A Residual-Based LM Test for Fractional Cointegration

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This version: January 18, 2002

Keywords: Long memory, LM test, single equations
JEL classification: C12, C22

Abstract

Nonstationary fractionally integrated time series may possibly be fractionally cointegrated. In this paper we propose a test for the null hypothesis of no cointegration. It builds on a static cointegration regression of the levels of the variables as a first step. In a second step, a univariate LM test known from the literature is applied to the single equation regression residuals. However, it turns out that the application of the LM test to residuals without further modifications does not result in a limiting standard normal distribution, which contrasts with the situation when the LM test is applied to observed series. Therefore, we suggest a simple modification of the LM test that accounts for the residual effect. At the same time it corrects for eventual endogeneity of the cointegration regression. The proposed modification guarantees a limiting standard normal distribution of the test statistic. Our procedure is completely regression based and hence easy to perform. Monte Carlo experiments establish its validity for finite samples.

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1 Introduction

Engle and Granger (1987) proposed in their pathbreaking paper to apply the Dickey-Fuller (DF) test to regression residuals in order to test the null hypothesis of no cointegration. A thorough asymptotic treatment was provided by Phillips and Ouliaris (1990). Both papers assume observed time series that are integrated of order one, I(1), and residuals that are I(1) under the null hypothesis. This test is designed against the alternative of integer integration, i.e. that the regression errors are I(0). Similarly, we assume time series that are all integrated of the same order d and suggest a residual-based test for the null hypothesis of no cointegration that the residuals are I(d) as well. The order of integration d is allowed to be noninteger. The residual-based LM test introduced here is constructed against the alternative of fractional cointegration, i.e. under H_1 the residuals are I($d - b$), $b > 0$.

In most economic applications it is believed that $d = 1$. Still, under cointegration the deviations must not necessarily be I(0) but may rather be fractionally integrated of order $d - b \neq 0$. Applied fractional cointegration analyses include Cheung and Lai (1993), Booth and Tse (1995) and Masih and Masih (1995, 1998), Baillie and Bollerslev (1994) and Dueker and Startz (1998). All those empirical studies treat small systems of mostly only two variables where a residual-based single equation approach seems appropriate. The residual-based test considered here does not rely on the DF statistic because it is well documented that the DF test has little power against fractional alternatives, see Sowell (1990), Diebold and Rudebusch (1991), Hassler and Wolters (1994) and Krämer (1998). We rather adopt the LM test pioneered by Robinson (1991) and further studied and extended by Robinson (1994), Agiakloglou and Newbold (1994), Tanaka (1999) and Breitung and Hassler (2002). Similarly as the DF test the latter variant is based completely on regressions and therefore particularly simple to compute. In case of observed time series all those variants follow under the null hypothesis a limiting normal distribution and can be performed as one-sided or two-sided tests.

The main contributions of this work are the following. First, we show that the LM test applied naively, i.e. without modifications, to regression residuals does not have a Gaussian limiting distribution. Second, we suggest a modification of the one-sided test by Breitung and Hassler (2002), which is in the spirit of Saikkonen (1991). It corrects for the residual effect as well as for the effect of eventual endogeneity of the regressors. It is shown that this modified residual-based LM test does have a standard normal asymptotic distribution under the null hypothesis of no cointegration.

The rest of the paper is organized as follows. After the intoduction our assumptions are presented and the LM test is introduced. In Section 3, it is shown that the LM test applied to regression residuals without modification results in a nonnormal limiting distribution depending on d. The fourth section deals with a simple regression based modification of the LM test applied to regression residuals that results in an asymptotic standard normal distribution. Finite sample properties of this fractional cointegration test are investigated by means of Monte Carlo experiments in Section 5. The final section concludes.

2 Preliminaries

Let the scalar $y_{1,t}$ and the m-dimensional vector $y_{2,t}$ be $I(d)$ processes, where $d > 0.5$ is known. More precisely, we assume the regression model

$$y_{1,t} = y_{2,t}' \beta + z_t, \; z_t \sim I(d-b), \; b \geq 0, \quad t = 1, 2, \ldots, T. \tag{1}$$

If $b > 0$ and $\beta \neq 0$, then $y_{1,t}$ and $y_{2,t}$ are called fractionally cointegrated. Notice that the observed series as well as the errors may be fractionally integrated, although in many economic applications $d = 1$ may hold. The null hypothesis to be tested is absence of cointegration,

$$H_0 : \, b = 0 \text{ vs. } H_1 : \, b > 0.$$

The error term z_t may correlate with (lags and leads of) the regressors. The assumptions on the stochastic processes are the following, where L denotes the usual lag operator.

Assumption 1: *Let $y_{2,t}$ and z_t from (1) be fractionally integrated processes,*

$$(1-L)^d y_{2,t} = v_{2,t} \,, \quad (1-L)^{d-b} z_t = v_{1,t} \,, \quad b \geq 0 \,,$$

with starting values $y_{2,t} = 0$ and $z_t = 0$ for $t \leq 0$, where $w_t' = (v_{1,t}, v_{2,t}')$ is a stationary and invertible process with zero mean, finite fourth moments and absolutely summable covariance function,

$$\sum_{j=0}^{\infty} \| E(w_t w_{t-j}') \| < \infty \,.$$

Here, $\| \cdot \|$ signifies the standard Euclidean norm. From this assumption it follows, see e.g. Saikkonen (1991), that there exists a stationary process x_t such that

$$v_{1,t} = \sum_{k=-\infty}^{\infty} v'_{2,t-k}\, \pi_k + x_t \,, \tag{2}$$

where

$$E\left(x_t v_{2,t-k}\right) = 0,\ k = 0, \pm 1, \pm 2, \dots\,, \quad \sum_{j=0}^{\infty} |E(x_t x_{t-j})| < \infty\,, \quad \sum_{k=-\infty}^{\infty} \|\pi_k\| < \infty\,.$$

If z_t was observable, an LM test against the alternative of fractional cointegration would build on the differences under H_0,

$$\zeta_t := (1 - L)^d z_t\,.$$

Under the alternative H_1, ζ_t is I$(-b)$, $(1 - L)^{-b}\zeta_t = v_{1,t}$. The LM variant proposed by Breitung and Hassler (2002) further requires the weighted sum of past values,

$$\zeta^*_{t-1} := -\left.\frac{\partial \zeta_t}{\partial b}\right|_{b=0} = \sum_{j=1}^{t-1} \frac{\zeta_{t-j}}{j}, \quad t = 2, 3, \dots, T. \tag{3}$$

Under the null, $\zeta_t = v_{1,t}$ is I(0), and due to the choice of the weights $1/j$, ζ^*_{t-1} is asymptotically stationary.

Let us assume for the moment that $v_{1,t}$ is white noise. Then ζ_t and ζ^*_{t-1} are uncorrelated under H_0. Therefore, the ordinary least squares (OLS) regression,

$$\zeta_t = \widehat{\phi}\, \zeta^*_{t-1} + \widehat{e}_t \tag{4}$$

results in a limiting normal distribution when testing for $\phi = 0$,

$$t_\phi = \widehat{\phi}\, \frac{\sqrt{\sum (\zeta^*_{t-1})^2}}{\sqrt{\frac{1}{T-1} \sum \zeta_t^2}} = \frac{\sum \zeta_t\, \zeta^*_{t-1}}{\sqrt{\sum (\zeta^*_{t-1})^2}\, \sqrt{\frac{1}{T-1} \sum \zeta_t^2}} \xrightarrow{d} \mathcal{N}(0, 1), \tag{5}$$

where $Var(\zeta_t)$ is estimated under the null hypothesis[1] ($\phi = 0$). The cointegration test against $b > 0$ should be performed as a one-sided test rejecting H_0 for too small values of t_ϕ, see Tanaka (1999, Theorem 1).

If $\zeta_t = v_{1,t}$ is a stable AR(p) process under H_0, then an appropriately augmented version of (4) might be used, see Breitung and Hassler (2002).

In practice, the coefficient vector β is usually unknown and, therefore, the error z_t is not observable. The traditional residual-based cointegration test suggested by Engle and Granger (1987) uses the residuals of the cointegration regression instead of the unobservable errors. In the following section we investigate the asymptotic properties of a test based on the regression residuals

$$\widehat{z}_t = y_{1,t} - \beta'\widehat{y}_{2,t} = z_t - \widehat{\beta}' y_{2,t} \, (\widehat{\beta} - \beta) \ .$$

instead of the unobservable errors z_t.

3 Effect of residuals

Even under very restrictive assumptions the effect of using residuals $\widehat{z}_t$ instead of true errors z_t is not negligible. The test statistic t_ϕ from (5) with ζ_t and ζ^*_{t-1} replaced by

$$\widehat{\zeta}_t := (1-L)^d \widehat{z}_t, \quad \widehat{\zeta}^*_{t-1} := \sum_{j=1}^{t-1} \frac{\widehat{\zeta}_{t-j}}{j}, \tag{6}$$

does not result in a limiting normal distribution. In fact, the asymptotic distribution depends on d, and hence percentiles are difficult to tabulate. Those statements will be made more precise now.

For the sake of simplicity the analysis in this section is restricted to simple regressions ($m = 1$). This special case suffices to make our point clear, and it allows to shorten the derivation by relying on a corresponding result by Cappuccio and Lubian (1997).

[1]Breitung and Hassler (2002) consider the usual t statistic with $\frac{1}{T-2}\sum \widehat{e}_t^2$ instead of $\frac{1}{T-1}\sum \zeta_t^2$. In large samples the difference is negligible, in small samples we collected experimental evidence that the present version of the test statistic is superior in terms of size.

Assumption 2: *Let* $y_{2,t}$ *and* z_t *from (1) with* $m = 1$, $\beta = 0$ *and* $0.5 < d < 1.5$ *be fractionally integrated noise processes,*

$$\begin{aligned} (1-L)^d y_{1,t} = (1-L)^d z_t &= v_{1,t} \sim iid(0, \sigma_1^2) \\ (1-L)^d y_{2,t} &= v_{2,t} \sim iid(0, \sigma_2^2)\,, \end{aligned}$$

where $v_{1,t}$ *and* $v_{2,s}$ *are independent of each other for all* t *and* s*.*

Assumption 2 corresponds exactly the spurious regression setup of Cappuccio and Lubian (1997). They prove that the OLS regression of (1) results in

$$\widehat{\beta} \xrightarrow{d} \frac{\sigma_1}{\sigma_2} \overline{\beta}_d\,, \tag{7}$$

where $\overline{\beta}_d$ is a nondegenerate random variable given as simple functional of socalled fractional Brownian motions of order d.

In the Appendix we will prove the following theorem.

Theorem 1: *Let the null hypothesis of no cointegration and Assumption 2 hold true. Replacing* ζ_t *and* ζ_{t-1}^* *in (5) by the residual-based variables from (6), the limiting distribution of the test statistic* t_ϕ *as* $T \to \infty$ *is given by*

$$\frac{\mathcal{Z}_1 - \sqrt{2}\,\overline{\beta}_d\,\mathcal{Z}_2 + \overline{\beta}_d^2\,\mathcal{Z}_3}{1 + \overline{\beta}_d^2}\,,$$

where $\mathcal{Z}_i$, $i = 1, 2, 3$ *are standard normal distributions independent of each other.*

Even under the highly restrictive assumption that errors and regressor are stochastically independent the asymptotic normality does not carry over if t_ϕ is simply computed from residuals of a cointegrating regression. This is due to the fact that $\widehat{\beta}$ does not converge to zero in case of spurious regressions. If the limit $\overline{\beta}_d$ from (7) was zero, then a limiting $\mathcal{N}(0, 1)$ distribution would arise, as we can see from Theorem 1. If we allow for general $m \geq 1$ in Assumption 2, then it can be shown by generalizing the result in (7) to multiple regressions that the limit of t_ϕ depends not only on d but also on the number of regressors, m.

For practical purposes the theorem implies that standard normal inference is not a valid guideline when the LM test is applied without further modifications to OLS residuals in order to test for the null of no cointegration. According to the Monte Carlo evidence in Section 5, size distortions establishing cointegration too often will result if the LM test is applied naively to regression residuals.

4 Correcting for residual effect and endogeneity

Under H_0 the variables from (6) are

$$\begin{aligned} \widehat{\zeta}_t = (1-L)^d \left(z_t - y_{2,t}'(\widehat{\beta} - \beta) \right) &= v_{1,t} - v_{2,t}'(\widehat{\beta} - \beta) \,, \\ \widehat{\zeta}_{t-1}^* = \sum_{j=1}^{t-1} \frac{v_{1,t-j} - v_{2,t-j}'(\widehat{\beta} - \beta)}{j} &= v_{1,t-1}^* - {v_{2,t-1}^*}'(\widehat{\beta} - \beta) \,, \end{aligned}$$

where $v_{1,t-1}^*$ and $v_{2,t-1}^*$ are defined analogously to (3). The residual-based LM test without further modifications does not result in limiting normality because $\widehat{\zeta}_t$ and $\widehat{\zeta}_{t-1}^*$ correlate. There are two sources for that. First, if $v_{1,t}$ and $v_{2,s}$ are iid and independent of each other for all t and s, then $v_{1,t}$ and $v_{2,t}$ are independent of $v_{1,t-1}^*$ and $v_{2,t-1}^*$. Nevertheless, $\widehat{\zeta}_t$ and $\widehat{\zeta}_{t-1}^*$ correlate via $\widehat{\beta}$, and this is the effect studied in Section 3. One may correct for this effect by projecting $\widehat{\zeta}_t$ on $(1-L)^d y_{2,t} = v_{2,t}$. Second, if $v_{1,t}$ and $v_{2,s}$ are iid and independent of each other for all $t \neq s$ but $E(v_{1,t} v_{2,t}) \neq 0$, then $v_{1,t}$ and $v_{2,t}$ are still independent of $v_{1,t-1}^*$ and $v_{2,t-1}^*$. This means that contemporaneous correlation of $v_{1,t}$ and $v_{2,t}$ does not cause additional correlation between $\widehat{\zeta}_t$ and $\widehat{\zeta}_{t-1}^*$. But it is reasonable to assume with economic applications that endogeneity introduces correlation of $v_{1,t}$ and $v_{2,s}$ even for $t \neq s$. Therefore, we consider, third, that $v_{1,t}$ and $v_{2,s}$ are iid but $E(v_{1,t} v_{2,t-j}) \neq 0$ and $E(v_{1,t} v_{2,t+j}) \neq 0$. If $j > 0$, this implies that $v_{1,t}$ and $v_{2,t-1}^*$ as well as $v_{2,t}$ and $v_{1,t-1}^*$ correlate, which causes, in addition to the residual effect, correlation between $\widehat{\zeta}_t$ and $\widehat{\zeta}_{t-1}^*$ due to endogeneity. Therefore, we now propose to correct for the residual effect as well as for endogeneity in order to obtain a simple modification of the residual-based LM test that has an asymptotic Gaussian distribution.

With (2) one may approximate for large K

$$\begin{aligned}\widehat{\zeta}_t &= x_t + \sum_{k=-\infty}^{\infty} v'_{2,t-k}\pi_k - v'_{2,t}(\widehat{\beta} - \beta) \\ &\approx x_t + \sum_{k=-K}^{K} v'_{2,t-k}\pi_k - v'_{2,t}(\widehat{\beta} - \beta)\,,\end{aligned}$$

because $\|\pi_k\| \to 0$ for $|k| \to \infty$. Motivated by the work of Saikkonen (1991), this suggests to project $\widehat{\zeta}_t$ on K leads and lags of $v_{2,t} = (1-L)^d y_{2,t}$ in order to obtain scalar residuals $\widetilde{x}_t$ that are approximately iid. To this end define the $(2K+1)m$ vector W_t,

$$W'_t = (v'_{2,t-K}, \ldots, v'_{2,t}, \ldots, v'_{2,t+K})\,.$$

After computing in a first step cointegration regression residuals $\widehat{\zeta}_t$, consider as a second step the regression of $\widehat{\zeta}_t$ on W_t:

$$\widehat{\zeta}_t = W'_t\,\widetilde{\theta} + \widetilde{x}_t\,,\ t = K+1, \ldots, T-K\,, \tag{8}$$

with residuals $\widetilde{x}_t \approx x_t$ that are used to compute analogously to (3)

$$\widetilde{x}^*_{t-1} := \sum_{j=1}^{t-1} \frac{\widetilde{x}_{t-j}}{j}\,.$$

In a third step we perform the LM regression of $\widetilde{x}_t$ on $\widetilde{x}^*_{t-1}$ analogously to (4) in order to compute the test statistic

$$\widetilde{t}_\phi = \frac{\sum \widetilde{x}_t\,\widetilde{x}^*_{t-1}}{\sqrt{\sum(\widetilde{x}^*_{t-1})^2}\,\sqrt{\frac{1}{T-2K}\sum \widetilde{x}^2_t}}\,. \tag{9}$$

If K in (8) grows with an appropriate rate to infinity, then the residual and endogeneity effects will be negligible. We adopt from Saikkonen (1991), Said and Dickey (1984) and Berk (1974) the assumption

$$K \to \infty\,, \quad K^3/T \to 0\,. \tag{10}$$

In practice, the choice of K is not at all obvious and might require some experimentation. Large values of K will reduce the power of the test. In the following section some Monte Carlo evidence is provided that already $K = 1$ may be a good choice in finite samples. Further, the truncation of the infinite sum from (2) in the regression (8) requires that π_k die out sufficiently fast. To become precise, we maintain the following assumption also employed in Saikkonen (1991):

$$T^{0.5} \sum_{|k|>K} \|\pi_k\| \to 0\,. \tag{11}$$

The appendix contains the proof of the following result.

Theorem 2: *Let model (1), the null hypothesis of no cointegration, Assumption 1 and (11) hold true. If furthermore x_t from (2) is iid, then for $\widetilde{t}_\phi$ from (9) as $T \to \infty$ and under (10): $\widetilde{t}_\phi \xrightarrow{d} \mathcal{N}(0,1)$.*

Remark A: We suggest to compute $\widetilde{t}_\phi$ in three steps: Run regression (1), correct for residual effect and endogeneity by means of regression (8), and compute the test statistic $\widetilde{t}_\phi$ from the LM regression analogous to (4). Alternatively, one may consider a two step procedure where the correction for residual and endogeneity effects is performed in the LM regression. Hence, the second and last step could be

$$\widehat{\zeta}_t = \widetilde{\phi}\,\widehat{\zeta}^*_{t-1} + W_t'\widetilde{\theta} + \widetilde{e}_t\,,$$

from that the t statistic for $\phi = 0$ could be used to test for no cointegration. Experimentally, however, we found the performance of the proposed three step procedure superior in finite samples.

Remark B: The assumption in Theorem 2 that x_t is iid was made to facilitate the proof. If x_t follows a stable AR(p) process one may correct for that as proposed in Breitung and Hassler (2002), which requires a simple fourth step. We take the residuals from the second step (8) and perform, thirdly, an autoregression,

$$\widetilde{x}_t = \widehat{\rho}_1 \widetilde{x}_{t-1} + \cdots + \widehat{\rho}_p \widetilde{x}_{t-p} + \widehat{\varepsilon}_t\,,$$

and compute

$$\widehat{\varepsilon}^*_{t-1} = \sum_{j=1}^{t-p-1} \frac{\widehat{\varepsilon}_{t-j}}{j}\,.$$

The last step then consists of an augmented LM regression of $\widehat{\varepsilon}_t$ on $\widehat{\varepsilon}^*_{t-1}$ and $\widetilde{x}_{t-i}$:

$$\widehat{\varepsilon}_t = \phi\, \widehat{\varepsilon}^*_{t-1} + \sum_{i=1}^{p} \psi_i\, \widetilde{x}_{t-i} + e_t\,.$$

The t statistic for $\phi = 0$ from this fourth step may again be used to test for no cointegration with percentiles from the standard normal distribution.

5 Monte Carlo evidence

We first consider the situation of errors independent of regressors and investigate the residual-based LM test without any correction and with the projection on $(1-L)^d y_{2,t}$ correcting for the residual effect ($K = 0$). The special case studied is a bivariate OLS regression of $I(1)$ variables, i.e. $m = 1$, and $d = 1$. More precisely, the Monte Carlo model is given by

$$y_{1,t} = y_{2,t} + z_t, \quad t = 1, 2, \ldots, T, \tag{12}$$

$$(1-L)\, y_{2,t} = v_{2t}\,,\ v_{2,t} \sim iid\mathcal{N}(0,1), \tag{13}$$

$$(1-L)^{d-b} z_t = v_{1,t}\,,\ v_{1,t} \sim iid\mathcal{N}(0,1), \tag{14}$$

where $v_{1,t}$ is independent of $v_{2,s}$. We, first, compute OLS residuals,

$$\widehat{z}_t = y_{1,t} - \widehat{\alpha} - \widehat{\beta}\, y_{2,t}.$$

Then we run the LM regression corresponding to (4) with and without correcting for the residual effect. Finally, t statistics are computed as in (5) or (9). Table 1 reports the number of rejections at the 5% level based on 5000 trials. The column for $b = 0$ under the null hypothesis includes in brackets the rejection frequencies of the LM test applied naively to regression residuals without correction, where the second step of the projection on $\Delta y_{2,t}$ is omitted.

Table 1 supports that the correction introduced in the previous section is essential even with exogenous regressors. Without this correction moderate size distortions are observed. However, if the LM test is applied to the residuals after projecting on $\Delta y_{2,t}$, the size distortions are marginal. Moreover,

Table 1: 5% size and power under exogeneity ($K = 0$)

	$b = 0$	$b = 0.1$	$b = 0.2$	$b = 0.3$	$b = 0.4$
T=100	0.053 (0.111)	0.335	0.716	0.932	0.990
T=250	0.053 (0.096)	0.601	0.983	1.00	1.00

Given are the percentages of rejection at the nominal 5% level. The simulated model is (12) - (14). Under H_0 we include in brackets rejection frequencies of the LM test applied to residuals without correction. Further information is given in the text.

the experimental evidence reveals that the correct LM test is very powerful even for alternatives close to the null hypothesis.

Next, we study two extensions. First, the error is allowed to correlate with the regressor, and second, the error term as well as the regressor may have a short memory autoregressive structure in addition to be integrated. To become precise, we add to (14)

$$v_{1,t} = c\, v_{2,t} + \sqrt{1 - c^2}\, u_t\,,\ u_t = a_1\, u_{t-1} + \varepsilon_{1,t},\ \varepsilon_{1,t} \sim iid\mathcal{N}(0, 1)\,, \tag{15}$$

and to (13)

$$v_{2,t} = a_2\, v_{2,t-1} + \varepsilon_{2,t},\ \varepsilon_{2,t} \sim iid\mathcal{N}(0, 1)\,,$$

where $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are independent of each other. Note that for $c \neq 0$ and $a_2 \neq 0$, the innovations $v_{1,t}$ are correlated with all regressor innovations $v_{2,t-k}$, $k \geq 0$. Hence, in theory, K from (8) would have to grow with the sample size to infinity. In order to save power, we nevertheless fix $K = 1$. Due to the autoregressive serial correlation of $v_{1,t}$ and $v_{2,t}$, the LM test requires one further step to correct for autocorrelation. It was performed as described in Remark B above.

Table 2 reports the number of rejections at the 5% level based on 5000 trials for $c = 0.5$. Almost identical findings are observed (but not reported here) for $c = -0.5$, while the case of exogeneity ($c = 0$) results in slightly better size and power properties. The autoregressive parameters a_1 and a_2 are taken from $\{0.5, 0, -0.5\}$; a closer correspondence of the nominal size with the experimental one was observed for the set $\{0.7, 0, -0.7\}$.

Table 2: 5% size and power under endogeneity, $K = 1$

	$T = 100$			$T = 250$		
b	$a_1 = 0.5$	$a_1 = 0$	$a_1 = -0.5$	$a_1 = 0.5$	$a_1 = 0$	$a_1 = -0.5$
	$a_2 = 0.5$					
0	0.062	0.064	0.071	0.063	0.064	0.061
0.1	0.129	0.143	0.086	0.171	0.246	0.208
0.2	0.190	0.283	0.221	0.322	0.582	0.564
0.3	0.291	0.467	0.430	0.537	0.834	0.846
0.4	0.410	0.666	0.667	0.750	0.965	0.972
	$a_2 = 0$					
0	0.065	0.071	0.069	0.066	0.063	0.055
0.1	0.133	0.205	0.211	0.169	0.316	0.412
0.2	0.193	0.391	0.465	0.346	0.740	0.865
0.3	0.301	0.618	0.737	0.606	0.949	0.983
0.4	0.439	0.806	0.891	0.816	0.996	1.000
	$a_2 = -0.5$					
0	0.067	0.067	0.073	0.064	0.060	0.054
0.1	0.098	0.191	0.250	0.134	0.326	0.475
0.2	0.142	0.383	0.553	0.294	0.736	0.904
0.3	0.221	0.613	0.811	0.541	0.964	0.996
0.4	0.324	0.807	0.938	0.760	0.997	1.000

Given are the percentages of rejection at the nominal 5% level. The simulated model is (12), (13), (14), (15) and (16) with $c = 0.5$. Further information is given in the text.

From Table 2, a couple of conclusions can be drawn. First, the test is slightly oversized. Second, the case $a_2 = 0$, where the error does not correlate with *past* values of the regressor, is more powerful than the case $a_2 \neq 0$. Third, if the autocorrelation of the innovations is of opposite sign, $a_2 = -a_1$, then the power is poor for b close to zero, while for $b = 0.4$ highest rates of rejection are observed. In general, there is less power than reported in Table 1 for $K = 0$.

6 Summary

We propose a test against the alternative that nonstationary time series integrated of order d are fractionally cointegrated, i.e. a linear combination is integrated of order $d-b$ with $b > 0$. Neither d nor $d-b$ have to be integer. Under the null hypothesis of no cointegration $b = 0$.

In a first step, residuals from a static cointegration regression of the levels are computed. Second, the LM test of Breitung and Hassler (2002) in the tradition of Robinson (1991, 1994), Agiagloklou and Newbold (1994) and Tanaka (1999) is applied to these single equation residuals. However, while the limiting distribution of the LM test applied to observed series is standard normal, this is no longer true when it is applied to regression residuals without further modifications. Moreover, the distribution arising from regression residuals is shown to depend on d, and hence it is cumbersome to tabulate. Therefore, we propose a simple modification of the LM test applied to residuals. It accounts for the residual effect and corrects for eventual endogeneity of the cointegration regression at the same time. This modified residual-based LM test follows again an asymptotic standard normal distribution. Our procedure relies solely on regression techniques and is hence easy to use. Its validity in finite samples is established through Monte Carlo experiments.

Appendix

Proof of Theorem 1:

Under the assumptions we obtain

$$\widehat{\zeta}_t = v_{1,t} - v_{2,t}\,\widehat{\beta}\,, \quad \widehat{\zeta}^*_{t-1} = \sum_{j=1}^{t-1} \frac{v_{1,t-j} - v_{2,t-j}\widehat{\beta}}{j} = v^*_{1,t-1} - v^*_{2,t-1}\,\widehat{\beta}\,,$$

and

$$T^{-1} \sum \left(\widehat{\zeta}^*_{t-1}\right)^2 = T^{-1} \sum \left(\left(v^*_{1,t-1}\right)^2 + \left(v^*_{2,t-1}\,\widehat{\beta}\right)^2 - 2\,v^*_{1,t-1}\,v^*_{2,t-1}\,\widehat{\beta} \right)\,,$$

$$T^{-0.5} \sum \widehat{\zeta}_t\,\widehat{\zeta}^*_{t-1} = T^{-0.5} \sum \left(v_{1,t}v^*_{1,t-1} - v_{1,t}v^*_{2,t-1}\widehat{\beta} - v_{2,t}v^*_{1,t-1}\,\widehat{\beta} + v_{2,t}v^*_{2,t-1}\,\widehat{\beta}^2 \right)\,.$$

Given that $v_{1,t}$ and $v_{2,s}$ are iid and independent of each other, if follows with (7), see the proof of Theorem 1 in Breitung and Hassler (2002),

$$T^{-1} \sum \left(\widehat{\zeta}^*_{t-1}\right)^2 \xrightarrow{d} \sigma_1^2\,\frac{\pi^2}{6} + \sigma_2^2\,\frac{\pi^2}{6}\,\frac{\sigma_1^2}{\sigma_2^2}\,\overline{\beta}_d^2 + 0 = \sigma_1^2\,\frac{\pi^2}{6}\,(1 + \overline{\beta}_d^2)\,,$$

$$\begin{aligned} T^{-0.5} \sum \widehat{\zeta}_t\,\widehat{\zeta}^*_{t-1} \quad &\xrightarrow{d} \quad \mathcal{N}_1\left(0, \pi^2\sigma_1^4/6\right) - \mathcal{N}_2\left(0, \pi^2\sigma_1^2\sigma_2^2/6\right)\,\frac{\sigma_1}{\sigma_2}\,\overline{\beta}_d \\ &- \quad \mathcal{N}_3\left(0, \pi^2\sigma_1^2\sigma_2^2/6\right)\,\frac{\sigma_1}{\sigma_2}\,\overline{\beta}_d + \mathcal{N}_4\left(0, \pi^2\sigma_2^4/6\right)\,\frac{\sigma_1^2}{\sigma_2^2}\,\overline{\beta}_d^2\,. \end{aligned}$$

The serial independence and the independence of the processes implies

$$E\left(v_{1,t}v^*_{2,t-1}v_{2,t}v^*_{1,t-1}\right) = E\left(v_{1,t}v^*_{1,t-1}\right)\,E\left(v_{2,t}v^*_{2,t-1}\right) = 0\,.$$

Hence, the second and the third normal distributions are independent of each other. Similarly, pairwise independence of all Gaussian random variables can be established. Thus we arrive at

$$\frac{\sum \widehat{\zeta}_t\,\widehat{\zeta}^*_{t-1}}{\sqrt{\sum(\widehat{\zeta}^*_{t-1})^2}} \xrightarrow{d} \frac{\sigma_1\,\left(\mathcal{Z}_1 - \sqrt{2}\,\overline{\beta}_d\,\mathcal{Z}_2 + \overline{\beta}_d^2\,\mathcal{Z}_3\right)}{\sqrt{1 + \overline{\beta}_d^2}}\,.$$

It is straightforward to show that

$$T^{-1} \sum \widehat{\zeta}_t^2 \xrightarrow{p} \sigma_1^2(1 + \overline{\beta}_d^2)\,,$$

which completes the proof.

Proof of Theorem 2:

We first establish a technical Lemma A.1 that is used to proof Lemma A.2. Theorem 2 will be an immediate consequence of the latter lemma.

Define the matrix norm $\|A\|_M$ of a square matrix defined as Euclidean norm with a conformable vector b, see Berk (1974), Said and Dickey (1984) and Saikkonen (1991):

$$\|A\|_M = \sup \{\|A\, b\|\,,\ \|b\| \le 1\}\ .$$

Lemma A.1: *Under the assumptions of Theorem 2:*

$$\begin{array}{ll}
(i) & \|(T-2K)^{-0.5}\sum W_t x_t\| = O_p(K^{0.5})\,, \\
(ii) & \|\left((T-2K)^{-1}\sum W_t W_t'\right)^{-1}\|_M = O_p(1)\,, \\
(iii) & \|(T-2K)^{-0.5}\sum W_{t-1}^* x_t\| = O_p(K^{0.5})\,, \\
(iv) & \|(T-2K)^{-0.5}\sum W_t x_{t-1}^*\| = O_p(K^{0.5})\,, \\
(v) & \|(T-2K)^{-0.5}\sum W_{t-1}^* x_{t-1}^*\| = O_p(K^{0.5})\,, \\
(vi) & \|(T-2K)^{-1}\sum W_{t-1}^* {W_{t-1}^*}'\|_M = O_p(1)\,, \\
(vii) & \|(T-2K)^{-1}\sum W_{t-1}^* W_t'\|_M = O_p(1)\,.
\end{array}$$

Proof: (i) Consider

$$\begin{aligned}
E\left\|(T-2K)^{-0.5}\sum_{t=K+1}^{T-K} W_t x_t\right\|^2 &= E\left[\sum_{k=-K}^{K}\sum_{i=1}^{m}(T-2K)^{-1}\left(\sum_{t=K+1}^{T-K} v_{2,t-k}^{(i)} x_t\right)^2\right] \\
&= (T-2K)^{-1}\sum_{k=-K}^{K}\sum_{i=1}^{m} E\left(\sum_{t=K+1}^{T-K} v_{2,t-k}^{(i)} x_t\right)^2,
\end{aligned}$$

where $v_{2,t-k}^{(i)}$, $i = 1,\dots,m$, stands for the ith component of the m-dimensional vector $v_{2,t-k}$. Notice that

$$\begin{aligned}
E\left(\sum_{t=K+1}^{T-K} v_{2,t-k}^{(i)} x_t\right)^2 &= \sum_{t=K+1}^{T-K} E\left(v_{2,t-k}^{(i)} x_t\right)^2 + 2\sum_{t=K+1}^{T-K-1}\sum_{s=t+1}^{T-K} E\left(v_{2,t-k}^{(i)} x_t v_{2,s-k}^{(i)} x_s\right) \\
&= O(T-2K)
\end{aligned}$$

because

$$E\left(v_{2,t-k}^{(i)} x_t\right)^2 < \infty\,, \quad \left|\sum_{s=t+1}^{T-K} E\left(v_{2,t-k}^{(i)} x_t v_{2,s-k}^{(i)} x_s\right)\right| < \infty\,.$$

This proves

$$E\left\|(T-2K)^{-0.5}\sum_{t=K+1}^{T-K} W_t x_t\right\|^2 = O((2K+1)m)\,,$$

and hence the required result.

(ii) Define $\widehat{R} = (T-2K)^{-1}\sum_{t=K+1}^{T-K} W_t W_t'$ and $R = E(W_t W_t')$ with

$$\left\|\widehat{R}^{-1}\right\|_M = \left\|\widehat{R}^{-1} + R^{-1} - R^{-1}\right\|_M \leq \left\|R^{-1}\right\|_M + \left\|\widehat{R}^{-1} - R^{-1}\right\|_M\,.$$

With R^{-1} being the inverse of the autocovariance matrix of a stationary and invertible process, $\|R^{-1}\|_M$ is bounded, see Berk (1974, (2.14)). Moreover, we can adopt the proof of Berk (1974, Lemma 3) or Said and Dickey (1984, Theorem 4.1) and show that under (10)

$$\left\|\widehat{R}^{-1} - R^{-1}\right\|_M = o_p(K^{-0.5})\,.$$

This proves the result.

(iii), (iv) and (v) Similar to (i).

(vi) Define $\widehat{R} = (T-2K)^{-1}\sum_{t=K+1}^{T-K} W_{t-1}^* W_{t-1}^{*\prime}$ and $R = E(W_{t-1}^* W_{t-1}^{*\prime})$ with

$$\left\|\widehat{R}\right\|_M = \left\|\widehat{R} + R - R\right\|_M \leq \|R\|_M + \left\|\widehat{R} - R\right\|_M\,.$$

With R being the autocovariance matrix of a stationary and invertible process, $\|R\|_M$ is bounded, see Berk (1974, (2.14)). Moreover, we can adopt the proof of Said and Dickey (1984, Lemma 4.1) and show that under (10): $\|\widehat{R} - R\|_M = o_p(K^{-0.5})$. This proves the result.

(vii) Similar to (vi).

■

Lemma A.2: *Under the assumptions of Theorem 2 with $x_t \sim iid(0, \sigma^2)$:*

$$\begin{array}{ll} (i) & (T-2K)^{-1} \sum \widetilde{x}_t^2 = (T-2K)^{-1} \sum x_t^2 + o_p(1)\,, \\ (ii) & (T-2K)^{-0.5} \sum \widetilde{x}_t\, \widetilde{x}_{t-1}^* = (T-2K)^{-0.5} \sum x_t\, x_{t-1}^* + o_p(1)\,, \\ (iii) & (T-2K)^{-1} \sum \left(\widetilde{x}_{t-1}^*\right)^2 = (T-2K)^{-1} \sum \left(x_{t-1}^*\right)^2 + o_p(1)\,. \end{array}$$

PROOF: Define

$$\pi' = (\pi'_{-K}, \ldots, \pi'_{-1}, \pi'_0 + (\widehat{\beta} - \beta)', \pi'_1, \ldots, \pi'_K)\,,$$

$$\rho_t = \sum_{|k|>K} v'_{2,t-k} \pi_k\,,$$

where π_k are from (2). Hence, the residuals from the cointegration regression may be rewritten as

$$\widehat{\zeta}_t = x_t + W_t' \pi + \rho_t\,.$$

The residuals from the projection on W_t thus yield

$$\begin{aligned} \widetilde{x}_t &= x_t - W_t' \left(\sum W_t W_t'\right)^{-1} \sum W_t x_t \\ &+ \rho_t - W_t' \left(\sum W_t W_t'\right)^{-1} \sum W_t \rho_t\,, \end{aligned}$$

where according to Saikkonen (1991) under (11)

$$E\left(\rho_t^2\right) = o(T^{-1})\,. \tag{16}$$

(i) First, it should be noted that ρ_t does not affect the probability limit of $(T-2K)^{-1} \sum \widetilde{x}_t^2$. To illustrate this consider e.g.

$$\sum \rho_t W_t' \left(\sum W_t W_t'\right)^{-1} \sum W_t x_t\,.$$

With the Cauchy-Schwarz inequality

$$E \left\| \sum_{t=K+1}^{T-K} \rho_t W_t \right\| \leq \sum_{t=K+1}^{T-K} E|\rho_t|\, \|W_t\| \leq \sum_{t=K+1}^{T-K} \left(E(\rho_t^2)\, E\|W_t\|^2\right)^2\,.$$

Because of (16) and $E\|W_t\|^2 = O(m(2K+1))$ we arrive at

$$(T-2K)^{-1} E \left\| \sum_{t=K+1}^{T-K} \rho_t W_t \right\| = o\left((K/T)^{0.5}\right) .$$

Together with Lemma A.1, (i) and (ii), this results in

$$\begin{aligned}(T-2K)^{-1} \sum \rho_t W_t' \left(\sum W_t W_t'\right)^{-1} \sum W_t x_t &= o_p\left((K/T)^{0.5}\right) O_p\left((K/(T-2K))^{0.5}\right) \\ &= o_p(1) .\end{aligned}$$

With ρ_t being thus negligible on obtains

$$\sum \widetilde{x}_t^2 = \sum x_t^2 - \sum x_t \, W_t' \left(\sum W_t \, W_t'\right)^{-1} \sum W_t \, x_t + o_p(T-2K) .$$

With Lemma A.1, (i) and (ii), this reduces to

$$\begin{aligned}(T-2K)^{-1} \sum \widetilde{x}_t^2 &= (T-2K)^{-1} \sum x_t^2 - (T-2K)^{-1} O_p(K^{0.5}) \, O_p(1) \, O_p(K^{0.5}) + o_p(1) \\ &= (T-2K)^{-1} \sum x_t^2 - O_p\left(K/(T-2K)\right) + o_p(1) ,\end{aligned}$$

which is the required result because $K/(T-2K) \to 0$.

(ii) With

$$\begin{aligned}\widetilde{x}_{t-1}^* &= x_{t-1}^* - W_{t-1}^{*}{}' \left(\sum W_t W_t'\right)^{-1} \sum W_t x_t \\ &+ \rho_{t-1}^* - W_{t-1}^{*}{}' \left(\sum W_t W_t'\right)^{-1} \sum W_t \rho_t\end{aligned}$$

we obtain

$$\begin{aligned}\sum \widetilde{x}_t \, \widetilde{x}_{t-1}^* &= \sum x_t \, x_{t-1}^* - \sum x_t \, W_t' \left(\sum W_t \, W_t'\right)^{-1} \sum x_t \, W_{t-1}^* \\ &+ \sum x_t \, W_t' \left(\sum W_t \, W_t'\right)^{-1} \sum W_{t-1}^* \, W_t' \left(\sum W_t \, W_t'\right)^{-1} \sum x_t \, W_t \\ &- \sum x_t \, W_t' \left(\sum W_t \, W_t'\right)^{-1} \sum x_{t-1}^* \, W_t + o_p((T-2K)^{0.5}) ,\end{aligned}$$

and Lemma A.1 yields

$$(T-2K)^{-0.5} \sum \widetilde{x}_t \, \widetilde{x}_{t-1}^* = (T-2K)^{-0.5} \sum x_t \, x_{t-1}^* + O_p(K/(T-2K)^{0.5}) + o_p(1) ,$$

which proves the stated result because $K/(T-2K)^{0.5} \to 0$ by (10).

(iii) Similar to (ii).

■

Given Lemma A.2, Theorem 1 in Breitung and Hassler (2002) implies under (10)

$$
\begin{aligned}
(T-2K)^{-1} \sum \widetilde{x}_t^2 &\xrightarrow{p} \sigma^2 , \\
(T-2K)^{-0.5} \sum \widetilde{x}_t \, \widetilde{x}_{t-1}^* &\xrightarrow{d} \mathcal{N}(0, \sigma^4 \pi^2/6) , \\
(T-2K)^{-1} \sum \left(\widetilde{x}_{t-1}^*\right)^2 &\xrightarrow{p} \sigma^2 \pi^2/6,
\end{aligned}
$$

which in turn implies Theorem 2.

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