Network Formation and Coordination: Bargaining the Division of Link Costs

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Abstract

This paper presents a model of network formation in which links are costly. We endogeneize the part of the cost supported by each of the players involved in a bilateral link. In this sense we consider that these sharings result from bargaining. We study this process in a context of coordination games. We show that, if this cost is not too high, players coordinate either in the risk-dominant action or in the efficient one: if costs of forming links are higher than the risk-dominance premium the efficient action is selected; meanwhile, if they are lower, the risk-dominant action prevails.

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1 Introduction

There are social and economic situations in which the existence of some kind of connections between the agents is necessary to interact. We can think for example on information transmission: agents need some way of communication in order to be able to exchange information. In many cases the establishment and maintenance of these connections is costly. We model a situation in which the benefits from interacting is related to coordination, that is, any two players who establish a link benefit if they are coordinated in the same action. The main feature of our model is how the agents who form a link share the cost it involves. We propose that this division results from bargaining; in this sense, we make the agents' shares of the link cost endogenous.

The model deals with the choice of a standard in a population (e.g. PC vs. Mackintosh) and with the network formation, that is, given the choices on standards, each agent decides who she wants to interact with (i.e. to form links). The earnings of the interaction between two agents (i.e. of forming a link) are represented by the payoffs of a 2x2 symmetric coordination game in which we identify the actions with the standards chosen. This game is characterized by two pure strategy Nash equilibria, one efficient and the other risk-dominant. The formation of a link is costly and we consider that the part of the cost each of the involved agent supports results from bargaining.

Thus, in this model each player will first decide a standard and then, each possible pair of players enters in a bargaining process in which they have to agree on how to share the cost (and form a link), or reach the outside option of not forming the link. We propose the Nash Bargaining solution to distribute the cost. We find that this game presents multiplicity of equilibria. Therefore we introduce a dynamics in which, from any initial state, each period players receive revision opportunities. We assume a best-response adjustment to update strategies. We analyze the set of limit states of this process and we get that the initial multiplicity persists. To deal with the equilibrium selection we use stochastic stability techniques. We find a treshold for the link cost that coincides with the risk-dominance premium. If the cost of the link is lower than this treshold we get that all the population coordinates in the (inefficient) risk-dominant standard. If the cost of the link is higher, efficiency is achieved, provided that the cost is not so high that no link can be profitable for both players involved.

The study of networks has been increasingly considered in the literature in the last years. Specially relevant in this field is the work by Jackson and Wolinsky (1996), who study the stability and efficiency of social and economic networks; in their work they do not formally model the procedure through which a graph is formed. There are also studies that explicitly analyze the dynamic process of network formation. Consider for example Bala and Goyal (2000), Jackson and Watts (1998), and Watts (2001). In these models agents only decide about link formation and there are no other actions that influence their payoffs. Bala and Goyal (2000) develop a noncooperative model of network formation considering both directed and non-directed networks. They show convergence to strict Nash networks.

Specifically, if we consider models in which the formation of links is costly, in the literature we find two ways to tackle with the link cost: (i) the one-sided links models, that are characterized by the fact that the agent who proposes to form a link will completely cover the cost; (ii) the two-sided links models, in which each of the two agents involved in the link will share the cost in equal amount. In the setup of social coordination games (where our research fits) Goyal and Vega-Redondo (2000) is framed in the first kind of models; among the second ones we find Jackson and Watts (1999) and Droste, Gilles and Johnson (2000), in which a spatial location of agents is introduced. Both kind of models (one-sided and two sided) seem to be questionable since it is reasonable to argue that when two players have the possibility to form (or maintain) a link, the one who will get a greater payoff from it will be willing to support a higher part of the cost it involves. We propose the Nash solution to distribute the cost of a link in this setup of bilateral coordination games. This endogeneization of the distribution of the cost provides two important advantages to our model over the former ones: (i) now, whenever a link is profitable (the sum of the link-payoffs of the two agents is higher than the link-cost) the link will be formed, which in fact will result in a higher connectivity; and (ii) the cost supported by each player in a certain link will depend on the relative payoff of both agents involved in it. We get that our results are related to the first kind of models.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 provides an adaptation of the Nash solution to our model. Section 4 analyses the static game. Section 5 introduces a dynamic process of network formation and studies the stochastic stability. Section 6 concludes.

2 The model

Let $N = \{1, 2, \ldots, n\}$ be the set of players, where $n \geq 3$. The structure of the model is as follows: each player $i \in N$ chooses an action a_i from his available action space $A_i = A = \{\alpha, \beta\}$ $\forall i \in N$. Then, after observing the pattern of actions, link formation follows from the Nash solution. To characterize the conditions for a link to form, first we have to define what are the benefits and costs it implies for the involved players. Given the action chosen by any two players i and j, the establishment of a link ij between them conveys:

(i) a gross payoff for the agents given by the symmetric function $\pi : A \times A \to \Re_+$.This function is represented by the following matrix:

	α	β
α	d	e
β	f	b

Where the entries are all positive and represent the payoff for player i, $\pi(a_i; a_j)$, and for player j, $\pi(a_j; a_i)$, given the actions $a_i \in A$ and $a_j \in A$ they chose respectively. The following conditions hold:

$$d > f; \; b > e; \; d > b; \; b + f > d + e \tag{1}$$

The conditions $d > f$ and $b > e$ reflect that this represents the payoff matrix of a 2x2 symmetric coordination game; $d > b$ means that $(\alpha; \alpha)$ is the Pareto efficient equilibrium of this game; and $b + f > d + e$ that $(\beta; \beta)$ is the risk dominant equilibrium as defined by Harsanyi and Selten (1988).

(ii) Additionally, the establishment of the link ij implies a fixed cost $c > 0$ that has to be supported in some feasible way, $c = c_{ij} + c_{ji}$, between the involved players. c_{ij} represents the part of the cost paid by player i in the link ij and c_{ji} the share paid by player j. We impose the following conditions: $c_{ij} \geq 0$ and $c_{ji} \geq 0$; meaning that payoffs are not transferable in this model[1]. We assume that the quantities c_{ij} and c_{ji} are determined by the Nash bargaining solution[2], as detailed in next section.

As stated earlier, we propose that both link formation and cost sharing result from the Nash solution. Given a pattern of actions $a = (a_1; a_2; ::: ; a_n)$, for any $i; j \in N$, we define $g_{i;j}(a) \in \{0; 1\}$ as follows: $g_{i;j}(a) = 1$ if, according to the Nash solution, a link is formed between players i and j and $g_{i;j}(a) = 0$ otherwise[3]. Note that we fix $g_{ii} = 0; \forall i \in N$ as in this model it makes no sense that a player forms a link with himself. Given this notation, for any a, the set of links actually formed in which a player i is involved would be represented by the following vector: $g_i(a) = (g_{i;1}(a); ::: ; g_{i;i-1}(a); g_{i;i+1}(a); ::: ; g_{in}(a))$. Note that $g(a) = (g_1(a); g_2(a); ::: ; g_n(a))$ results in a Network Structure. The expression for $g_{i;j}(a)$ will be obtained in next section when we apply the Nash solution to our model.

Given a pattern of actions, the total payoff, $\Pi_i(a_i; a_{-i})$, a player $i \in N$ gets in this model is the sum of the payoffs she gets from all her formed links (according to the Nash solution) net of the shares of the cost she pays, that is[4],

$$\Pi_i(a_i; a_{-i}) = \sum_{j \neq i} g_{i;j}(a) \cdot [\pi(a_i; a_j) - c_{ij}] \tag{2}$$

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[1] Note that $c_{ij} < 0$ would convey that player i gets additional profits from the link ij which, as $c_{ij} + c_{ji} = 0$, she would receive from player j.

[2] This axiomatic aproach could be conceived as a simplification of the fact that when two players enhance the possibility of forming a link they strategically bargain the distribution of the cost by way of an infinite sequence of alternating offers (Rubinstein model), say whith some decay or risk of breakdown. Whe know that the limit of the outcome when this factor tends to dissapear coincides with the Nash bargaining solution (cf. Osborne and Rubinstein (1990)).

[3] Note that $g_{i;j}(a) = g_{j;i}(a)$ for any $i; j \in N$.

[4] Note that c_{ij} is not defined when the link ij is not formed. In order to precisely state expression (2), assume without loss of generality that in this case $c_{ij} = \frac{c}{2}$. This does not have any influence in the results, since it will belong to a zero-component of the expression ($g_{ij} = 0$).

To give an intuition to our model consider the following example.

Example 1 We have a group $N = \{1, 2, \ldots, n\}$ of researchers. Each one has to decide a type of software to work with (a_i). Suppose they have to choose between using Mathematica ($a_i = \alpha$) or Matlab ($a_i = \beta$). Once each agent has chosen her software, which can be observed by all of them, each possible pair of researchers decide if they are going to work together. The earnings from interacting for two agents [$\pi(a_i; a_j)$ and $\pi(a_j; a_i)$] are represented by the payoffs of a bilateral symmetric coordination game as described above, where the actions are identified with the kind of computer chosen. We consider that two researchers using the same software can exchange more information, which benefits them. We consider, for example, that the choice of Mathematica (action α) is efficient, meaning that two agents using this software, who combine the information, get the maximum possible earnings. We say the choice of Matlab (action β) is risk-dominant in the sense that it provides a higher expected payoff to an agent who has no information about which software will be used by the others (which in fact happens by the time she takes her decision on computer), and therefore assigns equal probability to both kinds, . Interaction between two researchers has a cost (c); think for example in costs of phone calls, of travelling to meet, of mail... So any two researchers have to cover this cost if they want to work together. They bargain the share of the cost each one is going to pay (c_{ij} and c_{ji}), where they know which are the earnings they would get and the fact that if they do not reach an agreement they will not interact. Finally, the total payoff a researcher gets is the sum of earnings over all her interactions net of the shares of the costs she supports.

In next section, taken as given a pattern of actions a, we specify how the Nash solution provides the network formation and the sharing of costs. That is, we will determine under which conditions a link ij forms (define $g_{i;j}(a)$), and how the involved players share the cost (c_{ij} and c_{ji}).

3 Bargaining the distribution of the link cost

In this section we analyze when any two agents form a link and how they share the fixed cost c that it conveys. We assume that these sharings will depend on the value of the cost c that takes on the link, the values of the entries of the payoff matrix of the bilateral coordination game, and the corresponding actions chosen by any of the two players in that game.

3.1 The Nash bargaining solution in the division of the link cost

We propose the Nash bargaining solution as the most appropriate one to apply in this model, as it is commonly accepted and it can be easily applied to our model. Additionally this solution presents the feature that the resultant cost division will be dependent on the payoff each agent can get, that is, we will see that the agent receiving a higher benefit from the link supports a higher part of the cost. We have to adapt our model to be able to apply the Nash solution, because this bargaining process is defined to study the sharing of positive benefits among two players and, however, what we have to analyze here is how players share costs. We will see below that we can use this process, applying a transformation on our payoffs.

3.2 The Nash bargaining solution

First we will briefly explain how the Nash bargaining solution works. This bargaining problem is characterized by a pair $(X; d)$ in which X represents the feasible payoff pairs and d is the pair of payoffs resulting from disagreement. The problem is defined for the set of pairs $(X; d)$ that satisfy that X is convex, closed and upper-bounded (and free disposal is allowed). This set is denoted by B. A bargaining solution is a function $F : B \rightarrow \Re^2$ with the property that $F(X; d) \in X$. $F(X; d)$ is interpreted as the pair of payoffs in which any two rational players would agree when they face the bargaining problem $(X; d)$.

The Nash Bargaining Solution is characterized as follows,

$$s = \arg\max_{x \in X;\, x \geq d} (x_1 - d_1) \cdot (x_2 - d_2)$$

For more details on this matter see e.g. Osborne and Rubinstein (1990). A graphical illustration is provided in Figure 1.

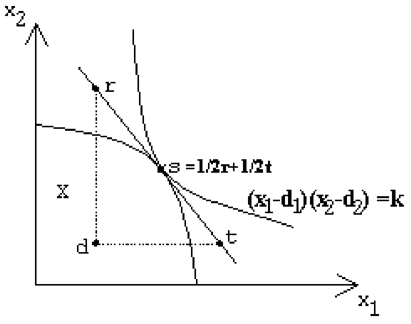

Figure 1: the Nash bargaining solution

We will restrict our attention to the cases in which the frontier of X is linear or piece-wise linear and to the disagreement value d = (0; 0) -note that this is the payoff vector from a particular link when two players decide not to form it. This structure, as we will see below, will fully characterize all the cases in our model. In the examples of Figure 2, some of these cases are graphically illustrated.

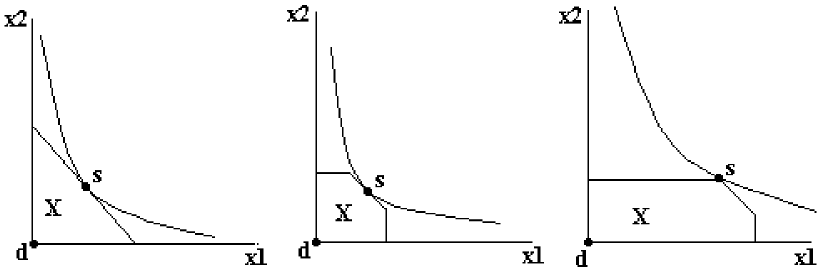

Figure 2: the linear case

Now we turn to our original model, where players do not have to share payoffs but costs.

3.3 The bilateral coordination game

The link has a cost c, which will be split in some way between the two agents involved in case the link is formed (i.e. if players i and j are planning to establish a link, they will pay respectively c_{ij} and $c_{j,i}$ s.t. $c_{ij} + c_{ji} = c$). In addition each of these agents has chosen an action $a_i \in \{\alpha, \beta\}$ and $a_j \in \{\alpha, \beta\}$.

The net benefit for the agents from this link will be:

Player i: $\Pi(a_i, a_j, c_{ij}) = \pi(a_i, a_j) - c_{ij}$ in case the link is formed and 0 otherwise;

Player j: $\Pi(a_j, a_i, c_{ji}) = \pi(a_j, a_i) - c_{ji}$ in case the link is formed and 0 otherwise; Where $\pi(a_i, a_j)$ and $\pi(a_j, a_i)$ represent the payoffs obtained respectively by each of them from the 2x2 symmetric coordination game (as defined in section 2).

We are interested in the study of the conditions under which the link will form and, in this case, how the cost c will be divided between the two agents (i.e. what the values c_{ij} and c_{ji} will be) using the Nash Bargaining Solution. This solution will depend on: the value of the link cost, c, and the values of the entries of the payoff matrix $\{b, d, e, f\}$.

3.4 Adaptation of the Nash Bargaining Solution to the cost-sharing model

We have a situation where we have a cost c that, in case the link is formed, will be split between the two agents involved. Note that players i and j have the option of forming the link, which imply to get their respective payoffs ($\pi(a_i, a_j)$ and $\pi(a_j, a_i)$) and bargaining the distribution of the cost c in some way ($c_{i,j}$ and $c_{j,i}$); or they have the outside option of not forming the link, which gives them a null payoff. Thus, they will be interested in establishing the link if, and only if, given the action chosen by each of the two players involved in the 2x2 game, the following inequality holds:

$$\pi_i(a_i, a_j) + \pi_j(a_i, a_j) \geq c. \tag{3}$$

That is, when the aggregate payoff for both players from the link exceeds the cost it conveys. In this situation they will have to agree on how to divide c into c_{ij} and c_{ji}. Obviously, the agent i would not let the link to be formed if $c_{ij} \geq \pi(a_i; a_j)$ and the passive agent would not in case $c_{ji} \geq \pi(a_j; a_i)$.

So, if (3) is satisfied, both players will have incentives to establish the link, sharing the cost in a way such that

$$\begin{aligned} \pi(a_i; a_j) &\geq c_{ij} \geq 0, \text{ and} \\ \pi(a_j; a_i) &\geq c_{ji} \geq 0 \end{aligned} \tag{4}$$

In addition, as we have got the condition under which a link forms, we can state an explicit expression for the term $g_{i;j}(a)$ defined in Section 2:

$$\text{For any } i; j \in N; i \neq j \qquad g_{i;j}(a) = \begin{cases} 1 & \text{if} \quad \pi(a_i; a_j) + \pi(a_j; a_i) \geq c \\ 0 & \text{otherwise} \end{cases}$$

Then, suppose that (3) is satisfied (otherwise the link would not form); then they will want to form the link supporting the cost in a manner that (4) holds, because in this case both of them would get non-negative profit from it, whereas if the link does not form they both would get zero. Given these conditions under which the link will be established, they will want to support the lowest part of the cost as possible. As mentioned, the values of c_{ij} and c_{ji} $(c_{ij} = c - c_{ji})$ will result from bargaining.

We propose to calculate those values using the Nash Bargaining Solution in the following way: as both players get zero payoff if the link is not formed, we can set the disagreement payoffs $d = (0; 0)$. Regarding to the set of feasible payoffs pairs X, it will be determined by the points

$$\begin{aligned} X &= \left\{ \begin{array}{c} (\hat{\rho}_i; \hat{\rho}_j) \in \Re^2_+ \quad s{:}t{:} \\ \hat{\rho}_i + \hat{\rho}_j \leq \pi(a_i; a_j) + \pi(a_j; a_i) - c \\ \hat{\rho}_i \leq \pi(a_i; a_j) \\ \hat{\rho}_j \leq \pi(a_j; a_i) \end{array} \right\}; \\ d &= (0; 0) \end{aligned} \tag{5}$$

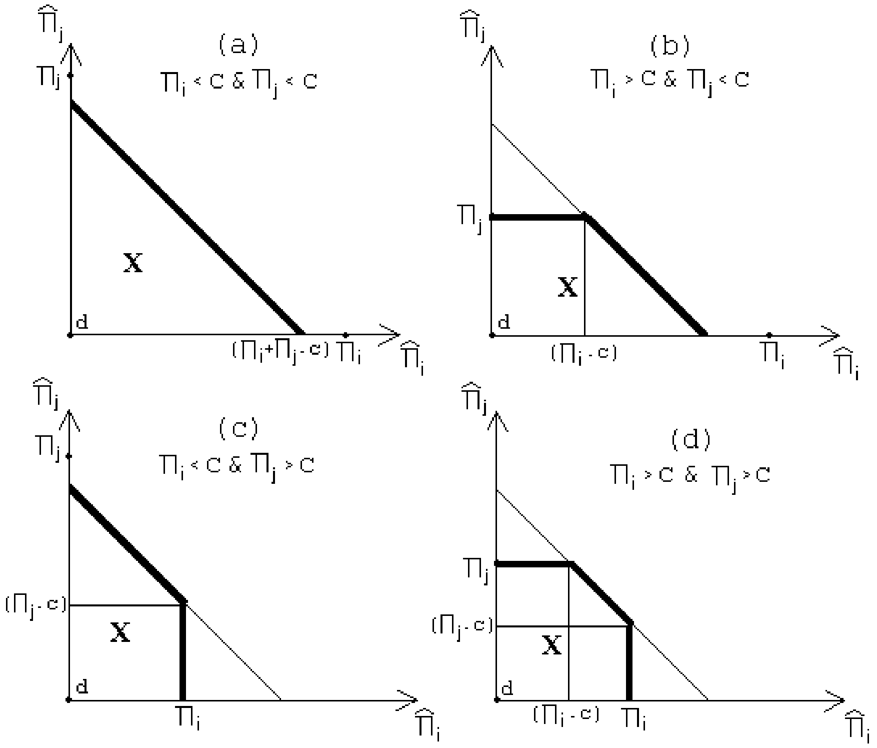

Figure 3: possible feasible payoff sets depending on the cost and players' payoffs

where, for notational convenience, we have denoted $\hat{\pi}_i = \hat{\pi}(a_i; a_j; c_{ij})$ and $\hat{\pi}_j = \hat{\pi}(a_j; a_i; c_{ji})$. Recall that players bargain how to split the cost of the link, but in this model we do not allow transfers of payoffs among the agents. For this reason a player cannot get from a link a net payoff exceeding her gross payoff (i.e. $\hat{\pi}_i \leq \pi(a_i; a_j)$ and $\hat{\pi}_j \leq \pi(a_j; a_i)$)[5]. That is the maximal (net) payoff agent i can get from the link ij (given the respective actions) is $\pi(a_i; a_j)$, that corresponds with the situation in which agent j completely covers the cost of the link. In Figure 3 we can see the set of feasible payoffs X, for the different possible relations among $\pi(a_i; a_j)$, $\pi(a_i; a_j)$ and c[6].

Now we turn to calculate the value of c_{ij} and c_{ji} resulting from the disagreement value

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[5] Otherwise, if we allowed for transfers, the form of the set X will be always as depicted in figure 3(a), irrespective of whether the values of $\pi_i(a_i; a_j)$ and $\pi_i(a_i; a_j)$ are higher or lower than c.

[6] In the figure we use the following notation: $\pi_i = \pi(a_i; a_j)$ and $\pi_j = \pi(a_i; a_j)$:

$d = (0, 0)$ and the set X as defined by (5). First we characterize the payoffs net of costs ($\hat{\pi}_i$ and $\hat{\pi}_j$) that players i and j respectively obtain from the link ij in case it is formed. We apply the Nash solution:

$$\hat{\pi}^N = \arg\max_{\hat{\pi}^N \in X,\, \hat{\pi}^N \geq d} (\hat{\pi}_i - d_i) \cdot (\hat{\pi}_j - d_j)$$

Proposition 1 Consider any $i, j \in N$

(i) If $\pi(a_i, a_j) + \pi(a_j, a_i) \geq c$, the link ij will form and the net benefits for the agents ($\hat{\pi}_i$ and $\hat{\pi}_j$) will be as follows:

$$\begin{aligned} \hat{\pi}_i &= \max\{\pi(a_i, a_j) - c, \min\{\tfrac{1}{2}(\pi(a_i, a_j) + \pi(a_j, a_i) - c), \pi(a_i, a_j)\}\} \\ \hat{\pi}_j &= \max\{\pi(a_j, a_i) - c, \min\{\tfrac{1}{2}(\pi(a_j, a_i) + \pi(a_i, a_j) - c), \pi(a_j, a_i)\}\} \end{aligned}$$

(ii) If $\pi(a_i, a_j) + \pi(a_j, a_i) < c$, the link do not form (therefore $\hat{\pi}_i = \hat{\pi}_j = 0$).

The proof is shown in the appendix.

Then, as a direct consequence of this proposition, we can state the next result, which specifies how the cost of a link is split between the two players who form it.

Corollary 1 If the link ij forms, the cost c will be split between agents i and j ($c_{ij} + c_{ji} = c$) in the following way: $c_{i,j} = \min\{c, \max\{\frac{1}{2}[\pi(a_i, a_j) + c - \pi(a_j, a_i)], 0\}\}$ and $c_{j,i} = \min\{c, \max\{\frac{1}{2}[\pi(a_j, a_i) + c - \pi(a_i, a_j)], 0\}\}$.

We get this directly from the fact that $c_{i,j} = \pi(a_i, a_j) - \hat{\pi}_i$ and $c_{j,i} = \pi(a_j, a_i) - \hat{\pi}_j$.

So, the cost an agent will pay for a link will be weakly increasing in the payoff she will get from it and weakly decreasing in the payoff of the other player, in the sense that the higher (lower) her payoff (the other player's payoff), the non-lower (non-higher) her share of the cost. That is, $c_{i,j}$ is weakly increasing in $\pi(a_i, a_j)$ and c, and weakly decreasing in $\pi(a_j, a_i)$. Note that, in any case, $c_{ij} + c_{ji} = c$.

As we have seen, due to the characteristics of the Nash Bargaining Solution, the outcome (part of the cost of a link that each player will support) will heavily depend of the gross payoff she is going to obtain from it.

4 The reduced game: translation to payoff matrices

Let us remember how the game works. First, each player chooses an action a_i from his available action space $A = \{\alpha; \beta\}$. Then, the formation of each possible link is bargained according to the Nash solution, where the set of feasible agreements X is as defined by (5) and the outside option d consists in not forming the link. Thus, given the pattern of actions a, we obtain $\{g_{ij}(a); c_{ij}\}$ for any $i; j \in N$. Therefore, using backward induction, we can construct new bilateral games that include all the former considerations and analyze which the Nash equilibria of these games will be[7]. The form of the resultant game will depend on the parameters of our model $\{d; b; f; e; c\}$.

In the following result we characterize the equilibrium outcomes of this model. First of all we have to mention that each possible combination of players actions $a = (a_1 \times \ldots \times a_n) \in S \equiv A^n$ defines a state. In this model each state results in a pattern of actions and a network architecture (pattern of links formed according to the Nash solution) ,i.e. $(a; g(a))$. Let us introduce some convenient notation. Let g^e denote the empty network, i.e. the case in which no link is formed. Let S^* be the set of equilibrium states. Denote by s^α and s^β the states in which all players are choosing the same action (all α or all β respectively) and the complete network (denoted by g^c) forms (i.e. the architectures in which all possible pairwise links are formed), i.e. $s^\alpha = ((\alpha; \ldots; \alpha); g^c)$ and $s^\beta = ((\beta; \ldots; \beta); g^c)$. Now we can state the following proposition.

Proposition 2 (i) If $c < \min\{f - e; 2(b - e)\}$, then $S^* = \{s^\alpha; s^\beta\}$

(ii) If $2(b - e) < c < f - e$; then $S^* = \{s^\alpha\}$

(iii) If $\{[f - e < c < 2b] \wedge [f + e < 2b]\}$, then $S^* = \{s^\alpha; s^\beta\}$

(iv) If $\{[f - e < c < f + e] \wedge [f + e > 2b]\}$, then $S^* = \{s^\alpha\}$

(v) If $\max\{2b; f + e\} < c < 2d$, then $S^* = \{\{s^\alpha\} \cup \{((\beta; \ldots; \beta); g^e)\}\}$

(vi) If $c > 2d$, then $S^* = \{s \in S : g(s) = g^e\}$

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[7] Note that if we consider the second stage of network formation as a subgame, we are in essence analysing the Subgame Perfect Equilibria of the original game.

Proof:

Let us consider the case in which

$$c < f - e$$

In this case, resulting from the Nash Solution (from corollary 1), we see that: (i) when both agents involved in a link are choosing the same action ($a_i = a_j$), and therefore $\pi(a_i, a_j) = \pi(a_j, a_i)$, the cost of a link will be equally shared: $c_{i,j} = c_{j,i} = c/2$ (provided that $2b > c$, in case this action is $a_i = a_j = \beta$, as otherwise the link wouldn't form[8] and both players would get a zero-payoff from that "potential" link); and (ii) when the two players involved in a link are choosing different actions (for example $a_i = \alpha$ and $a_j = \beta$), the part of the cost each of them will support (applying the Nash Bargaining solution) is[9]:

$$\begin{aligned} c_\alpha &= 0 \\ c_\beta &= c \end{aligned}$$

And therefore, the net-payoff matrix of the bilateral game resulting from this process will be:

	α	β
α	$\frac{2d-c}{2}$	e
β	$f - c$	$\max\{\frac{2b-c}{2}, 0\}$

game L (low cost): when $c < f - e$

This game (game L) has two pure-strategy (strict) Nash equilibria -(α, α); (β, β)- only when $c < 2(b - e)$. As $e > 0$, this proves (i).

When $2(b - e) < c < f - e$ game L has a unique (strict) Nash equilibria (α, α): this proves (ii).

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[8] Note that in that case $\pi(a_i, a_j) + \pi(a_j, a_i) = 2b$.

[9] In this case we denote by c_α the part of the cost of the link supported by the α-player and c_β the share of the β-player. So assume without loss of generality that player i is the α-player in the link ij (and player j the β-player), then $c_\alpha = c_{ij}$ and $c_\beta = c_{ji}$.

In case the cost c is such that:

$$f - e < c$$

then, again from Corollary 1, we get that: (i) if both agents are choosing the same action, each of them will cover half of the cost $c_{i;j} = c_{j;i} = c/2$ (provided that $2b > c$ and $2d > c$, in case this action is β or α respectively, as otherwise the link wouldn't form and both players would get a zero-payoff from that "potential" link); and (ii) when the two players involved in a link are choosing different actions (and provided that $f + e > c$, as otherwise the link wouldn't form), the part of the cost each of them will support (applying the Nash Bargaining solution) is:

$$\begin{aligned} c_\alpha &= \frac{c}{2} - \frac{f - e}{2} \\ c_\beta &= \frac{c}{2} + \frac{f - e}{2} \end{aligned}$$

And therefore, the net-payoff matrix of the bilateral game resulting from this process will be:

	α	β
α	$\max\{\frac{2d-c}{2};0\}$	$\max\{\frac{e+f-c}{2};0\}$
β	$\max\{\frac{e+f-c}{2};0\}$	$\max\{\frac{2b-c}{2};0\}$

⟨game H (high cost): when $c > f - e$⟩

This game (game H) has two pure-strategy (strict) Nash equilibria -$(\alpha;\alpha)$; $(\beta;\beta)$-, and therefore is a coordination game, only when $\{[f - e < c < 2b] \wedge [f + e < 2b]\}$. This proves (iii).

When $\{[f - e < c < f + e] \wedge [f + e > 2b]\}$ then game H has a unique (strict) Nash equilibrium $(\alpha;\alpha)$. This proves (iv).

To show (v) we just have to note that, given that $\max\{2b; f + e\} < c < 2d$, Game H takes the following form:

	®	¯
®	$\frac{2d ¡ c}{2}$	0
¯	0	0

thus, it presents a strict Nash equilibrium (®; ®) and the equilibrium (in weakly dominated strategies) (¯; ¯): The latter equilibrium corresponds to the situation in which both players choose action ¯ and the link do not form. Only in case all the population is choosing action ¯ the resulting state would be an equilibrium.

Part (vi) is trivial since, whenever $c > 2d$ no proposed link will form as there is no way to split the cost among two players without making (at least) one getting negative profits.

To complete the proof it is necessary to show that there are no equilibrium states in which some agents are using action ® and others action ¯. That is, to show that there are no equilibrium states in which the proportion of agents using each action in the population replicates a mixed strategy Nash equilibria of the 2x2 game (game H or game L). This is shown by next lemma:

Lemma 1 Assume $c < 2d$. Consider an equilibrium state of the game $s^{¤} = (a^{¤}; g(a^{¤}))$ 2 $S^{¤}$: Then $a_i^{¤} = a_j^{¤}$ for any $i; j$ 2 N:

The proof of the lemma is shown in the appendix. This result completes the proof. ¤

The above Proposition indicates that for high values of the cost, but not as high as no link would form, all the agents would connect in the efficient action of the coordination game (®), and for low values of the cost, given parameter conditions, it is both possible that either all agents connect in the efficient action (®) or in the risk dominant action (¯) of the coordination game as a subgame perfect equilibrium of the one-shot game.

Next corollary characterizes the network architecture that results in equilibrium. We see that it is either empty or complete, depending on the value of the cost.

Corollary 2 (i) If $c < 2d$, then an equilibrium network is complete. (ii) If $c > 2d$, then an equilibrium network is empty.

This results shows that in this model, either all the agents get connected (if the cost is not too high) or nobody establishes any link. There are no equilibrium states with coexistence of different components (groups of players completely connected among themselves, but where there is no connection between players of different groups). This relies on the fact that when $c < 2d$ any equilibrium would imply every agent choosing the same action.

5 Dynamics

Time is considered discrete, $t = 1, 2, \ldots$ At each t, the state of the system is given by $s(t) = (a(t), g(a(t)))$, where $a(t)$ represents the pattern of actions chosen by the agents in this period and $g(a(t))$ is the resulting network architecture of formed links according to the Nash solution. Assume the system starts from any initial state $s(0) \in S = \{(a, g(a)) : a \in A^n\}$, where S represents the finite set of all possible states of this system. Every period each player receives a chance to revise her action with a positive independent probability $p \in (0, 1)$. The structure of a player's revision is: first he chooses action, and then all her links (already existing and potential ones) are updated according to the Nash solution. In the sense that those links satisfying (3) are maintained/formed and those which do not are severed/not formed, being the cost sharing determined by the current chosen actions. Given that, we assume that a player revising will select an action

$$a_i(t) \in \arg\max_{a_i \in A} \Pi_i(a_i, a_{-i}(t-1)).$$

That is, her optimal action given the strategy configuration resulting from last period (i.e. a myopic best response), and the fact that she knows that links cost will be (re)negotiated. When there are more that one best response we will consider that any of it is chosen with equal probability[10]. This defines a Markov chain on S, being Q the transition matrix induced by the proposed dynamics, and $\mu_t \in \Delta(S)$ the probability measure over the state space S in period t, we have $\mu_t = \mu_{t-1} \cdot Q = \mu_0 \cdot Q^t$. We define the set of limit states of this process as $\overline{S} = \{s \in S : \lim_{t \to \infty} [\mu \cdot Q^t](s) > 0 \text{ for some } \mu \in \Delta(S)\}$ where $[\cdot](s)$ denotes the probability associated to state s by the probability measure $[\cdot]$ over the state space. Next result characterize the set of limit states of this unperturbed system.

Proposition 3 (i) If $c < \min\{f - e; 2(b - e)\}$, then $\overline{S} = \{s^\alpha; s^\beta\}$

(ii) If $\{[f - e < c < 2b] \wedge [f + e < 2b]\}$, then $\overline{S} = \{s^\alpha; s^\beta\}$

(iii) If $c < 2d$ and neither (i) nor (ii) holds, then $\overline{S} = \{s^\alpha\}$

(vi) If $c > 2d$, then $\overline{S} = \{s \in S : g(s) = g^e\}$

Now we turn to the proof. First note that the statement of the proposition is equivalent to say that when $c < 2d$, the set of limit states of the (unperturbed) dynamics coincide with the set of "strict" equilibrium states of the reduced game, and when $c > 2d$ this set is conformed by all the possible equilibrium states (in this case there are no strict equilibrium states). As throughout the dynamics we assume, that the conditions under which a link is formed result from the Nash solution players are maximizing, this game would be equivalent to think that any two players are in fact playing either game H or game L as defined above, depending on the parameter conditions. In the cases in which this resulting game has a unique Nash equilibrium -$(\alpha; \alpha)$- it is obvious that the system would converge to the state in which all player connects to everybody, all choosing the action corresponding to that equilibrium. In the cases in which, given the parameter conditions, the set of equilibrium states is $\{\{s^\alpha\} \cup \{((\beta; \ldots; \beta); g^e)\}$, it is clear that the system will converge to the strict equilibrium state s^α, i.e. the state $\{((\beta; \ldots; \beta); g^e)\}$ is ruled out by the dynamics as any player revising her action will change to α with positive probability (cf. footnote 9), and, once one player has switched to α, the following players to revise will do the same. The more problematic case would arise when the resulting game (H or L), given parameter conditions, has two (strict) Nash equilibria –$(\alpha; \alpha)$ and $(\beta; \beta)$-

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[10] Note that this implies that if the player revising was actually chosing an action that was a best response, not neccesarily will she continue to use the same one.

but then, it can be easily checked that there exists a treshold ξ for the number of players choosing action α in the initial state $s(0)$ such that, if the number of $\alpha-$ players in $s(0)$ is greater than ξ, the system would converge to s^{α}, meanwhile if it is lower, the system would converge to s^{β}. To see this note that each player updating her strategy would choose the action maximizing her profit for that period (given the strategies followed by the other players in the previous one and considering that link formation will follow as stated above), and would connect at least to those players using that action. But then it follows that all the revision of players' strategies would go into the same direction, i.e. consecutively choosing the same action and connecting to those ones who are using it. This must be clear as, if one player revised in the direction of an action, the next one to revise would have even more incentives to do the same (otherwise the first reviser wouldn't have maximized). This process at some period converges to an equilibrium state of the stage game and remains there hereafter. Note that this results heavily depends in the fact that in this model once players take their decisions about actions, the link formation process directly follows.

We observe then that, under some parameter conditions, we have a multiplicity of limit states of the unperturbed dynamics, in the sense that there are parameter configurations for which both s^{α} and s^{β} conform $\overline{S}$. To handle with the problem of equilibrium selection in those cases, and so, to study the nature of long run outcomes, we will characterize the set of stochastic stable states of this model. To do so we rely on the approach proposed by Kandori, Mailath and Rob (1993), and Young (1993). We allow for mutations, interpreted as the possibility of players making errors when implementing their choices on actions. We suppose that occasionally players make mistakes; i.e. with a small probability $\varepsilon > 0$, a player chooses his action at random when revising his strategy[11]. For any $\varepsilon > 0$, the process defines a Markov Chain on S (with transition matrix Q_{ε}) that is aperiodic and irreducible and, therefore, has a unique invariant probability distribution (cf. Freidlin and Wentzell (1984)). Let μ_{ε} stand for this invariant distribution ($\mu_{\varepsilon} \cdot Q_{\varepsilon} = \mu_{\varepsilon}$), and define

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[11] Note that we do not allow for mutation in link formation, as we are considering an axiomatic approach for this process. In any case the consideration that with small probability $\gamma > 0$, the formation of a link is reversed wouldn't affect the results.

$\hat{\mu} = \lim_{\varepsilon \to 0} \mu^{\varepsilon}$, i.e. the limit of the invariant probability distribution when the probability of mistake converges to zero. We say that a state $s \in S$ is stochastically stable when it is in the support of $\hat{\mu}$, i.e. when $\hat{\mu}(s) > 0$.

Let the set of stochastically stable states be denoted by $\hat{S} \equiv \{s \in S : \hat{\mu}(s) > 0\}.$ In the following proposition we characterize this set[12].

Proposition 4 (i) If $c < (b + f) - (d + e)$, then $\hat{S} = s^{\beta}.$

(ii) If $(b + f) - (d + e) < c < 2d$, then $\hat{S} = s^{\alpha}.$

(iii) If $c > 2d$, then $\hat{S} = \{s \in S : g(s) = g^{e}\}.$

Let's see the proof. Part (iii) is trivial since when $c > 2d$; it is not profitable to form any possible link. Note that when $c < (b + f) - (d + e)$, the resulting bilateral game is Game L as defined above ($(b + f) - (d + e) < f - e$). We can see that under this parameter configuration the situation is completely equivalent to that in which each agent is playing game L with every other player in the population (i.e. like a round robin context). It can be checked that, when $c < (b + f) - (d + e)$, Game L has two pure-strategy Nash Equilibria: $(\alpha; \alpha)$, that is the efficient one, and $(\beta; \beta)$; that is the risk dominant equilibrium. Under this setting, in which total connectivity is guaranteed, of 2x2 symmetric games with two Nash Equilibria in which one of them is risk dominant, Kandori, Mailath and Rob (1993) show that the risk dominant equilibrium prevails as the stochastically stable state. We then relate to this result to get the prove of (i). Finally if $(b + f) - (d + e) < c < 2d$, we can have either a situation equivalent to one in which everybody is connected with the rest of agents, playing either Game H or Game L (depending on whether the cost c is greater or lower than $f - e$, respectively); anyway, it can be checked that in any case $(\alpha; \alpha)$ represents either the unique Nash equilibrium or the both efficient and risk dominant one, and therefore the unique candidate to conform stochastically stable states. It follows then that (ii) applies.

We get a treshold for the cost $\tilde{c} = (b + f) - (d + e)$ above which the efficient action is chosen and below which the risk-dominant action is selected. This result is quite intuitive

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[12] Note that $\hat{S} \subseteq \overline{S}.$

as the treshold coincides with the risk-dominance premium, that is, for a cost higher than the difference between the sum of payoffs of the risk dominant action $(b + f)$ and the sum of payoffs of the efficient action $(d + e)$ the efficient action is selected, meanwhile otherwise the selected action is the risk dominant one. This implies that the higher this difference is (what heuristically we could interpret as the higher risk dominance of action ¯ in the bilateral coordination game) the higher the treshold is; and therefore the higher the link cost must be for the selection of the efficient outcome in the long run. Finally note that this treshold always exists (i.e. $\mathbf{e} > 0$) as we are assuming that (1) holds.

6 Conclusions

We have proposed a model in which each and everyone of the agents decide strategically on taking an action for a coordination game, then, each possible pair of players enter in a bargaining process in which they decide if they form a link, and in this case, how they share the cost it involves. To this aim, we use the Nash solution, which results in a network architecture of formed pairwise links between any two agents whose chosen actions provide an aggregate payoff in the symmetric coordination game exceeding the cost of the link. Then each player gets a total payoff equals to the sum of her corresponding payoffs from all the links she gets involved in, net of the parts of the costs she pays in each of them. The bilateral coordination game has two Nash equilibria, one of them efficient and the other one risk-dominant. We have adapted the model so that the Nash solution can be applied. In Corollary 1 we have stated how, given their respective actions, two players who form a link share the cost it involves. This division of the cost is of great importance for the results of this paper and the main contribution to the literature on network formation.

We have characterized the equilibrium states of this model, depending on the parameter conditions (Proposition 2), and we have observed that there exist cases, when the link cost is not too high, that present multiplicity. Finally, to deal with the problem of equilibrium selection we have introduced a dynamics and analyzed the sets of stochastically stable states. We have characterized a treshold for the cost of the link and showed that for values of the link cost higher than this treshold, but not so high that no link is formed, the efficient equilibrium is selected and the complete network arises, whereas for lower values the risk-dominant equilibrium is selected and each player gets connected to everybody (Proposition 4). The value of the treshold in our model coincides with the risk-dominance premium, i.e. it is directly connected to the "degree" of risk dominance of the inefficient action in the coordination game, in the sense that, the higher the aggregate payoff of the risk dominance action is with respect to the aggregate payoff of the efficient action, the higher this treshold will be, what increases the space of link costs for which the (inefficient) risk dominant outcome is selected in the long run.

We can relate our result to Goyal and Vega-Redondo (2000) in which one-sided links structure for the costs is assumed, and to Jackson an Watts (1999), who suppose a two-sided links model. Both get that when the costs of link formation are low the risk dominant equilibrium is selected as stochastically stable, but when costs are high the first one selects the efficient equilibrium but in the second one both the risk-dominant and the efficient one coexist. Surprisingly the results we get are closer to Goyal and Vega-Redondo (2000), when, at least intuitively, the distribution of the cost we propose is closer to the two-sided than to the one-sided links model[13]. However, on the one hand, we have to note that the out of equilibrium behavior is crucial when stochastic stability is analyzed as it determines the probabilities of transitions between different equilibrium states, and our model is much more flexible in this respect (as the two-sided links model maintains the equally shared costs and in our model those quantities are payoff-dependent). On the other hand Jackson an Watts (1999) propose a model in which actions and links are revised independently whereas in our model when actions are revised links are updated automatically.

Now we have guided our research to change the structure of the stages of the game, i.e. considering that players first take decisions about link formation and, after that, they decide on actions; that is, we suppose that links decisions are more "rigid". We are also working in the extension of the features present in our model to networks in which indirect links are considered. Another aspect to be analyzed is the rate of convergence

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[13] In our setup, in equilibrium, the distribution of the cost coincides with the two sided links model $(\frac{c}{2}; \frac{c}{2})$, and would be the opposite to the one-sided one (either $(c; 0)$ or $(0; c)$).

of the dynamics in our setup. Other interesting projects we want to develop include the extension of our results to generic 2x2 games and the study of economic applications of our model.

7 Appendix

Proof of Proposition 1

Part (ii) is straightforward in the sense that if $\pi(a_i;a_j)+\pi(a_j;a_i)<c$, any possibility of splitting the cost between the two agents ($c=c_{ij}+c_{ji}$), makes at least one of the two players get a net payoff lower than the outside option zero. Thus, the link would not form.

Let's go now to part (i). When $\pi(a_i;a_j)+\pi(a_j;a_i)\geq c$, it is always possible to find at least one distribution $c_{ij}+c_{ji}=c$ such that $\pi(a_i;a_j)-c_{ij}\geq 0$ and $\pi(a_j;a_i)-c_{ji}\geq 0$. This implies that players would prefer this possibility to disagreement (that conveys zero-payoff for both of them) and so the link will form. Regarding to the values $\bar{\pi}_i$ and $\bar{\pi}_j$ (c_{ij} and c_{ji} are directly obtained -see Corollary 1 above), we get them applying the Nash Bargaining Solution

$$(\bar{\pi}_i^N;\bar{\pi}_j^N)=\arg\max_{(\bar{\pi}_i^N;\bar{\pi}_j^N)\in X;\ (\bar{\pi}_i^N;\bar{\pi}_j^N)\geq d}(\bar{\pi}_i-d_i)\cdot(\bar{\pi}_j-d_j)$$

with $d=(0;0)$ and the feasible set X as defined by (5). Let's see how they are calculated.

Let's suppose without loss of generality that $\pi(a_i;a_j)\geq\pi(a_j;a_i)$

We have to consider three possible situations:

- $\pi(a_i;a_j)\geq c$ and $\pi(a_j;a_i)\geq c$
- $\pi(a_i;a_j)\geq c$ and $\pi(a_j;a_i)\leq c$
- $\pi(a_i;a_j)\leq c$ and $\pi(a_j;a_i)\leq c$

But first note that will study maximization problems of the kind

$$\left\{\begin{array}{c}\max_{\bar{\pi}_i;\bar{\pi}_j} H(\bar{\pi}_i;\bar{\pi}_j)=\bar{\pi}_i\cdot\bar{\pi}_j\\ s:t:\\ \bar{\pi}_i+\bar{\pi}_j\leq\pi(a_i;a_j)+\pi(a_j;a_i)-c\\ \bar{\pi}_i\leq\pi(a_i;a_j)\\ \bar{\pi}_j\leq\pi(a_j;a_i)\\ \bar{\pi}_i;\bar{\pi}_j\geq 0\end{array}\right\}$$

In all cases we are going to study, the strictly Pareto-efficient points of X are included in the line $\hat{\pi}_i + \hat{\pi}_j = \hat{\pi} \;(\Rightarrow \frac{d\hat{\pi}_j}{d\hat{\pi}_i} = -1)$. As the Nash bargaining solution must be a (strictly) Pareto-efficient point, if the solution is interior it must satisfy:

$$\frac{d\hat{\pi}_j}{d\hat{\pi}_i} = -\frac{\frac{\partial H(\hat{\pi}_i,\hat{\pi}_j)}{\partial \hat{\pi}_i}}{\frac{\partial H(\hat{\pi}_i,\hat{\pi}_j)}{\partial \hat{\pi}_j}} = -\frac{\hat{\pi}_j}{\hat{\pi}_i} = -1 \Rightarrow \hat{\pi}_i = \hat{\pi}_j$$

And if we have a corner solution it will be either $(\pi(a_i,a_j), \pi(a_j,a_i) - c)$ or $(\pi(a_i,a_j) - c, \pi(a_j,a_i))$

Let's go now to analyze the three mentioned cases:

(a) In the case $[\pi(a_i,a_j) \geq c$ and $\pi(a_j,a_i) \geq c]$ and the feasible set would be defined by

$$X_1 = \left\{ \begin{array}{c} (\hat{\pi}_i,\hat{\pi}_j) \in \Re^2_+ \quad s.t.: \\ \hat{\pi}_i + \hat{\pi}_j \leq \pi(a_i,a_j) + \pi(a_j,a_i) - c \\ \hat{\pi}_i \leq \pi(a_i,a_j) \\ \hat{\pi}_j \leq \pi(a_j,a_i) \end{array} \right\}$$

and so, the values of $\hat{\pi}_i$ and $\hat{\pi}_j$ would result from the Nash Bargaining Solution as

$$(\hat{\pi}_i^N,\hat{\pi}_j^N) = \begin{cases} \left(\frac{\pi(a_i,a_j)+\pi(a_j,a_i)-c}{2}, \frac{\pi(a_i,a_j)+\pi(a_j,a_i)-c}{2}\right) & \text{if the solution is interior} \\ (\pi(a_i,a_j) - c, \pi(a_j,a_i)) & \text{otherwise} \end{cases} \quad (1)$$

This case is represented graphically in Figure 4.

(b) In the case $[\pi(a_i,a_j) \geq c$ and $\pi(a_j,a_i) \leq c]$ the feasible set X_2 would be defined by

$$X_2 = \left\{ \begin{array}{c} (\hat{\pi}_i,\hat{\pi}_j) \in \Re^2_+ \quad s.t.: \\ \hat{\pi}_i + \hat{\pi}_j \leq \pi(a_i,a_j) + \pi(a_j,a_i) - c \\ \hat{\pi}_j \leq \pi(a_j,a_i) \end{array} \right\}$$

and so, the values of $\hat{\pi}_i$ and $\hat{\pi}_j$ would derive from the Nash Bargaining Solution as

$$(\hat{\pi}_i^N,\hat{\pi}_j^N) = \begin{cases} \left(\frac{\pi(a_i,a_j)+\pi(a_j,a_i)-c}{2}, \frac{\pi(a_i,a_j)+\pi(a_j,a_i)-c}{2}\right) & \text{if the solution is interior} \\ (\pi(a_i,a_j) - c, \pi(a_j,a_i)) & \text{otherwise} \end{cases}$$

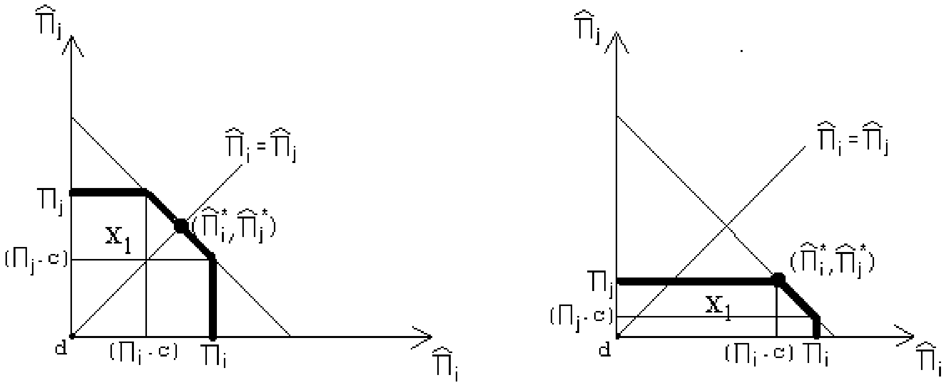

Figure 4: the Nash solution, graphical illustration

(c) Finally, in the case [$¼_i(a_i; a_j)$ · c and $¼_j(a_i; a_j)$ · c] the feasible set X_3 would be defined by

$$X_3 = \begin{array}{c} 8 \\ < \\ : \end{array} \begin{array}{c} (½_i; ½_j) \ 2 \ <^2_+ \quad s{:}t{:} \\ ½_i + ½_j \ · \ ¼(a_i; a_j) + ¼(a_j; a_i) \ ¡ \ c) \end{array} \begin{array}{c} 9 \\ = \\ ; \end{array}$$

and so, the values of $½_i$ and $½_j$ would derive from the Nash Bargaining Solution as

$$(½^N_i; ½^N_j) = µ\frac{¼(a_i; a_j) + ¼(a_j; a_i) \ ¡ \ c}{2}; \frac{¼(a_i; a_j) + ¼(a_j; a_i) \ ¡ \ c}{2}¶$$

Finally, consider the results from (a), (b) and (c) for the case $¼(a_i; a_j)$ ¸ $¼(a_j; a_i)$, and the symmetric ones we would obtain for the case $¼(a_i; a_j)$ · $¼(a_j; a_i)$. We get the proof from the combination of all of them.

Proof of Lemma 1

First, note that it would only be possible to find equilibrium population configurations with coexistence of conventions (different actions chosen) when, given parameters, the resultant game (game L or game H) has a mixed strategy equilibrium in which players assign positive probability to both actions. So we restrict the analysis to those cases and show that in none of them a profile with coexistence of different actions is an equilibrium.

Thus, we have to consider three situations:

(i) $c < \min\{f - e, 2(b - e)\}$ (game L)

(ii) $\{[f - e < c < f + e] \wedge [f + e < 2b]\}$ (game H)

(iii) $\{[f + e < c < 2b] \wedge [f + e < 2b]\}$ (game H)

Let us consider case (i), we will show, to the contrary, that it doesn't exist equilibrium configurations in which part of the population is playing action α and the rest playing action β: To this aim, suppose, to the contrary, that there is an equilibrium state in which a number of players $n_\alpha \in (0, N)$ is choosing action α and the remaining $N - n_\alpha$ are $\beta-$ players.

Equilibrium conditions would require that neither $\alpha-$ players nor $\beta-$ players want to deviate, that is:

$$(n_\alpha - 1)\frac{2d - c}{2} + (N - n_\alpha)\cdot e \geq (N - n_\alpha)\frac{2b - c}{2} + (n_\alpha - 1)(f - c) \qquad (6)$$

$$(N - n_\alpha - 1)\frac{2b - c}{2} + n_\alpha(f - c) \geq (N - n_\alpha - 1)\cdot e + n_\alpha\frac{2d - c}{2} \qquad (7)$$

But note that (6) and (7) result in:

$$\left.\begin{array}{l} n_\alpha \geq N\left(\frac{2b-2e-c}{2[(d+b)-(e+f)]}\right) + \frac{2d-2f+c}{2[(d+b)-(e+f)]} \\ n_\alpha \leq N\left(\frac{2b-2e-c}{2[(d+b)-(e+f)]}\right) - \frac{2b-2e-c}{2[(d+b)-(e+f)]} \end{array}\right. \text{ which are incompatible expressions.}$$

Using the same kind of arguments for case (ii), we get the conditions:

$$\left.\begin{array}{l} n_\alpha \geq N\left(\frac{2b-(e+f)}{2[(d+b)-(e+f)]}\right) + \frac{2d-(e+f)}{2[(d+b)-(e+f)]} \\ n_\alpha \leq N\left(\frac{2b-(e+f)}{2[(d+b)-(e+f)]}\right) - \frac{2b-(e+f)}{2[(d+b)-(e+f)]} \end{array}\right. \text{ which are incompatible expressions}$$

And finally, for case (iii), we get that the conditions such that neither $\beta-$ players nor $\alpha-$ players want to deviate:

$$\left.\begin{array}{l} n_\alpha \geq N\left(\frac{2b-c}{2[(d+b)-c]}\right) + \frac{2d-c}{2[(d+b)-c]} \\ n_\alpha \leq N\left(\frac{2b-c}{2[(d+b)-c]}\right) - \frac{2b-c}{2[(d+b)-c]} \end{array}\right. \text{ are also incompatible.}$$

This completes the proof.

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