to $\lambda = 0.9$. In the corresponding market equilibrium the consumers obtain the utility $u_{0.1}^1(2, 0.9) \approx 2.4969$ and $u_{0.1}^2(2, 0.9) \approx 2.4036$, respectively. Now consider the sidepayment $\tau^1(0.1)$ that is necessary to keep consumer 1 at the utility level $u_{0.1}^1(2, 0.9)$ if the ray 0.9 is replaced by the ray $\lambda = 0.1$. Let $x^1(2 + \tau^1, \lambda)$ be consumer 1's optimal consumption plan at ray λ after the sidepayment has increased consumer 1's endowment at $t = 0$ to $2 + \tau^1$. Then $\tau^1(0.1)$ is given by

$$U_{0.1}^1(x^1(2 + \tau^1(0.1), 0.1)) = u_{0.1}^1(2, 0.9)\,.$$

Let $\tau^2(0.1)$ be defined in a similar way.

Calculation shows that $\tau^1(0.1) \approx 0.0664$ and $\tau^2(0.1) \approx -0.0681$. Thus $\tau^1(0.1) + \tau^2(0.1) \approx -0.0017 < 0$. Hence, the Drèze equilibrium corresponding to $\lambda = 0.9$ is Pareto dominated by a constrained feasible allocation in which the production ray is $\lambda = 0.1$. It follows from the Remark in Section 3 that the same statement holds for every $a > 0$. Due to symmetry, the Drèze equilibrium corresponding to the production ray $\lambda = 0.1$ is Pareto dominated by a constrained feasible allocation corresponding to the production ray 0.9. We conclude that the existence of a constrained efficient market equilibrium in the quasilinear case is destroyed when arbitrarily small income effects are introduced.

3 Minimal Constrained Efficiency

In the example a planner who can implement constrained efficient allocations can Pareto dominate all market equilibria. In this section we investigate whether the planner can be weakened such that at least one market equilibrium is Pareto undominated.

The planner and the firm in the market economy can both determine the available asset. However, the planner is more powerful than the market, since the planner is not bound to respect individual budget constraints. We weaken the planner by taking away the right to choose consumers' investment levels. This task is now performed by the market. That is to say, after the planner has chosen the normalized production plan, consumers make their optimal investments subject to their budget constraints. This determines each individual's consumption at $t = 1$.

More precisely, we proceed as follows. After the planner has chosen the normalized production plan $(-1, \lambda, 1 - \lambda)$, the stock market opens and each consumer i chooses the investment $\alpha^i(\lambda)$ such that the resulting consumption bundle $(x_0^i, x_{\mathbf{1}}^i) = (x_0^i(\lambda), x_{\mathbf{1}}^i(\lambda)) = e^i + \alpha^i(\lambda)(-1, \lambda, 1 - \lambda)$ maximizes the consumer's utility in the budget set. Then the stock market is closed and nobody, including the planner, can change $x_{\mathbf{1}}^i(\lambda)$. The only possibility still available to the planner is to redistribute total consumption $\sum_i x_0^i(\lambda)$ at $t = 0$.

A constrained feasible allocation is minimally constrained efficient if it is not possible for a planner, who is restricted by the market as explained above, to Pareto improve upon the allocation.

Definition . *A constrained feasible allocation is called* minimally constrained efficient *if there is no Pareto superior allocation* $(\lambda, (c_0^i, x_{\mathbf{1}}^i)_i)$ *satisfying*

(i) $x_{\mathbf{1}}^i = e_{\mathbf{1}}^i + \alpha^i(\lambda)(\lambda, 1 - \lambda)$*, where* $\alpha^i(\lambda)$ *is i's optimal investment given the ray* λ*,*

(ii) $\sum_i c_0^i = \sum_i e_0^i - \sum_i \alpha^i(\lambda)$*, and*

(iii) $\sum_i \alpha^i(\lambda)(-1, \lambda, 1 - \lambda) \in Y$*.*

In contrast to the definition of constrained efficiency, the investments in condition (i) depend on the distribution of initial endowments and the ray λ. The condition says that, after the planner has chosen the production ray, individual consumption at $t = 1$ is determined by the market. Condition (ii) states that the planner can redistribute the aggregate consumption $\sum_i e_0^i - \sum_i \alpha^i(\lambda)$ at

$t = 0$. Condition (iii) says that the planner adjusts the level of production to the consumers' aggregate investment.

The concept is called minimal constrained efficiency for the following reason. The planner must have the power to select the asset at $t = 0$. Hence, after we have deprived the planner of the right to choose shares, we cannot take away the only remaining tool, that is to say, the power to make compensations at $t = 0$.

Our method of defining minimal constrained efficiency can also, in principle, be used if there are several goods in each state. In this case, even equilibria with respect to fixed assets are typically constrained inefficient due to price effects. Therefore, Grossman (1977) weakened the definition of constrained efficiency by introducing a central planner with incomplete coordination. In Grossman's concept of a social Nash optimum, the planner cannot act simultaneously in different states. Grossman shows that, in the model with fixed assets, an equilibrium is a social Nash optimum. Apart from the ability to choose λ, our planner is weaker than Grossman's, since the shareholdings and the individual consumption in each state s at $t = 1$ are determined by individual optimization on the asset and spot markets and cannot be altered by our planner. Also, at $s = 0$ our planner can only redistribute the resources that have not been used for production in accordance with consumers' investment decisions.

Numerical computation shows that the unique Drèze equilibrium in the example in Dierker, Dierker, and Grodal (2001) is minimally constrained efficient, but not constrained efficient. Therefore, one would like to know whether at least one Drèze equilibrium in a finance economy is minimally constrained efficient.

In order to answer this question we again analyze the perturbed quasilinear example from Section 2. We show that arbitrarily small income effects prevent the originally constrained efficient Drèze equilibria from being minimally constrained efficient.

Remark . *For arbitrarily small $a > 0$, no market equilibrium associated with some ray λ is minimally constrained efficient.*

It has been claimed in Section 2 that constrained inefficiency of all Drèze equilibria holds for any $0 < a \leq 1$. The claim follows from the Remark, since a minimally constrained inefficient market equilibrium is also constrained inefficient.

Proof. Consider any ray λ and the corresponding market equilibrium allocation. Clearly, the equilibrium corresponding to $\lambda = 0.5$ is not minimally constrained efficient. Therefore, let $\lambda \neq 0.5$. We show that the production ray $1 - \lambda$, together with a suitable reallocation of consumption at $t = 0$ is preferred to λ by both types of consumers. Due to symmetry we can assume $\lambda < 0.5$.

Agent i consumes $x^i(\lambda) \in B^i(\lambda)$ when the ray λ is chosen. If λ is replaced by $1-\lambda$, agent i consumes $x^i(1-\lambda)$ and achieves the utility level $U_a^i(x^i(1-\lambda)) \neq U_a^i(x^i(\lambda))$. Let τ^i be the amount of good 0 required in addition to $x^i(1-\lambda)$ in order to let i achieve the original utility level $U_a^i(x^i(\lambda))$. More precisely,

$$U_a^1(x^1(1-\lambda) + \tau^1(1,0,0)) = U_a^1(x^1(\lambda)) \tag{2}$$

and

$$U_a^2(x^2(1-\lambda) + \tau^2(1,0,0)) = U_a^2(x^2(\lambda))\,. \tag{3}$$

Since $\lambda < 0.5 < 1-\lambda$, we have $\tau^1 < 0$ and $\tau^2 > 0$. Moreover, by symmetry, $x_0^1(\lambda) = x_0^2(1-\lambda)$ and

$$U_a^1(x^1(\lambda)) = U_a^2(x^2(1-\lambda))\,, \qquad U_a^2(x^2(\lambda)) = U_a^1(x^1(1-\lambda))\,. \tag{4}$$

We add (2) and (3), use symmetry and the utility specifications (1), and obtain

$$(\tau^1+\tau^2) + a((\tau^1)^2 + (\tau^2)^2) + 2a(\tau^1 x_0^1(1-\lambda) + \tau^2 x_0^1(\lambda)) = 0\,. \tag{5}$$

Since calculation of consumer 1's optimal shares yields that the demand for good zero is strictly decreasing, we have $x_0^1(\lambda) > x_0^1(1-\lambda) > 0$.

Assume that $\tau^1+\tau^2 \geq 0$ and, hence, $\tau^2 \geq |\tau^1|$. Then $\tau^2 x_0^1(\lambda)) > |\tau^1 x_0^1(1-\lambda)|$. Therefore, the left hand side of (5) must be strictly positive for every $a > 0$, which is a contradiction. We conclude that $\tau^1 + \tau^2 < 0$. Hence, the equilibrium corresponding to λ is not minimally constrained efficient. □

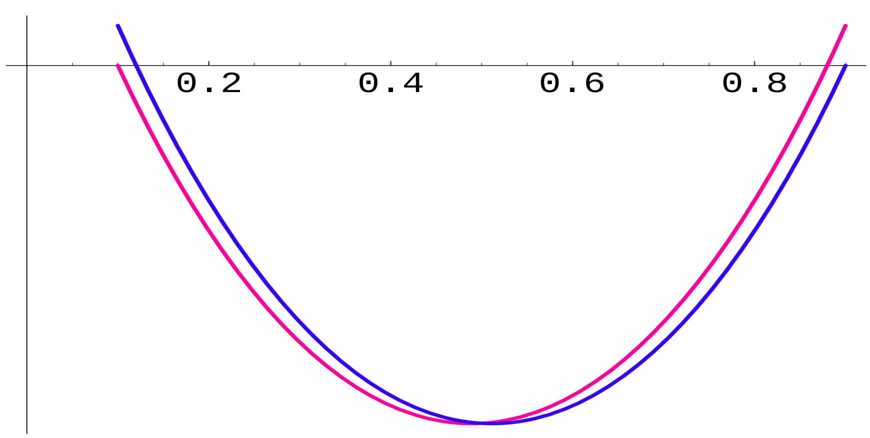

Figure 2: Intersecting total "saving" functions

Figure 2 illustrates the case $a = 0.1$. Take the equilibrium at 0.1 and consider the sidepayment $\tau^1(\lambda)$ necessary to keep consumer 1 at the utility level

$U_a^1(x^1(0.1))$ if the ray 0.1 is replaced by the ray λ. That is to say, $\tau^1(\lambda)$ is given by

$$U_a^1(x^1(\lambda) + \tau^1(\lambda)(1,0,0)) = U_a^1(x^1(0.1)) .$$

Let $\tau^2(\lambda)$ be defined in a similar way. Thus, $\tau^1(\lambda) + \tau^2(\lambda)$ specifies the total amount of compensation required to maintain the utility levels achieved at 0.1. The relationship to Figure 1 becomes clearer if the compensation is replaced by $-(\tau^1(\lambda) + \tau^2(\lambda))$, which is the amount of good 0 that can be saved at λ while keeping consumer i on the utility level $U_a^i(x^i(0.1))$. This total "saving" function becomes positive at $\lambda = 0.9$, which indicates that the equilibrium with respect to $\lambda = 0.1$ is not minimally constrained efficient. A similar saving function can be defined if the other boundary $\lambda = 0.9$ is taken as reference point. If a goes to 0, both curves in Figure 2 approach the social surplus curve depicted in Figure 1 (up to a constant).

The nonexistence of constrained efficient and minimally constrained market equilibria is caused by the following facts. First, the example is built upon a nonconvexity. In the unperturbed, quasilinear example, the nonconvexity can be described as follows. The amount of good 0 initially available in the economy just suffices to maintain the utility profile $(u^1(2,0.1),\ u^2(2,0.1))$ reached at the boundary point $\lambda = 0.1$, if the other boundary point $\lambda = 0.9$ is chosen. However, if the firm implements any ray λ strictly between 0.1 and 0.9, this amount is insufficient. Second, as soon as the perturbation parameter a becomes positive, the graphs of the two saving functions intersect each other. To maintain the profile $(u_a^1(2,0.1),\ u_a^2(2,0.1))$ at $\lambda = 0.9$, one can dispense with a positive amount of good 0. A similar statement holds, if the two boundary points are interchanged [cf. Figure 2]. These two features cannot be ruled out in general. Therefore, one cannot expect the market to be able to achieve minimally constrained efficient outcomes.[3]

The allocations attainable by the market depend on the initial allocation of endowments. To obtain a situation in which a constrained efficient market equilibrium exists in the perturbed example, a lump sum redistribution of initial endowments is required. Markets do not perform such redistributions and thus, are less powerful than even the very weak planner discussed in the context of minimal constrained efficiency. The importance of the initially determined distribution of wealth in nonconvex environments was first pointed out by Guesnerie (1975) in the framework of complete markets and nonconvex production sets. Guesnerie showed that all marginal cost pricing equilibria can be inefficient, even though Pareto efficiency requires prices to equal marginal costs.

[3]It has been emphasized in the literature on compensation criteria à la Hicks and Kaldor that intersecting utility possibility frontiers often entail inconsistent policy recommendations [see, e.g., Gravel (2001)].

4 Concluding Remarks

We have seen that shareholders' social surplus can reach its minimum at a Drèze equilibrium if all shareholders have quasilinear utilities. This is due to the fact that the definition of a Drèze equilibrium only takes welfare changes of first order into account. Thus, no distinction is made between an interior maximum and any other critical point.

In the quasilinear case, a constrained efficient Drèze equilibrium exists. Therefore, it is tempting to refine the Drèze equilibria in order to rule out constrained inefficient allocations. However, our example shows that this endeavor can fail to provide any solution as soon as one deviates from the quasilinear setting: Arbitrarily small income effects render all market equilibria constrained inefficient.

Moreover, even if the efficiency requirements are substantially reduced, they can remain unfulfilled at every market equilibrium in a finance economy. In our example the stock market cannot even achieve a minimally constrained efficient outcome if the quasilinear setting is abandoned. Hence, the existence of a constrained efficient equilibrium in the quasilinear economy should be viewed as an artifact lacking any robustness.

Clearly, there are economies in which the problem does not arise. For example, Drèze equilibria are constrained efficient, if there is only one firm and if every consumer's indirect utility function is quasiconcave. This function describes the maximum amount of utility the consumer can derive from a production decision at different levels of wealth at $t = 0$. The indirect utility functions underlying Figure 1 are not quasiconcave. This is due to the fact that the specification of the direct utility functions U^i makes optimal shareholdings sufficiently sensitive to changes in the production ray.[4] Since the indirect utility depends on how the optimal number of shares, that an individual holds, varies with the asset span and individual wealth at $t = 0$, it is, unless attention is restricted to particularly simple examples, quite difficult to state economically meaningful conditions ensuring the quasiconcavity of indirect utility functions. We do not think that imposing restrictions on consumers' characteristics presents a promising approach to overcome the problem of nonexistence of constrained efficient market equilibria.

Majority voting presents another way to overcome the social choice problems faced by shareholders. For properties of corporate control by majority voting, see DeMarzo (1993) and Geraats and Haller (1998). Apart from problems such as equilibrium existence, agenda control, non sincere voting etc., the following point deserves attention. Since the voting outcome depends on power, it need not reflect welfare appropriately. The point is easily understood in the context of

[4] If the power 0.6 in the definition of U^i is replaced by a number below 0.5, quasiconcavity of the indirect utility function u^i results.

the quasilinear example in Section 2. To break ties, a third quasilinear consumer with arbitrarily small weight is introduced, whose utility increases if the ray λ approaches 1/2. This additional consumer becomes the median voter. Due to symmetry, majority voting leads to $\lambda = 1/2$ if every shareholder has one vote. Moreover, it is not difficult to modify the example such that voting according to the one share-one vote rule yields the same outcome. The median voter, although of arbitrarily small weight, has overwhelming power. The median voter's optimal choice, though, is the welfare minimum. Thus, majority voting should be seen as a modeling device that is better suited for positive than for normative purposes.

Instead of examining whether a proposed production plan can be unanimously improved upon after sidepayments are made, one can compare the gains, expressed in units of good 0, that are obtained from any production plan in comparison to a given reference point. In the perturbed quasilinear example the point of zero production, that is to say, the allocation (e^1, e^2) of initial endowments, can be used for reference. Consumer i's surplus $S^i(\lambda)$ is given by the amount of good 0 consumer i needs in excess of e^i to obtain the same utility level as if the firm chose the ray λ. The total surplus associated with some market equilibrium can then be maximized. In the perturbed quasilinear example the maximum is taken at both boundary points $\lambda = 0.1$ and $\lambda = 0.9$. Thus, the same outcome as in the quasilinear case is obtained.

A major advantage of this approach lies in the fact that it relies on the maximization of continuous functions rather than maximization of incomplete, intransitive, and nonconvex relations. The surplus maximum is characterized as follows: It presents the minimum amount of good 0 needed in the absence of the firm in order to be able to compensate all consumers such that they can attain every utility profile that is induced by some production decision. Clearly, this type of surplus maximization, which is motivated by the lack of constrained efficient market equilibria, does not aim at achieving constrained efficiency and its theoretical foundation remains controversial.

The surplus function described above can be viewed as a particular social welfare function. To overcome the problem of the nonexistence of constrained (or even minimally constrained) efficient market equilibria, one might also resort to any other social welfare function. However, it is a priori unclear which welfare function is particularly well suited for this purpose.[5]

A less radical procedure suggesting itself in the perturbed quasilinear example is the choice of the boundary equilibria $\lambda = 0.1$ or $\lambda = 0.9$ on the basis that they are "less inefficient" than, say, $\lambda = 0.5$. To define the degree of inefficiency, interpersonal utility comparisons are not required.

[5] In another context involving lotteries, Dhillon and Mertens (1999) argue in favor of relative utilitarianism.

The last three approaches provide welfare oriented ways that may be used to overcome the problem laid out in this paper. In each case a particular function is optimized. These approaches require a large amount of information and are far more complex than the usual profit maximization in General Equilibrium Theory with complete markets. They would change the character of the theory considerably.

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