A BARGAINING APPROACH TO THE PROVISION OF PUBLIC GOODS*

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Abstract

Welfarism is a very strong and critizied axiom (see Roemer (1988)) and, in economies with public goods and agents with quasi-linear preferences, it characterizes a very special class of solutions known as monotone utility path. Here it is provided the characterization of four solutions, namely, Nash solution, lexicographic extension of Nash solution, equal-loss solution and rational equal-loss solution without making use of the welfarism axiom or equivalent axioms.

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1 Introduction

Axiomatic Bargaining Theory characterizes solutions to a problem in such a way that all the information not contained in the utility possibility set is ignored. One basic axiom in this theory is welfarism. Two enviroments with the same utility posibility set give rise to the same solution. In a previous paper, Ginés and Marhuenda (2000), it is shown that Kalai's characterization of the welfare egalitarian solution can be transplanted from classical Bargaining Theory into some meaningful environments, in our case the provision of public goods. We use an axiom called solidarity, which states that any improvement in the valuation of the public goods of any agent should not damaged for the other agents. But solidarity plus Pareto optimality implies welfarism. And moreover in this case, implied by welfarism, where the characterization is obtained using axioms related to the primitives, we found that individual contributions are irrelevant, the important data is the total surplus.

The present paper deals with solutions which take into account the individual contributions. First it is presented a new characterization of the welfare egalitarian solution without using the welfarism axiom or equivalent axiom to it as the solidarity axiom. Then, changing one by one the axioms characterizing the welfare egalitarian solution it is found the characterization of three more solutions which take into account the individual contributions to the total surplus. And those three solutions of the classical Bargaining Theory are the lexicographic extension of the Nash solution, the equal-loss solution, and the rational equal-loss solution.

The intersection of these solutions with the classical bargaining theory comes from the idea that they have their correspondence in the bargaining theory with claims where the claims are defined as the individual contributions to the surplus.

The lexicographic extension of the Nash solution appears when Nash solution does not fit with the claims. It equalizes gains from inactivity point as much as claims allow. The non-envy or equal-loss solution prescribes equal share of the cost of the public good provided. It corresponds to the equal-loss from the claims point. The rational equal loss avoids situations where the equal-loss prescribes a payment bigger than the individual valuation of the public good. That is, utilities below the inactivity point.

2 Model

Let $N = \{1, 2, 3, ..., n\}$ denote the set of agents. The space of public goods is $Y = \mathbb{R}^m_+ = \{y \in \mathbb{R}^m : y \geq 0\}$. The technology to produce those goods is jointly owned by all agents and it is described by a function $c : Y \rightarrow \mathbb{R}$ which measures the cost of producing each bundle of public goods in terms of the single private good of the economy. The set $X_i = \mathbb{R}$ represents the possible payments, in terms of the private good, made by agent $i \in N$. Let $X^n = \prod_{i=1}^n X_i$.

The preference relation of agent $i \in N$ is represented by a quasi-linear utility function $u_i(y; t_i) = b_i(y) - t_i$, with $(y; t_i) \in Y \times X_i$, which represents the utility obtained by agent $i \in N$ when bundle $y \in Y$ of public goods is implemented and he has to contribute the amount t_i towards its financing.

Assumption 2.1 *The cost function* $c : Y \rightarrow \mathbb{R}_+$ *is continuous and non-decreasing.*

Assumption 2.2 *For each* $i \in N$, $b_i : Y \rightarrow \mathbb{R}$ *is a continuous, non-decreasing function satisfying* $\lim sup_{||y|| \rightarrow \infty} b_i(y)/c(y) = 0$.

Given $N = \{1, ..., n\}$, $b_i(y) - t_i$ is interpreted as the net benefit agent $i \in N$ obtains when he has to contribute t_i units of his private good in order to enjoy the bundle y of public goods. Thus, $b_i(y) - t_i$ is the net contribution that agent $i \in N$ makes towards the net surplus $b_1(y) + ... + b_n(y) - c(y)$ that the society obtains from the consumption of a bundle $y \in Y$ of public goods. Since the utilities of the agents are always quasi-linear, it is identified the utility function with its non-linear part.

Let $b = (b_1, ..., b_n)$ be a vector of utilities, $y \in Y$ be a bundle of public goods and $t = (t_1, \ldots, t_n) \in X^n$ a vector of contributions, I will use the following notation $u_b(y;t) = (b_1(y) - t_1, ..., b_n(y) - t_n)$. The utility profile resulting from $b = (b_1, ..., b_i, ..., b_n)$ when utility function b_i is replaced by a new utility function v_i is denoted by $(b_{-i}, v_i) = (b_1, ..., b_{i-1}, v_i, b_{i+1}, ..., b_n)$. Given two different utility profiles b and v defined on Y, denote $b \geq v$ whenever $b(y) \geq v(y)$ for every $y \in Y$.

An economy $e = (N_0; b, c)$ is defined by a finite set of agents $N_0 \subset N = \{1, ..., n\}$, a utility profile b and a cost function c. Let E be the set of economies satisfying 2.1 and 2.2 and such that if $e \in E$ implies there is $y \in Y$ with $\sum_{i \in N_0} b_i(y) - c(y) \geq 0$.

An allocation $(y;t) = (y; t_1, ., t_{n_0}) \in Y \times X^{n_0}$ is feasible in economy $e = (N_0; b, c) \in E$ whenever $c(y) \leq \sum_{i=1}^{n_0} t_i$. $F(e) = \{(y; t_1, ., t_{n_0}) : c(y) \leq \sum_{i=1}^{n_0} t_i\}$ is the set of all feasible allocations and $U(e) = \{u_b(y;t) : (y;t) \in$

$F(e)\}$ denotes the set of feasible utilities. Denote by $PO(e)$ the set of Pareto optimal allocations, that is, those feasible allocations $(y;t) \in F(e)$ for which if $u_b(y;t) < u_b(z;r)$ then $(z;r)$ is not feasible. And $UPO(e) = \{u_b(y;t) : (y;t) \in PO(e)\}$ the set of vectors of utilities provided by the Pareto optimal allocations of this economy.

Now, for each economy $e \in E$, I define $V(e) = \text{Max}_{y \in Y} \sum_{i=1}^{n_0} b_i(y) - c(y)$ as the total surplus to share among the agents. Since agents have quasi-linear preferences, $UPO(e) = \{(a_1, ..., a_{n_0}) \in \mathbb{R}^{n_0} : \sum_{i \in N} a_i = V(e)\}$. And let $\bar{y} \in \arg max_{y \in Y} \sum_{i=1}^{n_0} b_i(y) - c(y)$ denote an optimal bundle of public goods.

A solution for the problem of the optimal provision of public goods is a function $S : E \rightarrow Y \times X$ which assigns to every economy $e \in E$ a feasible allocation $S(e)$.

Axiom 2.3 *(PO): A solution S satisfies Pareto optimality if $S(e) \in PO(e)$ for each $e \in E$.*

Since one of the aims of the paper is to obtain characterizations of the Nash or welfare egalitarian solution without the welfarism axiom, it is introduced a weaker one.

Axiom 2.4 *(WIIA): A solution S satisfies weak independence of irrelevant* *alternatives whenever, given* $e = (N_0; b, c)$ *and* $e' = (N_0; v, c')$ *such that* $b_i \geq v_i$ *for each* $i \in N$ *and* $c \leq c'$, $S(e) \in PO(e')$ *and* $u_b(S(e)) \in UPO(e')$ *implies that* $u_v(S(e')) = u_b(S(e))$.

It is demanded not only to keep the utility but also the allocation. This axiom, of course, is satisfied by welfare egalitarian solution.

Both, PO and WIIA, will be the basic axioms used to characterize the solutions. In this context of quasi-linear economies with public goods and when it is assumed that all agents have the same claim, that is the total surplus, most of the solutions collapse in the Nash solution (or welfare egalitarian solution).

Definition 2.5 *The Nash solution is defined as follows: Given an economy* $e \in E^n$, $N(e) = \arg max_{\{(y;t) \in F(e)\}} \{\prod_{i=1}^{n} (b_i(y) - t_i)\}$.

In order to characterize the Nash solution, first it is imposed that a solution should provide utilities bigger than the inactivity point and this property is called weak individual rationality

Axiom 2.6 *(WIR): A solution* S *satisfies weak individual rationality if* $u_b(S(e)) \geq b(0)$ *for each economy* $e = (N; b, c) \in E^n$.

Another axiom standard in the classical bargainig theory is equal translation invariance. With this property the solution is invariant under equal translation of the inactivity point.

Axiom 2.7 *(ETI): A solution S satisfies equal translation invariance if given an economy $e = (N; b, c) \in E^n$ and a real number $k \leq V(e)/n$, then $u_{b-k}(S(N; b-k, c)) = u_b(S(N; b, c)) - k$, where $b-k$ denotes the utility profile $(b_1(y) - k, ..., b_n(y) - k)$.*

Since fixed costs are allowed, the next property states how to share them. The independence of cost function's zero axiom (ICFZ) says that any fixed costs ($c^*(0) > 0$) or any subsidies ($c^*(0) < 0$) are shared equally among the agents.

Axiom 2.8 *(ICFZ): A solution S satisfies independence of cost function's zero whenever given an economy $e = (N; b, c) \in E^n$ with $S(e) = (\bar{y}; t)$ and a scalar $\beta \leq n \min_{i \in N} b_i(\bar{y})$ and $\beta \leq V(e)$, such that $c^* = c + \beta$, and $h = (\beta/n, ..., \beta/n)$ then $u_b(S(e)) - h = u_b(S(N; b, c^*))$.*

Now, a new characterization of the Nash solution is provided, without using the solidarity axiom or equivalent axioms to welfarism.

Theorem 2.9 *A solution S satisfies PO, WIIA, ETI, ICFZ and WIR axioms if and only if $S(e) \in N(e)$.*

Because sometimes it is important to take into account the individual contributions to the surplus, an axiom called No Private Transfers (NPT) is introduced. This axiom reflects the idea that the individual contributions to the surplus matter and cannot be sumarized by total surplus. NPT axiom was used in Moulin (1987) in order to characterize the egalitarian equivalent solution in the case of one public good. Although private transfers are allowed, nobody will receive in the solution more utility than its contribution to the surplus. This axiom provides a bound on the possible claims.

Axiom 2.10 *(NPT): A mechanism S satisfies the axiom of No Private Transfers if for every $e \in E$, $S(e) = (y; t_1, ..., t_n)$ is such that $t_i \geq 0$ for each $i = 1, ..., n$.*

Clearly this axioms is not satisfied by the Nash solution. If NPT axiom is added to the characterization then some other axiom should be dropped. If the choosed one is ICFZ axiom, the characterization of the lexicographic extension of the Nash solution is obtained. If WIR axiom is the one dropped, the characterization of the Equal-loss solution appears. And finally if ETI axiom is picked, it is obtained the characterization of the rational equal-loss solution.

Finally, an axiom called separability (Sp) is introduced to generalize some characterizations from two agents to a generic n. Under Sp it is possible to reduce the problem from n agent to $n-1$ agents under the condition that agent n, with the lower valuation of the public good should always end up with utility level $u_n(S(e))$.

Axiom 2.11 *(Sp): A solution S satisfies the separability axiom if given an economy $e = (N_0; b, c) \in E$ with $N_0 = \{1, \ldots, n_0\}$ $b_1 \geq b_2 \geq \ldots \geq b_{n_0-1} \geq b_{n_0}$, then $u_{b_{-(n_0)}}(S(N_0; b, c)) = u_b(N_1; b_{-(n_0)}, c')$ where $c'(y) = c(y) - b_{n_0}(y) + u_{n_0}(S(e))$ for all $y \in Y$ and $N_1 = \{1, ..., n_0 - 1\}$.*

The definition of the other three solutions and the characterization results are stated properly.

The lexicographic extension of the Nash solution coincides with the Nash solution in the case Nash is compatible with NPT axiom, alternatively some agents do not pay any money and the rest pay in order to attain the same utility. Agents are equalized as much as the axiom NPT allows.

Definition 2.12 *Given an economy $e \in E$, the lexicographic extension of the*

Nash solution is defined by $N^L(e) = \arg max_{\{(y;t)\in PO(e)\cap NPT(e)\}}\{\prod_{i=1}^{n}(b_i(y) - t_i)\}$[1]

The equal-loss solution or non-envy solution, in the present setting, means that all the agents pay the same amount of private good in order to provide the public goods. In the Axiomatic Bargaining literature it corresponds to the equal-loss solution from the claims point.

Definition 2.13 *Given an economy* $e \in E$*, the non-envy or equal-loss solution consists of all Pareto efficient allocations that prescribe equal contributions to the cost of the public goods. And then,* $EL(e) = \{(y;t) \in PO(e) : t_i = t_1 \text{ for all } i = 1, \ldots, n\}$

Finally to avoid situations where the equal-loss solution prescribes a result below the inactivity point, it appears the rational equal-loss.

Definition 2.14 *Let* $e = (N; b, c) \in E$ *be an economy,* $(\bar{y}; t_1, ..., t_n) \in REL(e)$ *if it is defined in the following way: Let* $\bar{y}$ *be a Pareto optimal bundle of public goods. Suppose, without loss of generality, that* $b_1(\bar{y}) \leq b_2(\bar{y}) \leq \ldots \leq b_n(\bar{y})$*. Denote by* $j = \min\{i \in N : b_i(\bar{y}) \geq (c(\bar{y}) - \sum_{h=1}^{j-1} b_h(\bar{y}))/(n - j + 1)\}$

[1] Since the utilities are quasi-linear the set $PO(e) \cap NPT(e) = \{(y;t) \in PO(e)/t_i \geq 0 \text{for} i \in N\}$ is always non-empty.

If $j = 1$ then $t_i = c(\bar{y})/n$ for all $i = 1, \ldots, n$. (That is equal share of the cost).

If $j \geq 2$, for $1 \leq p \leq j-1$ define $t_p = b_p(\bar{y})$ and for $n \geq p \geq j$ assign $t_p = (c(\bar{y}) - \sum_{h=1}^{j-1} b_h(\bar{y}))/(n-j+1)$.

Every agent pays as much as its valuation of the public goods allows. The solution tends to equalize the contributions of the agents to the cost of the public goods.

Now, the main result is stated.

Theorem 2.15 *A solution S satisfies PO,WIIA axioms and*

a)ICFZ, WIR, ETI axioms if and only if $S(e) \in N(e)$.

b)NPT, WIR, ETI and SP axioms if and only if $S(e) \in N^L(e)$.

c)NPT, ICFZ and ETI axioms if and only if $S(e) \in EL(e)$.

d)NPT, ICFZ, WIR and SP axioms if and only if $S(e) \in REL(e)$.

3 Appendix

The following lemma will be used in all the following results.

Lemma 3.1 *Given $e = (N; b, c) \in E^n$ an economy and S a solution satisfying PO and WIIA, there is a function $\bar{v} : Y \to \mathbb{R}$, scalars $(\alpha_1, ..., \alpha_n) \in \mathbb{R}^m_+$*

and a cost function c' such that if $v = (v_1, ..., v_n)$ is a utility profile with $v_i = \alpha_i \bar{v}$ the economy $e' = (N; v, c')$ satisfies $S(e) = S(e')$

Proof

Let $e = (N; b, c) \in E^n$ an economy and S a solution satisfying PO and WIIA and let $S(e) = (\bar{y}; t)$.

First define a function $c^*(y) = \bar{\lambda}(y)c(\bar{y})$ where $\bar{\lambda}(y) \in \arg max\{\lambda \in \mathbb{R}_+ : \lambda\bar{y} \leq y\}$. Now define a cost function $c'(y) = (c \vee c^*)(y) = \text{Max}\{c(y), c^*(y)\}$, which satisfies all the conditions and the function $\bar{v}(y) = 1$ if $y \geq \bar{y}$. Otherwise $\bar{v}(y) = \bar{\lambda}(y)$ where $\bar{\lambda}(y)$ is defined as above.

Also define for each $i \in N$, $\alpha_i = b_i(\bar{y})$ and $v_i(y) = \alpha_i \bar{v}(y)$. Let $v = (v_1, ..., v_n)$. Denote by $(b \wedge v)$ a preference profile such that $(b \wedge v)_i(y) = \min\{b_i(y), v_i(y)\}$. Then $V(N; (b \wedge v), c') \leq V(e)$ because $(b \wedge v) \leq b$ and $c' \geq c$. The fact that $c'(\bar{y}) = c(\bar{y})$, $(b \wedge v)_i(\bar{y}) = b_i(\bar{y})$ for all $i \in N$ clearly concludes that $V(N; (b \wedge v), c') = V(e)$. Applying PO and WIIA it is derived $S(N; (b \wedge v), c') = S(N; b, c)$.

Moreover also by applying WIIA and PO $S(N; (b \wedge v), c') = S(N; v, c')$. And, as a result, $S(N; v, c') = S(N; b, c)$.

□

In all the proofs it is easy to check that the different solutions satisfy the axioms which characterize them and, therefore, it is omitted.

Proof of Theorem 2.9

Let $e = (N_0; b, c) \in E$ an economy and S a solution satisfying PO, WIIA, WIR, ICFZ, ETI . Applying lemma 3.1, it is derived that $S(N_0; v, c') = S(e)$. Let $V = V(e)$, Now define the following economy $e'' = (N_0; v + V/n, c' + V)$. By ICFZ, $u_{v+V/n}(S(e'')) = u_{v+V/n}(S(N_0; v + V/n, c')) - V/n$ and $u_{v+V/n}(S(N_0; v + V/n, c')) = u_v(S(e')) + V/n$ by ETI. Then, $u_{v+V/n}(S(e'')) = u_v(S(e'))$ because of the above equalities. On the other hand, $u_{v+V/n}(S(e'')) = u_v(S(N_0; v, c' + V)) + V/n$ by ETI. Now, the economy $(N_0; v, c' + V)$ is analized. $V(N_0; v, c' + V) = 0$ and by definition $v(0) = 0$. Then, applying WIR it is obtained that $u_v(S(N_0; v, c' + V)) = 0$. To sum up, on one hand $u_{v+V/n}(S(e'')) = u_v(S(e')) = u_b(S(e))$, on the other hand $u_{v+V/n}(S(e'')) = V/n$, furthermore $u_b(S(e)) = V/n$. □

Proof of Theorem 2.15

b) Lexicographic extension of the Nash solution

Clearly, in the case of two agents, there are two possibilities for the lexicographic extension of the Nash solution. If the Nash solution exists and prescribes no transfers, both coincide. If that is not the case, one of the agents pays the entire cost of the efficient bundle of public goods. This can be generalized to n agents.

Let $e = (N_0; b, c) \in E$ an economy and S a solution satisfying PO, WIIA, WIR, NPT, ETI and Sp. Firts of all, applying lemma 3.1, $S(N_0; v, c') = S(e)$. Suppose, without loss of generality, that agent n is the agent with the lower valuation of the public goods. In our case because all utility functions in v are comparable by construction, this corresponds to the lower α_i.

If $\alpha_n \geq V(e)/n$ then $u_{v-V(e)}(S(N_0; v - V(e)/n, c')) = u_v(S(e')) - V(e')/n$ by ETI. Now, define $v'(y) = \max\{v(y) - V(e)/n, 0\}$, and $e'' = (N_0; v', c')$ a new economy. By contruction, $V(e'') = 0$ and $v(0) = 0$. Then by WIR it is obtained that $u_{v'}(S(e'')) = 0$. Applying WIIA to $(N_0; v - V(e), c')$ and e'' it is derived that $S(N_0; v - V(e)/n, c') = S(e'')$. Finally, $0 = u_{v'}(S(e'')) = u_{v-V(e)}(S(N_0; v - V(e)/n, c')) = u_v(S(e')) - V(e')/n$. And clearly, $u_b(S(e)) =$

$u_v(S(e')) = V(e')/n$

In other case, denote $k = v_n(\bar{y})$. Now apply ETI to this economy $e' = (N_0; v, c')$ with k. Then $u_{v-k}(S(N_0; v-k, c')) = u_v(S(e')) - k$. Next it is defined a preference profile v' similarly to the definition of v, such that $v'_i(0) = 0$ and $v'_i(\bar{y}) = v_i(\bar{y}) - k$ for each $i \in N_0$. Applying PO and WIIA $S(N_0; v', c') = S(N_0; v-k, c')$. And, in particular, $v'_n(\bar{y}) = 0$. By NPT and WIR $u_n(S(N_0; , v', c')) = 0$. Then $u_n(S(N_0; v-k, c')) = 0$ and $u_n(S(N_0; v, c')) = u_n(S(N_0; v-k, c')) + k = v_n(\bar{y})$. Let $N_1 = \{1, ..., n-1\}$. Applying consistency reduce the problem from n agents to $n-1$. The new economy is $e'' = (N_1; v, c'')$ where $c''(y) = c'(y) - v_n(y) + u_n(S(N_0; v, c'))$ Now repeat the process from the begining. All this process continues until there is an economy with equal division of the surplus or it is reduced to an economy with two agents where equal division does not meet the claims, that is, it implies violation of NPT axiom. In this case, applying ETI as above, the solution attained prescribes that the lower agent does not pay any private good. This solution corresponds to the lexicographic extension of the Nash solution compatible with NPT.

c) Equal-loss solution

Let $e = (N; b, c)$ an economy and S a solution satisfying PO, WIIA, NPT, ETI and ICFZ. Using lemma 3.1, $S(N; v, c') = S(e) = (\bar{y}; t)$. Denote $C = c'(\bar{y})$ Now, $u_{v+C/n}(S(N_0; v + C/n, c')) = u_v(S(e')) + C/n$ by ETI axiom. On the other hand aplying lemma 3.1 to the economy $(N_0; v + C/n, c')$ a new preference profile v' is obtained such that $v'(0) = 0$ and a cost function c'' such that $S(N_0; v', c'') = S(N_0; v + C/n, c')$. Define $c'''(y) = \max\{c''(y), C\}$. By WIIA axiom, $S(N_0; v', c''') = S(N_0; v', c'')$ Apply ICFZ axiom to the economy $(N_0; v', c''')$ with $\beta = C$, it is derived that $u_{v'}(S(N_0; v', c''' - C)) - C/n = u_{v'}(S(N_0; v', c'''))$

d)Rational equal-loss solution

Let $e = (N; b, c) \in E^{n+1}$ be an economy and a solution S satisfying PO, WIIA, NPT, WIR, ICFZ and Sp. Firstly,by PO and WIIA, there is $e' = (N; v, c')$ as in lemma 3.1. Now it is possible to distinguish two cases:

I) if $(c(\bar{y})/(n+1) \leq \min_{i \in N} v_i(\bar{y}))$, that is, if there is a equal-loss solution compatible with the WIR, define

$$\hat{c}(y) = \begin{cases} \hat{\lambda}(y)c'(\bar{y}) & \text{if } y \geq \bar{y} \\ c'(\bar{y}) & \text{otherwise} \end{cases}$$

where $\bar{\lambda} \in \arg max\{\lambda \in \mathbb{R}_+ : \lambda\bar{y} \leq y\}$.

Now $c'' = \hat{c} \vee c'$. Clearly $c' \leq c''$ and by WIIA and PO $S(e'') = S(e')$ where $e'' = (N; v, c'')$. But $e^* = (N; v, c^*)$ with $c^* = c'' - c'(\bar{y})$, satisfies by NPT and PO that $S(e^*) = (\bar{y}; 0)$ and by ICFZ $S(e'') = (\bar{y}; r)$ where $r_1 = ... = r_{n+1} = c'(\bar{y})/(n+1)$. And, finally, $S(e) = S(e') = S(e'')$.

II) Otherwise, assume without loss of generality that $v_1(\bar{y}) \geq v_2(\bar{y}) \geq ... \geq v_{n+1}(\bar{y})$. Pick $y^* = \lambda^* \bar{y}$ with $\lambda^* \leq 1$ such that $v_{n+1}(\bar{y}) = c'(y^*)/(n+1)$. Define a cost function

$$\hat{c}(y) = \begin{cases} c'(y^*) & \text{if } 0 \leq \lambda \leq \lambda^* \\ \dfrac{\lambda - \lambda^*}{1 - \lambda^*} c(\bar{y}) + \dfrac{1 - \lambda}{1 - \lambda^*} c(y^*) & \text{if } \lambda^* \leq \lambda \leq 1 \\ \lambda c(\bar{y}) & \text{if } \lambda \geq 1 \end{cases}$$

In each case $\lambda \in \arg max\{\lambda_1 \in \mathbb{R}_+ : \lambda_1 \bar{y} \leq y\}$. Define $c'' = c' \vee \hat{c}$, then $c' \leq c''$ and $c''(\bar{y}) = c'(\bar{y})$ by WIIA and PO $S(e'') = S(e')$, where $e'' = (N; v, c'')$. But if a new economy $e^* = (N; v, c^*)$ is constructed with $c^* = c'' - c'(y^*)$, applying ICFZ, if $z = (c'(y^*)/(n+1), ..., c'(y^*)/(n+1)) \in \mathbb{R}^{n+1}$ then $u_b(S(e^*)) - z = u_b(S(e''))$. In particular, $u_{n+1}(S(e'')) = u_{n+1}(S(e^*)) - c'(y^*)/(n+1)$. Moreover by NPT $u_{n+1}(S(e^*)) \leq v_{n+1}(\bar{y}) = c'(y^*)/(n+1)$, then it is possible to conclude by WIR that $u_{n+1}(S(e'')) = 0$. Furthermore, $u_{n+1}(S(e')) = 0$. Applying Sp to economy e' restrict the problem to n agents and with $c''(y) = c'(y) - v_{n+1}(y)$. Now, repeat, if there is the equal-loss consistent with WIR apply this case if not reduce the problem as before until $n = 2$ in which this process uniquely determines the rational equal-loss.

□

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