Non Cooperative Networks in Oligopoly

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Abstract

In an oligopoly, prior to competing in the market, firms have an opportunity to pick up externalities from other firms by setting links. We take the view that these links are costly in the sense that they take resources to create and maintain. The link formation decisions by firms define an industrial network. We study the incentives for firm to form links and the effect of this link formation on the architecture of the resulting networks. Our analysis shows that equilibrium networks differ dramatically depending on the nature of market competition (Cournot or Bertrand). More precisely, whereas in the case of Cournot oligopoly we should expect to see networks in which each firm gets access to externalities of all other firms, in the case of Bertrand oligopoly, we should expect to see networks in which one firm derives benefit from externalities of all other firms while these latter get no externality.

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Introduction

Empirical studies have emphasised the role played by access to information of competitors in competitive strength of the firms (Marshall A. (1890), Cohen, Levinthal (1990)). This information can concern various aspects, such as markets, products, or technologies. They often constitute non rival but non free externalities. More precisely, information which a firm is endowed with can be acquired by another firm in forming a link, whatever the number of firms who already has formed a link with this firm. However, in establishing and maintaining a link with another firm in order to get externality flows, a firm must incur costs, in terms of effort, time, and money. The notion of link setting is very large ; it is not restricted to entering into communication with another firm, but includes all the resources devoted by a firm to pick up externalities produced by another firm. Thus, it concerns the hiring of a worker of another firm who embodies the required information, the reorganisation of the firm to enhance his absorptive capacity of these externalities, ...

In fact, two types of links are distinguished in the existing literature. In the first instance, firms form pair-wise collaborative links (see Jackson M.O., Wolinski (1996)) with other firms, which involve a commitment of resources on the part of collaborators. This first instance has been extensively developed by scholars (Yi S. (1997, 1998), Kranton R., Minehart D. (2001), Goyal S., Joshi S. (2001)). As for the second instance, it has received much less attention, in spite of his growing empirical relevance. In this second instance, links setting are non cooperative in the sense that a firm can establish a link with another firm without seeking permission of the latter (Bala V., Goyal S. (2000[a]), Bala V., Goyal S. (2000[b])). Examples of activities assimilated to links setting in this second instance are economic intelligence activities, like looking at web-sites of competitors, consulting patents registered by competitors, reading articles in specialised reviews about competitors' practices, settling in the same area as competitors to benefit from "secrets in the air", ...

In this paper, we concentrate on this second instance. More specifically, we consider a set of firms potentially able to operate on a set of different markets, given their technology and organisational capabilities. We choose to focus our analysis on one of these markets, supposed to be defined as an (endogenous) asymmetric homogeneous oligopoly frictionless market.

We assume that each firm who operating on this market can form links with the other firms operating on this particular market or more generally on the set of the different markets under consideration. These links, through externality flows, generate cost economies. Thus the set of links of the different firms, present on this particular market, with other firms, located on this market or on the set of the different markets, defines a links network and induces a distribution of costs ac cross the firms in the industry.

Given these costs, firms then compete in the market. We consider two alternative approaches about firms' behaviours on this market and mechanisms by which individual consumers' demands are allocated among competing firms: the Cournot approach and the Bertrand approach. Recall that Cournot assumed each firm in the market sets quantity and that the market allocated sales equal to what any given firm produces at the market clearing price, whereas Bertrand assumed that each firm sets its price and that the firm with the lowest price, if there is only one of such firm, is allocated all sales.

We characterize the structure of the equilibrium networks. Litter ally, a given network is said to be an equilibrium network in the following way: holding the links of the other firms to be constant, no firm has an incentive to break a link or links it has already set or to establish new links with other firms present on the market or on the set of the different markets.

Formally, our model can be described as a multi-stage game with two stages. In stage I, each firm decides which links to establish with competitors. We suppose the results of stage I, i.e. the set of links formed by the firms, are common knowledge among competitors when firms make decisions about quantities (Cournot) or prices (Bertrand) in stage II. Given these links, in stage II, the firms choose the quantities they produce (Cournot) or the prices they set (Bertrand) simultaneously. In the spirit of equilibrium analysis, a natural conjecture is that the second stage output (Cournot) or price (Bertrand) choices will be those of a Cournot Equilibrium or Bertrand Equilibrium for the prevailing network architecture of the industry.

These assumptions have the following implication: in stage I, when a firm makes decision about her links, it takes into account not only the quantity (or price) the new configuration of links induces her to produce, but also the quantities (or prices) this new configuration of links induces her competitors to produce. More specifically, we suppose each firm knows the Cournot (or Bertrand) Nash equilibrium associated with every configuration of links and keeps these equilibria in mind in taking her decision in stage I. Hence the equilibrium networks can be described as subgame perfect equilibria since they are induced by Nash equilibria in every subgame of the original game.

We study both non transitive and transitive flows of externalities. In the former case, link setting with another firm is a condition to get externality flows from this firm. In the later case, link setting with another firm allows access not only to the set of information produced by this particular firm but also to the sets of information produced by the set of firms with which this firm is tied. Thus, in this case, in deciding whether to set a link there is a trade off between the externality flows that a firm gets due to this setting and the externalities flow that this firm gives to the other firms who have a direct or indirect link with it. Our results concerning equilibrium networks differ dramatically depending on the nature of market competition (Cournot or Bertrand) and on costs of forming links. However, the introduction of transitivity does not change the result. More precisely, unless the costs of links setting are excessively high, in the case of Cournot oligopoly, we should expect to see networks in which each firm picks up externalities of all other firms. In the case of Bertrand oligopoly, we should expect to see networks in which one firm gets externalities from all other firms while these latter get no externality. If costs are excessively high then firms have not an incentive to set links in order to get externality flows. The rest of this paper is organised as follows. Section 1 describes the basic model while sections 2 and 3 present the results. In section 2, we suppose that externality flows are non transitive. In section 3, we introduce transitivity in externality flows.

1 The Model

1.1 Definitions

Let $N = \{1, \ldots, i, j, \ldots, n\}$, with $n \geq 3$, denote a finite set of ex-ante identical firms. To describe externality flows, it is useful to define some concepts of directed network (graph) theory.

Definition 1 *A network* $\mathbf{g} = (N, E)$ *is a set of vertices connected by links. Formally, a network is a set of vertices and a relation between vertices, adjacency. We focus on directed networks whose links are ordered pairs of vertices. That is, each link can be followed from one* *vertex to the next. Let $N = \{1, \ldots, i, \ldots, n\}$ be the set of vertices and let $E \subset N \times N$ be the set of links.*

Let $\mathcal{G}$ be the set of directed networks.

Definition 2 *A network $\mathbf{g} = (N, E)$ is complete if for every pair of vertices $i \in N$ and $j \in N$, there is a link between $i \in N$ and $j \neq i \in N$. We note the complete network $\mathbf{g}^c$.*

Definition 3 *A strong network $\mathbf{g} = (N, E)$ is a network such that, for all $i \in N$ and $j \in N$, there exists at least a path $C_g[i, j]$ and a path $C_g[j, i]$ in $\mathbf{g} \in \mathcal{G}$.*
A network $\mathbf{g} = (N, E)$ is a minimally strong network if it is a strong network, and if for all $i, j \in E$, we have $\mathbf{g}' = (N, E \setminus \{i, j\})$ is not a strong network.

Definition 4 *A network $\mathbf{g} \in \mathcal{G}$ is an inward pointing star if there exists an unique vertex $i \in N$ such that i has formed a link with each $j \in N$, and every $j \neq i \in N$ has formed no link.*

Definition 5 *A network $\mathbf{g}^e \in \mathcal{G}$ is empty if $E = \emptyset$. We note $\mathbf{g}^e$ the empty network.*

There exists a one to one mapping from the set of firms and the set of vertices. Hence, we use indistinctly the words firm and vertex.
Let us define the following relation $\sim$. We say $i \in N$ gets externalities from $j \in N$ if and only if $i \sim j$. Let $N_i(\mathbf{g}) = \{j \mid i \sim j\}$ be the set of firms j such that $i \in N$ gets information from $j \in N$. Let $n_i(\mathbf{g})$ be the cardinal of $N_i(\mathbf{g})$.
Moreover we define $\hat{N}_i(\mathbf{g}) = \{j \mid i \in N_j(\mathbf{g})\}$ the set of firms who get externalities from $i \in N$. Let $\hat{n}_i(\mathbf{g})$ be the cardinal of $\hat{N}_i(\mathbf{g})$.

Let $G_i = \{0, 1\}^{n-1} \cup \{0\}$ be the set of links of firm $i \in N$. G_i is a space vector. Let $\mathbf{e}_1 = (1, 0, ..., 0, ..., 0)$, $\mathbf{e}_2 = (0, 1, ..., 0, ..., 0)$, $\mathbf{e}_k = (0, 0..., 1, ..., 0), \mathbf{e}_n = (0, 0..., 0, ..., 1)$ be the unit vectors in G_i, $\mathbf{0} = (0, 0..., 0, ..., 0)$ the zero vector in G_i and the vector $\mathbf{1} = (1, \ldots, 1, \ldots, 1)$. Moreover, we denote $\mathbf{e}'_k$ a vector such that $\mathbf{e}_k + \mathbf{e}'_k = \mathbf{1}$. We note $\mathbf{g}_i = (g_{i,1}, \ldots, g_{i-1}, 0, g_{i+1} \ldots, g_{i,j}, \ldots, g_{i,n}) \in G_i$ the $n-$uplet of links formed or not by $i \in N$ and $\mathbf{g}_{-i} = (\mathbf{g}_1, \ldots, \mathbf{g}_{i-1}, \mathbf{g}_{i+1}, \ldots, \mathbf{g}_n)$. We will frequently refer to all firm other than some given firm $i \in N$

as firm i's opponents and denote them by $-i$.
Let $G = \times_i G_i$ and $\mathbf{g} \in G$.
We note S_i the set of strategy of firm $i \in N$ in a network game with oligopoly. We note $S = \times_i S_i$ the set of strategy profile and $\mathbf{s} = (\mathbf{s}_1, \ldots, \mathbf{s}_i, \ldots, \mathbf{s}_n) \in S$ a strategy profile.
Let $Q_i = [0, \infty)$ be the quantity set of firm $i \in N$. We note $q_i \in Q_i$ the quantity chosen by firm $i \in N$. A strategy of firm $i \in N$ in an oligopoly 'à la Cournot' is $\mathbf{s}_i = (g_i, q_i(\mathbf{g}))$.
Let $P_i = [0, \infty)$ be the quantity set of firm $i \in N$. We note $p_i \in P_i$ the price chosen by firm $i \in N$. A strategy of firm $i \in N$ in an oligopoly 'à la Bertrand' is $\mathbf{s}_i = [g_i, p_i(\mathbf{g})]$.

1.2 Links Setting and Cost Reduction

Firms choose their links and compete in the market. A link setting is an investment in information search about technology, products or markets. The information obtained can be interpreted as externalities flows. We assume that flows are non symmetric: when a firm establishes a link with another firm, whereas the former benefits from this link, the converse is not true.

We will suppose that a link setting requires a fix investment cost given by δ. We assume that initially firms are symmetric with zero fix cost and identical cost functions. We consider that link setting is a way to reduce cost of production. More specifically, we assume that firm $i \in N$ marginal cost in the network $\mathbf{g} \in \mathcal{G}$ is a function of the number of firms whose she gets information.

A network $\mathbf{g} \in \mathcal{G}$ induces a marginal cost vector for the firms which is given by $c(\mathbf{g}) = \{c_1(\mathbf{g}), c_2(\mathbf{g}), ..., c_i(\mathbf{g}), ..., c_n(\mathbf{g})\}$. For all $i \in N$ and for all $\mathbf{g} \in \mathcal{G}$ we will always assume that $c_i(\mathbf{g}) \geq 0$. We suppose there is no fixed cost except the cost of forming links.
In a second stage, firms compete in the market. We study the textbook example of a market with homogeneous product and with price and quantity competition. More specifically, for reasons of tractability, we assume the inverse demand function in the product market is linear:

$$p = \alpha - \sum_{i \in N} q_i, \ \alpha > 0. \tag{1}$$

For every network, we suppose there is a well defined equilibrium in the second stage product market game. The profit of firm $i \in N$, gross of the cost of forming links, are given by $\pi_i(\mathbf{g}_i, \mathbf{g}_{-i})$.

1.3 Equilibrium Networks

A network $\mathbf{g} \in \mathcal{G}$ is said to be in equilibrium if, holding the set of links formed by the other firms to be constant, any firm that is linked to another firm in $\mathbf{g} \in \mathcal{G}$ has an incentive to maintain this link. Moreover, any firm that is not linked to another firm in $\mathbf{g} \in \mathcal{G}$ has no incentive to form a link with this firm. We note $\mathcal{G}^*$ the set of equilibrium networks.

Let $\mathbf{g}^* = (\mathbf{g}_i^*, \mathbf{g}_{-i}^*)$ and $\mathbf{g} = (\mathbf{g}_i, \mathbf{g}_{-i}^*)$ be two networks.
Lastly, we note $\Pi_i : S \to I\!R^+$, a payoff function which maps strategy profile to payoff of firm $i \in N$. Precisely, $\Pi_i(\mathbf{s}) = \pi_i(\mathbf{s}) - \sum_{j \in N} g_{i,j}$.
We define $q^*(\mathbf{g}) = (q_1^*(\mathbf{g}), \ldots, q_i^*(\mathbf{g}), \ldots, q_n^*(\mathbf{g}))$ the Cournot equilibrium quantity given the network $\mathbf{g}$, that is for all $i \in N$ we have: $q_i^*(\mathbf{g}) = \max_{\mathbf{q_i(g)} \in Q_i} \pi_i(q_i(\mathbf{g}), q_{-i}(\mathbf{g}))$. Likewise, we define $p^*(\mathbf{g}) = (p_1^*(\mathbf{g}), \ldots, p_i^*(\mathbf{g}), \ldots, p_n^*(\mathbf{g}))$ the Bertrand equilibrium price given the network $\mathbf{g}$ that is for all $i \in N$ we have: $p_i^*(\mathbf{g}) = \max_{\mathbf{p_i(g)} \in P_i} \pi_i(p_i(\mathbf{g})$ $, p_{-i}(\mathbf{g}))$.

Formally, an equilibrium network in a network game with oligopoly 'à la Cournot', $\mathbf{g}^* \in \mathcal{G}^*$ satisfies the following condition:
$\forall i \in N$, $\Pi_i(\mathbf{g}_i^*, \mathbf{g}_{-i}^*, q_i^*(\mathbf{g}^*)) \geq \Pi_i(\mathbf{g}_i, \mathbf{g}_{-i}^*, q_i^*(\mathbf{g}))$, $\forall \mathbf{g}_i \in G_i$.
Moreover, an equilibrium network in a network game with oligopoly 'à la Bertrand', $\mathbf{g}^* \in \mathcal{G}^*$ satisfies the following condition:
$\forall i \in N$, $\Pi_i(\mathbf{g}_i^*, \mathbf{g}_{-i}^*, p_i^*(\mathbf{g}^*)) \geq \Pi_i(\mathbf{g}_i, \mathbf{g}_{-i}^*, p_i^*(\mathbf{g}))$, $\forall \mathbf{g}_i \in G_i$.

2 Non Cooperative Networks and Cournot or Bertrand Oligopolies

In this section we assume that a firm gets information from another firm if and only if it has set a link with this firm. Let $\gamma_0 \in I\!R_+^*$, and $\gamma \in I\!R_+^*$. We suppose the marginal cost function of a firm $i \in N$ has the following form:

$$c_i(\mathbf{g}) = \gamma_0 - \gamma n_i(\mathbf{g}) \text{ such that } \gamma_0 > \gamma n_i(\mathbf{g}), \text{ for all } n_i(\mathbf{g}). \qquad (2)$$

2.1 Non Cooperative Networks and Cournot Oligopoly

In non cooperative networks and Cournot oligopoly each firm chooses first the links she sets, then the quantity it produces.
Given any network $\mathbf{g} \in \mathcal{G}$, the Cournot equilibrium quantity, denote q_i^*, is :

$$\forall i \in N, \; q_i^*(\mathbf{g}) = \frac{(a - \gamma_0) + n\gamma n_i(\mathbf{g}) - \gamma \sum_{j \neq i} n_j(\mathbf{g})}{n+1}. \tag{3}$$

We assume that $(\alpha - \gamma_0) - (n-1)(n-2)\gamma > 0$ in order to ensure that each firm produces a strictly positive quantity.
We observe that the equilibrium quantity of firm $i \in N$ depends on his proper links and on the links of others.
The equilibrium profits, denote $\pi_i^*(\mathbf{g})$, can be written as:

$$\forall i, \in N \; \pi_i^*(\mathbf{g}) = (q_i^*(\mathbf{g}))^2.$$

We now characterise equilibrium networks under quantity competition.

Proposition 1 *Suppose there is quantity competition among firms. Assume marginal demand satisfies equation (1) and cost satisfies equation (2). Then:*

1. *if* $\delta < (\frac{n\gamma}{n+1})^2$, $\mathbf{g}^c$ *is the unique equilibrium network;*
2. *if* $\delta > (n-1)(\frac{n\gamma}{n+1})^2$, $\mathbf{g}^e$ *is the unique equilibrium network;*
3. *if* $\delta \in [(\frac{n\gamma}{n+1})^2, (n-1)(\frac{n\gamma}{n+1})^2]$, $\mathbf{g}^e$ *and* $\mathbf{g}^c$ *are the unique equilibrium networks.*

Proof:

1. Let $\delta < (\frac{n\gamma}{n+1})^2$.We first show that $\mathbf{g}^c$ is an equilibrium network. Precisely, we show that no firm has an incentive to break a link in $\mathbf{g}^c$. We note that $\mathbf{g}^c = (\mathbf{g}_i, \mathbf{g}_{-i})$ where $\mathbf{g}_i = \mathbf{e}_i'$ and $\mathbf{g}_{-i} = (\mathbf{g}_1, \ldots, \mathbf{g}_j, \ldots, \mathbf{g}_n)$ where for all $j \in N$, $\mathbf{g}_j = \mathbf{e}_j'$. Let $N' \subset N$ be the set of firms $i \in N$ breaks a link with. Indeed, $\pi_i(\mathbf{g}_i - \sum_{j \in N'} \mathbf{e}_j, \mathbf{g}_{-i}) - \pi_i(\mathbf{g}_i, \mathbf{g}_{-i}) + |N'|\delta = |N'|\delta - (|N'|\frac{n\gamma}{n+1})^2 < 0$ because $|N'|(\frac{n\gamma}{n+1})^2 \geq (\frac{n\gamma}{n+1})^2 > \delta$.

   Next, we show that if $\mathbf{g} \neq \mathbf{g}^c$, then $\mathbf{g} \in \mathcal{G}$ is not an equilibrium network. Indeed, suppose there exists $\mathbf{g} \in \mathcal{G}$ such that $\mathbf{g} \neq \mathbf{g}^c$

is an equilibrium network. If $\mathbf{g} \neq \mathbf{g}^c$ there is a firm $i \in N$ such that $\mathbf{g}_i \neq \mathbf{e}_i'$. We assume without loss of generality that $g_{i,j} = 0$. Then $\pi_i(\mathbf{g}_i + \mathbf{e}_j, \mathbf{g}_{-i}) - \pi_i(\mathbf{g}_i, \mathbf{g}_{-i}) - \delta = (\frac{n\gamma}{n+1})^2 - \delta > 0$ because $(\frac{n\gamma}{n+1})^2 > \delta$. Hence, firm $i \in N$ has an incentive to modify its strategy. A contradiction. The result follows.

2. Let $\delta > (n-1)(\frac{n\gamma}{n+1})^2$. We note that $\mathbf{g}^e = (\mathbf{g}_i, \mathbf{g}_{-i})$ where $\mathbf{g}_i = \mathbf{0}$ for all $i \in N$. We first show that $\mathbf{g}^e$ is an equilibrium network. Precisely, we show that no firm has an incentive to set a link in $\mathbf{g}^e$. Let $N' \subset N$ be the set of firms $i \in N$ forms a link with. We have: $\pi_i(\mathbf{g}_i + \sum_{j \in N'} \mathbf{e}_j, \mathbf{g}_{-i}) - \pi_i(\mathbf{g}_i, \mathbf{g}_{-i}) - |N'|\delta = (|N'|\frac{n\gamma}{n+1})^2 - |N'|\delta < 0$ because $\delta > (n-1)(\frac{n\gamma}{n+1})^2 \geq |N'|(\frac{n\gamma}{n+1})^2$. Thus, g^e is an equilibrium network.

   Next, we show that the network $\mathbf{g} \neq \mathbf{g}^e \in \mathcal{G}$ is not an equilibrium network. To establish a contradiction, assume $\mathbf{g} \neq \mathbf{g}^e \in \mathcal{G}^*$ .Then there exists $i \in N$ and $j \in N$ such that $g_{i,j} = 1$. It is obvious that agent $i \in N$ has an incentive to sever the link i, j since :

$$\pi_i(\mathbf{g}_i - \mathbf{e}_j, \mathbf{g}_{-i}) - \pi_i(\mathbf{g}_i, \mathbf{g}_{-i}) + \delta = \delta - (\frac{n\gamma}{n+1})^2 > 0,$$

   and therefore, $\delta > n(\frac{n\gamma}{n+1})^2 > (\frac{n\gamma}{n+1})^2$. A contradiction.

3. Let $\delta \in ((\frac{n\gamma}{n+1})^2, n(\frac{n\gamma}{n+1})^2)$. It is straightforward, with similar arguments than in preceding parts, that $\mathbf{g}^c$ and $\mathbf{g}^e$ are equilibrium networks.
   Moreover, any other architecture is not an equilibrium network. To establish a contradiction, assume that the non empty network $\mathbf{g} \neq \mathbf{g}^c \in \mathcal{G}$ is an equilibrium network. Then there exists two firms $i \in N$ and $j \in N$ such that $g_{i,j} = 1$ and two firms $i' \in N$ and $j' \in N$ such that $g_{i',j'} = 0$. Given that $\mathbf{g} \in \mathcal{G}$ is an equilibrium network, we get:

$$\pi_i(\mathbf{g}_i - \mathbf{e}_j, \mathbf{g}_{-i}) - \pi_i(\mathbf{g}_i, \mathbf{g}_{-i}) + \delta = \delta - (\frac{n\gamma}{n+1})^2 < 0.$$

   and

$$\pi_{i'}(\mathbf{g}_{i'} + \mathbf{e}_{j'}, \mathbf{g}_{-i'}) - \pi_{i'}(\mathbf{g}_{i'}, \mathbf{g}_{-i'}) - \delta = (\frac{n\gamma}{n+1})^2 - \delta < 0.$$

   A contradiction. The result follows.

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The intuition behind this proposition is as follows: when a firm $i \in N$ sets a link, the positive effects on the profits of this firm is given by $\left(\frac{n\gamma}{n+1}\right)^2$ while the negative effects are given by δ. Thus, link formation is clearly profit enhancing if the cost of link setting is not too high, more precisely if $\left(\frac{n\gamma}{n+1}\right)^2 > \delta$. In this case the complete network is the resulting equilibrium network. On the contrary, if the cost of link setting is too high, $\left(\frac{n\gamma}{n+1}\right)^2 < \delta$ then a firm $i \in N$ has not incentive to form links, and the empty network is the resulting equilibrium network.

2.2 Non Cooperative Networks and Bertrand Oligopoly

In non cooperative networks and Bertrand oligopoly, each firms chooses first the links she establishes ,then the price she sets. Given the linear form of the cost function (2), a standard result of Bertrand is that a firm will make profits only if it is the unique minimal firms in the market. Indeed, for all $i \in N$ and for all $j \in N$:

$$\pi_i(\mathbf{g}_i, \mathbf{g}_{-i}) = 0 \text{ if } c_i(\mathbf{g}_i) \geq c_j(\mathbf{g}_j) \text{ for } i \neq j$$

and

$$\pi_i(\mathbf{g}_i, \mathbf{g}_{-i}) > 0 \text{ if } c_i(\mathbf{g}_i) < c_j(\mathbf{g}_j) \text{ for } i \neq j.$$

Proposition 2 *Suppose there is price competition among firms. Moreover, assume that marginal demand satisfies equation (1) and cost satisfies equation (2). Then:*
- if $\delta \geq \gamma(\alpha - \gamma_0)$, the empty network is the unique equilibrium network;
- if $\delta < \gamma(\alpha - \gamma_0)$, the inward pointing stars are the unique equilibrium networks.

Proof: We proceed in two steps.

First, we show that in an equilibrium network $g \in \mathcal{G}^*$ there is at most one firm who has set links. To establish a contradiction, suppose that in an equilibrium network $g = (\mathbf{g}_j, \mathbf{g}_{-j}) \in \mathcal{G}^*$ there exists $i \in N$ and $j \in N$ who has formed links. Without loss of generality, assume $n_i \geq n_j$. We get:

$$\pi_j(\mathbf{0}, \mathbf{g}_{-j}) - \pi_j(\mathbf{g}_j, \mathbf{g}_{-j}) + n_j\delta = n_j\delta > 0.$$

A contradiction. Hence, in equilibrium there is at most one firm $i \in N$ who has established links whereas all the other firms has formed no links.

Second, we show that in an equilibrium network $g \in \mathcal{G}^*$ the number of links established by the firm $i \in N$ is 0 if $\delta > \gamma(\alpha - \gamma_0)$ while this number is $n-1$ if $\delta \leq \gamma(\alpha - \gamma_0)$. Indeed, suppose a network $\mathbf{g} = (\mathbf{g}_i, \mathbf{g}_{-i}) \in \mathcal{G}$ such that one firm, say $i \in N$, has formed $n_i(\mathbf{g}) \in \{1, \ldots, n-1\}$ links whereas the other firms have established no links. Let $\epsilon \in I\!R_+^*$ such that $\epsilon \to 0$, we have:

$$\pi_i(\mathbf{g}_i, \mathbf{g}_{-i}) - n_i(\mathbf{g})\delta = n_i(\mathbf{g})(\gamma(\alpha - \gamma_0 + \epsilon) - \delta) - \epsilon(\alpha - \gamma_0 + \epsilon).$$

We can distinguish two cases:

1. if $\delta > \gamma(\alpha - \gamma_0)$, then $\pi_i(\mathbf{g}_i, \mathbf{g}_{-i})$ is decreasing with n_i. This implies that the only equilibrium network is the empty network;
2. if $\delta \leq \gamma(\alpha - \gamma_0)$, then $\pi_i(\mathbf{g}_i, \mathbf{g}_{-i})$ is increasing with n_i. This implies that the inward pointed star networks are the only equilibrium networks.

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It is interesting to compare our result with that of Goyal and Joshi (2001) who under a similar specification of demand and marginal cost derives the empty network as the only equilibrium network. This difference in our result can be explained by the non symmetry property of externality flows. Indeed, in the model of the authors, externality flows are symmetric. Then, the star networks can not be equilibrium networks for the following reason: in the star, all firms have formed links and get the same externality. Hence, their profits gross of the cost of forming links are null and their net profits are negative. Therefore each firm has an incentive to sever her links and star networks are not equilibrium networks.

3 Non Cooperative Transitive Networks and Cournot or Bertrand Oligopolies

3.1 Non Cooperative Transitive Networks and Cournot Oligopoly

The introduction of transitivity makes appreciation of links setting effects more complex. Indeed, when a firm sets a link it is possible this change does not increase its information stock, hence does not decrease its marginal cost. It is the case for example when a firm $i \in N$ who already has a link with say $j \in N$ sets a link with $k \in N$ whereas j itself has a link with $k \in N$.
Formally, with transitivity, we get:

$$i \sim j, \; j \sim k \Rightarrow i \sim k.$$

Hence, if in $\mathbf{g} = (\mathbf{g}_i, \mathbf{g}_{-i}) \in \mathcal{G}$ we have $i \sim j$, $j \sim k$ and $g_{i,k} = 0$ then $c_i(\mathbf{g}_i + \mathbf{e}_k, \mathbf{g}_{-i}) = c_i(\mathbf{g}_i, \mathbf{g}_{-i})$.
It is straightforward that if $j \notin N_i(\mathbf{g})$ and $\mathbf{g} = (\mathbf{g}_i, \mathbf{g}_{-i})$ then $c_i(\mathbf{g}_i + \mathbf{e}_j, \mathbf{g}_{-i}) \leq c_i(\mathbf{g}_i, \mathbf{g}_{-i}) + \gamma n_j(\mathbf{g})$.
Moreover, when a firm $i \in N$ sets a link i, j it must take into account the fact it changes not only her externality stock, but also the externality stocks of the firms $j \in \hat{N}_i(\mathbf{g})$. Nevertheless, the Cournot equilibrium quantity (q^*) (and profit) remains the same as in equation 3.

Proposition 3 *Suppose there is quantity competition among firms. Moreover, assume that marginal demand satisfies equation (1) and cost satisfies equation (2), then:*

1. *if* $\delta < (\frac{-2\gamma}{n+1})^2$, *only minimally strong networks* $\mathbf{g} \in \mathcal{G}$ *are equilibrium networks;*
2. *if* $\delta > (\frac{(n-1)^2\gamma}{n+1})^2$, *the empty network* $\mathbf{g}^e \in \mathcal{G}$ *is the unique equilibrium network;*
3. *if* $\delta \in [(\frac{-2\gamma}{n+1})^2, (\frac{(n-1)^2\gamma}{n+1})^2]$, *only the empty network and minimally strong networks are equilibrium networks.*

Proof:

1. Let $\delta < (\frac{-2\gamma}{n+1})^2$.

First, we show that a minimally strong network $\mathbf{g} = (\mathbf{g}_i, \mathbf{g}_{-i})$ is an equilibrium network. In a minimally strong network $\mathbf{g} = (\mathbf{g}_i, \mathbf{g}_{-i})$ each firms has set at least one link. Assume without loss of generality that, $g_{i,j} = 1$. Let $N' \subset N$ be the set of firms $i \in N$ forms a link with. Let $k \leq n_j$. We get:

$$\pi_i(\mathbf{g}_i, \mathbf{g}_{-i}) - \pi_i(\mathbf{g}_i - \sum_{j \in N'} \mathbf{e}_j, \mathbf{g}_{-i}) = |N'|(\delta - (\gamma k(n - \hat{n}_i))^2).$$

It is straightforward that $|N'|(\delta - (\gamma k(n - \hat{n}_i))^2) < 0$ since $\delta < (\frac{-2\gamma}{n+1})^2$.

Second, we show that non strong networks $\mathbf{g} \in \mathcal{G} = (\mathbf{g}_i, \mathbf{g}_{-i})$ are not equilibrium networks. To establish a contradiction, assume that a non strong network $\mathbf{g} \in \mathcal{G} = (\mathbf{g}_i, \mathbf{g}_{-i})$ is an equilibrium network. Then there exists two firms $i \in N$ and $j \in N$ such that $j \notin N_i$. We get:

$$\pi_i(\mathbf{g}_i + \mathbf{e}_j, \mathbf{g}_{-i}) - \pi_i(\mathbf{g}_i, \mathbf{g}_{-i}) = (\gamma k(n - \hat{n}_i))^2 - \delta.$$

It is straightforward that $(\gamma k(n - \hat{n}_i))^2 - \delta > 0$ since $\delta < (\frac{-2\gamma}{n+1})^2$. A contradiction. Hence an equilibrium network is a strong network.

Moreover, it is obvious that a strong network $\mathbf{g}$ who is not minimally connected is not an equilibrium network, since at least one firm $i \in N$ has an incentive to delete a link.

2. Let $\delta > (\frac{(n-1)^2\gamma}{n+1})^2$.

First, we show that the empty network $\mathbf{g}^e = (\mathbf{g}_i, \mathbf{g}_{-i})$ is an equilibrium network. Let $N' \subset N$ be the set of firms $i \in N$ forms a link with. We get:

$$\pi_i(\mathbf{0}, \mathbf{g}_{-i}) - \pi_i(\mathbf{0} + \sum_{j \in N'} \mathbf{e}_j, \mathbf{g}_{-i}) = |N'|(\delta - (\gamma k(n - \hat{n}_i))^2).$$

It is straightforward that $|N'|(\delta - \gamma k(n - \hat{n}_i))^2) > 0$ since $\delta > (\frac{(n-1)^2\gamma}{n+1})^2$.

Second, we show that non empty networks $\mathbf{g} \in \mathcal{G} = (\mathbf{g}_i, \mathbf{g}_{-i})$ are not equilibrium networks. To establish a contradiction, assume that a non empty network $\mathbf{g} \in \mathcal{G} = (\mathbf{g}_i, \mathbf{g}_{-i})$ is an equilibrium network. Then there exists two firms $i \in N$ and $j \in N$ such that $j \in N_i$. We get:

$$\pi_i(\mathbf{g}_i - \mathbf{e}_j, \mathbf{g}_{-i}) - \pi_i(\mathbf{g}_i, \mathbf{g}_{-i}) = \delta - (\gamma k(n - \hat{n}_i))^2.$$

It is straightforward that $\delta - (\gamma k(n - \hat{n}_i))^2 > 0$ since $\delta > (\frac{(n-1)^2\gamma}{n+1})^2$. A contradiction. Hence an equilibrium network is a strong network.

3. Let $\delta \in [(\frac{-2\gamma}{n+1})^2, (\frac{(n-1)^2\gamma}{n+1})^2]$. By a similar reasoning than in preceding parts, it is obvious that the empty network and minimally strong networks are equilibrium networks.
   Now, we show that a non empty and non minimally strong network $\mathbf{g} \in \mathcal{G}$ is not an equilibrium network. Without loss of generality, we assume $i \in N$ such that $\hat{n}_i \geq \hat{n}_j$ for all $j \in N$.

   First, we show that if $n_i = 0$ in $\mathbf{g} \in \mathcal{G}^*$, then $\sum_{k \in N} g_{k,i} \leq 1$.
   To establish a contradiction assume there exists $k \in N$ and $j \in N$ such that $g_{k,i} = g_{j,i} = 1$ and $g_{j,k} = 0$. By transitivity, we get:

$$\pi_j(\mathbf{g}_j - \mathbf{e}_i + \mathbf{e}_k, \mathbf{g}_{-i}) - \pi_j(\mathbf{g}_j, \mathbf{g}_{-j}) = (\frac{\gamma(n - n_j)}{n+1})^2 > 0.$$

   A contradiction.

   Second, we show that if $n_i(\mathbf{g}) = 0$ in $\mathbf{g} = (N, E) \in \mathcal{G}^*$, then $\hat{n}_i = 0$. Indeed, to establish a contradiction, assume $\hat{n}_i \neq 0$, $\mathbf{g} = (N, E) \in \mathcal{G}^*$ and $n_i(\mathbf{g}) = 0$. In this case, there exists $j \in N$ such that $j \in \hat{N}_i$ and $g_{j,i} = 1$. Therefore, we have:

$$\left(\gamma\left(\frac{n - \hat{n}_j}{n+1}\right)\right)^2 > \delta.$$

   Let $\mathbf{g}' = (N, E \cup \{i, j\})$. If firm $i \in N$ sets a link with $j \in N$, we get $\hat{n}_i(\mathbf{g}') - \hat{n}_i(\mathbf{g}) = 0$ and $n_i(\mathbf{g}') = n_j(\mathbf{g}) + 1$. Then we have:

$$\left(\gamma\left(\frac{n(n_j(\mathbf{g}) + 1)}{n+1}\right)\right)^2 \geq \left(\gamma\left(\frac{n - \hat{n}_j(\mathbf{g})}{n+1}\right)\right)^2 > \delta.$$

   A contradiction.
   Hence, if $\hat{n}_i \neq 0$ then $n_i(\mathbf{g}_i) \neq 0$. The result follows.

Third, we show that $\hat{n}_i(\mathbf{g}_i) = n - 1$. To establish a contradiction assume $j \notin N_i$ and $g \in \mathcal{G}^*$. It is straightforward that $N_i \cap N_j = \emptyset$. Moreover,

$$\begin{aligned} \pi_j(\mathbf{g}_j + \mathbf{e}_i, \mathbf{g}_{-j}) - \pi_j(\mathbf{g}_j, \mathbf{g}_{-j}) &= \left(\frac{\gamma n_i(\mathrm{g}(n-\hat{n}_j(\mathrm{g})))}{n+1}\right)^2 \\ &\geq \left(\frac{\gamma n_i(\mathrm{g}(n-\hat{n}_i(\mathrm{g})))}{n+1}\right)^2 \\ &> \delta \end{aligned}$$

A contradiction. The result follows.

Fourth, we show that firm $i \in N$ has an incentive to form link with every other firm $j \in N$. Assume $\mathbf{g} \in \mathcal{G}^*$. We know that $\hat{n}_i = n - 1$ and $n_i \neq 0$. So, there exists $j \in N_i$. Hence:

$$\frac{\gamma n_j(n - \hat{n}_i)}{n+1} > \delta^{\frac{1}{2}}.$$

To establish a contradiction, assume that there exists $k \in N$ such that $g_{i,k} = 0$. Given that $n_k \geq 1$, we have:

$$\frac{\gamma n_k n}{n+1} < \delta^{\frac{1}{2}}.$$

Given that $n_k n > n_j$, $\mathbf{g} \notin \mathcal{G}^*$. A contradiction.

The result follows.

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It is interesting to compare our result with that of Goyal and Joshi (2001) who under a similar specification of demand and marginal cost derives a complete network as the only equilibrium network. This difference in our result, can be explained by the transitivity property of externality flows. Indeed, the first part of our proposition explains why the complete network can not be an equilibrium network under transitivity of externality flows.

3.2 Non Cooperative Transitive Networks and Bertrand Oligopoly

In the case of Bertrand oligopoly, the introduction of transitivity does not change the result. Indeed, the empty network is the only equilibrium network.

Proposition 4 *Suppose there is price competition among firms. Moreover assume that marginal demand satisfies equation (1) and cost satisfies equation (2). Then:*
- if $\delta \geq \gamma(\alpha - \gamma_0)$, the empty network is the unique equilibrium network;
- if $\delta < \gamma(\alpha - \gamma_0)$, the inward pointing stars are the unique equilibrium networks.

Proof: We use the same arguments than in proposition 2 to establish this proposition.

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A star network is a relatively counter-intuitive equilibrium network. Indeed, when a network is a star, the firm $i \in N$ at the centre incurs the cost of forming and maintaining $n-1$ links. Given the transitivity property of externality flows, a firm $j \in N$ located at the periphery can access to the same externalities that $i \in N$ by setting a link with the latter and by incurring only the cost of forming one link. So, it is relatively amazing to state that firm j does not have an incentive to set a link with i. In fact, this behaviour can be explained in the following way. When firms decide which prices to set, they only take into account the variable cost and not the fix cost. More specifically, when a firm decides about her price, she already has incurred the cost of forming and maintaining links. So, this cost constitutes a fix cost. Therefore, the price behaviours of two firms with the same variable cost will be identical, whatever the number of links each has formed. It implies that in the case where there are at least two firms who derive the same benefit from externality flows, the profits of each will be null. Hence, being aware of this fact, when a network is a star, no firm at the periphery has an incentive to form a link with the firm at the centre. Then the star network is an equilibrium network.

4 Conclusion

In this paper we have presented the endogenous formation of directed networks in an oligopoly with either price or quantity competition. A distinctive aspect of our approach is that the costs of forming links are incurred only by the firm who sets the links. We have characterised a set of equilibrium networks. Our results show that the equilibrium networks have simple architectures. In particular, when the costs of link formation are not too high, then in the case of Cournot oligopoly the complete (under non transitivity) and the strong (under transitivity) networks are the only stable networks. In the case of Bertrand oligopoly the inward pointing stars are the only equilibrium networks. When the costs of link formation are too high the empty network is the only equilibrium Network whatever the nature of market competition. Our findings are very different from those derived by Goyal and Joshi (2000) who have used the notions of cooperative links and stable networks (notions introduced by Jackson and Wolinski (1996)).

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