Existence of Mixed Strategy Equilibria in a Class of Discontinuous Games with Unbounded Strategy Sets.

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Abstract.

We prove the existence of mixed strategy equilibria for a class of discontinuous two-player games with non-compact strategy sets. We apply this result to obtain continuum of new mixed strategy equilibria in Bertrand game with two firms. Moreover, we construct continuum of mixed strategy equilibria in the first- and second-price auctions with toeholds.

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1. Introduction

The aim of this paper is to show the existence of Nash equilibria in mixed strategies for a class of two-player discontinuous games with complete information in which the strategy sets are non-compact. The problem of the existence of equilibria in discontinuous games has been already addressed by Dasgupta and Maskin (1986a, 1986b), Maskin (1986), Simon (1987), Simon and Zame (1990), and, more recently by Reny (1999). Unlike the existing papers, we construct Nash equilibria in mixed strategies whenever the players' strategy set coincides with the set of real numbers.

We provide a class of games which fit into our theoretical framework: Auctions with toeholds. Two bidders compete for an object. Each of them owns a (strictly positive) share of the object. Their valuations and their shares are common knowledge. Both bidders submit simultaneously sealed bids, the higher bidder gets the object and buys her competitor's share at the selling price. The relevant feature of this game is that each bidder is a buyer and a seller at the same time. Discontinuity comes from a tie breaking rule. If ties are broken through any random device such that a bidder gets the object with probability strictly less than one, players' best responses are not well defined. We suggest a way out of the resulting discontinuity by "opening" the players' strategy space. We let players submit any real number. However, the existence of mixed strategy Nash equilibria is not guaranteed by any fixed point theorem since the strategy set is not compact. We show that a continuum of mixed strategy equilibria do exist in our model.

An interesting co-product of our analysis is that we construct a new class of mixed strategy equilibria in the classical Bertrand game. This class of equilibria differ from the one proposed by Klemperer (2000). Moreover, the expected profit of both firms is strictly positive in all our mixed strategy equilibria.

The rest of the paper is organized as follows. The next section describes our assumptions and states the main existence results. We provide some examples in Section3. Section 4 concludes.

2. The Model

We consider two classes of games and prove existence of mixed equilibria in each of them.

Let $\Gamma_A = (\{i, j\}, \mathsf{R} \times R, (u_i, u_j))$ be a two-player game. Assume that if player i chooses strategy $x \in \mathsf{R}$ and player j plays strategy $y \in \mathsf{R}$, then payoff functions $u_k : \mathsf{R} \times \mathsf{R} \to \mathsf{R}$, $k = i, j$, are

$$(u_i(x,y), u_j(x,y))_A = \begin{cases} (v_i(y), w_j(y)), & \text{if } x > y \\ (w_i(x), v_j(x)), & \text{if } x < y \\ (\alpha v_i(x) + [1-\alpha] w_i(x), [1-\alpha] w_j(x) + \alpha v_j(x)) & \text{if } x = y \end{cases},$$

where $\alpha \in [0,1]$. Here α is a tie breaking rule. The model allows any tie breaking rule. We make the following assumptions about functions $v_k(t)$ and $w_k(t)$, $k = i, j$.

$$A1.\ w_k(t) \text{ is differentiable, } k = i, j.$$

$$A2.\ w'_k(t) \geq 0,\ k = i, j.$$

This assumption shows that if a player chooses a number (strategy) less than the opponent's number (strategy) then she would prefer to pick up a number as closed as possible to the opponent's strategy. This folds, for example, in the auctions with toeholds, when each player is a buyer and a seller at the same time.

$A3$. There exists $\underline{t} \in \mathsf{R}$, such that $w_k(t) - v_k(t) > 0$, for all $t \geq \underline{t}$ and $k = i, j$.

The explanation for this assumption is as follows. If one player chooses a very high number (a price in the Bertrand model or a bid in the auction with toeholds), then the other player will prefer to choose a smaller number.

$$A4.\ \int_{\underline{t}}^{+\infty} \frac{w'_k(t)\,dt}{w_k(t) - v_k(t)} = +\infty, \text{ for } k = i, j.$$

Assumptions A1-A4 guarantee our main result.

Theorem 1. Suppose that assumptions A1-A4 hold. Then there exists continuum of equilibria in mixed strategies in game Γ_A. Moreover, for any $t^* \geq \underline{t}$, the following probability distribution constitutes a mixed equilibrium:

$$F_j(t) = \begin{cases} 0, & \text{if } t < t^* \\ 1 - \exp\left[-\int_{t^*}^{t} \frac{w'_i(s)ds}{w_i(s) - v_i(s)}\right] & \text{if } t \geq t^* \end{cases}, \tag{1}$$

where $i \neq j$.

Proof. Notice first that distribution function $F_j(t)$ is a positive, strictly increasing function which satisfies $F_j(t^*) = 0$ and $F_j(+\infty) = 1$, because of assumption A4. We show now that the distribution functions from (1) constitute a mixed strategy equilibrium. Suppose that player j, $i \neq j$, uses the c.d.f. $F_j(t)$ above, then there are two cases.

Case 1. If player i chooses a strategy $x \in [t^*, +\infty)$, then i's expected payoff is:

$$E[u_i(x, F_j(y))] = \int_{t^*}^{x} v_i(s) f_j(s)\,ds + \int_{x}^{+\infty} w_i(x) f_j(s)\,ds, \tag{2}$$

where the first integral in the right-hand side is player i's expected payoff if her strategy x is greater than the opponent's strategy y, and the second integral is player i's expected payoff if her strategy x is smaller than the opponent's strategy y.

From the probability distribution function $F_j(t)$, from (1), it is immediate to get the density function $f_j(t)$:

$$f_j(t) = \begin{cases} 0, & \text{if } t < t^* \\ \frac{w_i'(t)}{w_i(t)-v_i(t)} \exp\left[-\int_{t^*}^{t} \frac{w_i'(s)ds}{w_i(s)-v_i(s)}\right], & \text{if } t \geq t^* \end{cases}.$$

The expected i's payoff (2) can be rewritten as

$$\begin{aligned} E\left[u_i\left(x, F_j(y)\right)\right] = & \int_{t^*}^{x} v_i(y) \frac{w_i'(y)}{w_i(y)-v_i(y)} \exp\left[-\int_{t^*}^{y} \frac{w_i'(z)\,dz}{w_i(z)-v_i(z)}\right] dy + \\ & + w_i(x) \int_{x}^{+\infty} \frac{w_i'(y)}{w_i(y)-v_i(y)} \exp\left[-\int_{t^*}^{y} \frac{w_i'(z)\,dz}{w_i(z)-v_i(z)}\right] dy. \end{aligned}$$

Using assumption A4, we get

$$E\left[u_i\left(x, F_j(y)\right)\right] = w_i(x)\left(\exp\left[-\int_{t^*}^{x} \frac{w_i'(z)\,dz}{w_i(z)-v_i(z)}\right]\right) +$$

$$+ \int_{t^*}^{x} \left[v_i(y) - w_i(y) + w_i(y)\right] \frac{w_i'(y)}{w_i(y)-v_i(y)} \exp\left[-\int_{t^*}^{y} \frac{w_i'(z)\,dz}{w_i(z)-v_i(z)}\right] dy.$$

Player i's expected payoff becomes:

$$E\left[u_i\left(x, F_j(y)\right)\right] = w_i(x)\left(\exp\left[-\int_{t^*}^{x} \frac{w_i'(z)\,dz}{w_i(z)-v_i(z)}\right]\right) -$$

$$-\int_{t^*}^{x} w_i'(y) \exp\left[-\int_{t^*}^{y} \frac{w_i'(z)\,dz}{w_i(z)-v_i(z)}\right] dy + \int_{t^*}^{x} w_i(y) \frac{w_i'(y)}{w_i(y)-v_i(y)} \exp\left[-\int_{t^*}^{y} \frac{w_i'(z)\,dz}{w_i(z)-v_i(z)}\right]$$

or

$$E\left[u_i\left(x, F_j(y)\right)\right] = w_i(x)\left(\exp\left[-\int_{t^*}^{x} \frac{w_i'(z)\,dz}{w_i(z)-v_i(z)}\right]\right) - w_i(x) \exp\left[-\int_{t^*}^{x} \frac{w_i'(z)\,dz}{w_i(z)-v_i(z)}\right] +$$

$$+w_i(t^*) - \int_{t^*}^{x} w_i(y) \frac{w_i'(y)\,dz}{w_i(y) - v_i(y)} \exp\left[-\int_{t^*}^{y} \frac{w_i'(z)\,dz}{w_i(z) - v_i(z)}\right] dy +$$
$$+\int_{t^*}^{x} w_i(y) \frac{w_i'(y)}{w_i(y) - v_i(y)} \exp\left[-\int_{t^*}^{y} \frac{w_i'(z)\,dz}{w_i(z) - v_i(z)}\right] dy,$$

and we finally obtain

$$E\left[u_i(x, F_j(y))\right] = w_i(t^*) \text{ for any } x \in [t^*, +\infty).$$

Case 2. If player i chooses a strategy $x \in (-\infty, t^*)$, then i's expected payoff is:

$$E\left[u_i(x, F_j(y))\right] = w_i(x), \tag{3}$$

because $x < y$ in this case. From assumption A2, it follows that $w_i(x) \leq w_i(t^*)$ for any $x \in (-\infty, t^*)$. Hence, any $x \in [t^*, +\infty)$ is a best reply to the probability distribution function $F_j(t)$, from (1).

The same argument is valid for player j. Hence the distribution functions from (1) characterizes a mixed equilibrium. End of proof.

We turn to the second possibility now. Consider the game $\Gamma_B = (\{i, j\}, \mathsf{R} \times \mathsf{R}, (u_i, u_j))$. Assume that if player i chooses strategy $x \in \mathsf{R}$ and player j plays strategy $y \in \mathsf{R}$, then payoff functions $u_k : \mathsf{R} \times \mathsf{R} \to \mathsf{R}$, $k = i, j$, are

$$(u_i(x,y), u_j(x,y))_B = \begin{cases} (v_i(x), w_j(x)), & \text{if } x > y \\ (w_i(y), v_j(y)), & \text{if } x < y \\ (\alpha v_i(x) + [1-\alpha] w_i(x), [1-\alpha] w_j(x) + \alpha v_j(x)) & \text{if } x = y \end{cases},$$

where $\alpha \in (0, 1)$. We make the following assumptions

$$B1.\ v_k(t) \text{ is differentiable, } k = i, j.$$

$$B2.\ v_k'(t) \leq 0,\ k = i, j.$$

This assumption describes property of the first price auction: given that you win, you would like to announce your bid as small as possible.

$B3$. There exists $\bar{t} \in \mathsf{R}$, such that $v_k(t) - w_k(t) > 0$, for all $t \leq \bar{t}$ and $k = i, j$.

This assumption describes the fact that if your opponent chooses a number (strategy) lower than some level, then it is always better to play slightly higher than her strategy, than to play lower than her number. This situation takes place in auctions.

$$B4.\ \int_{-\infty}^{\bar{t}} \frac{v_k'(t)\,dt}{w_k(t) - v_k(t)} = +\infty, \text{ for } k = i, j.$$

Theorem 2. Suppose that assumptions B1-B4 hold. Then there exists continuum of equilibria in mixed strategies in game Γ_B. Moreover, for any $t^{**} \leq \bar{t}$, the following probability distribution constitutes a mixed equilibrium:

$$F_j(t) = \begin{cases} \exp\left[-\int_t^{t^{**}} \frac{v_i'(s)\,ds}{w_i(s) - v_i(s)}\right], & \text{if } t \leq t^{**} \\ 1, & \text{if } t \geq t^{**} \end{cases}, \tag{4}$$

where $i \neq j$.

Proof. Notice first that distribution function $F_j(t)$ is a positive, strictly increasing function which satisfies $F_j(-\infty) = 0$ and $F_j(t^{**}) = 1$, because of assumption B4. We show now that the distribution functions from (4) constitute a mixed equilibrium. Suppose that player j, $i \neq j$, uses the c.d.f. $F_j(t)$ above, then there are two cases.

Case 1. If player i chooses a strategy $x \in (-\infty, t^{**}]$, then i's expected payoff is:

$$E[u_i(x, F_j(y))] = \int_{-\infty}^{x} v_i(x)\, f_j(y)\, dy + \int_x^{t^{**}} w_i(y)\, f_j(y)\, dy. \tag{5}$$

From the probability distribution function $F_j(t)$, from (4), it is obvious to get the density function $f_j(t)$:

$$f_j(t) = \begin{cases} \frac{v_i'(t)}{w_i(t) - v_i(t)} \exp\left[-\int_t^{t^{**}} \frac{v_i'(s)\,ds}{w_i(s) - v_i(s)}\right], & \text{if } t \leq t^{**} \\ 0, & \text{if } t \geq t^{**} \end{cases}.$$

The expected i's payoff (5) can be rewritten as

$$E[u_i(x, F_j(y))] = v_i(x)\, F_j(x) +$$

$$+ \int_x^{t^{**}} [v_i(y) + w_i(y) - v_i(y)] \frac{v_i'(y)}{w_i(y) - v_i(y)} \exp\left[-\int_y^{t^{**}} \frac{v_i'(s)\,ds}{w_i(s) - v_i(s)}\right] dy.$$

It is equivalent to

$$E[u_i(x, F_j(y))] = v_i(x)\, F_j(x) + \int_x^{t^{**}} v_i'(y) \exp\left[-\int_y^{t^{**}} \frac{v_i'(s)\,ds}{w_i(s) - v_i(s)}\right] dy +$$

$$+ \int_x^{t^{**}} v_i(y) \frac{v_i'(y)}{w_i(y) - v_i(y)} \exp\left[-\int_y^{t^{**}} \frac{v_i'(s)\,ds}{w_i(s) - v_i(s)}\right] dy,$$

or

$$E\left[u_i\left(x,F_j\left(y\right)\right)\right]=v_i\left(x\right)\exp\left[-\int\limits_x^{t^{**}}\frac{v_i'\left(s\right)ds}{w_i\left(s\right)-v_i\left(s\right)}\right]-v_i\left(x\right)\exp\left[-\int\limits_x^{t^{**}}\frac{v_i'\left(s\right)ds}{w_i\left(s\right)-v_i\left(s\right)}\right]+$$

$$+v_i\left(t^{**}\right)-\int_x^{t^{**}}v_i\left(y\right)\frac{v_i'\left(y\right)}{w_i\left(y\right)-v_i\left(y\right)}\exp\left[-\int\limits_y^{t^{**}}\frac{v_i'\left(s\right)ds}{v_i\left(s\right)-w_i\left(s\right)}\right]dy+$$

$$+\int_x^{t^{**}}v_i\left(y\right)\frac{v_i'\left(y\right)}{w_i\left(y\right)-v_i\left(y\right)}\exp\left[-\int\limits_y^{t^{**}}\frac{v_i'\left(s\right)ds}{v_i\left(s\right)-w_i\left(s\right)}\right]dy,$$

and we obtain

$$E\left[u_i\left(x,F_j\left(y\right)\right)\right]=v_i\left(t^{**}\right)\ \text{ for any } x\in\left(-\infty,t^{**}\right].$$

Case 2. If player i chooses a strategy $x\in\left(t^{**},+\infty\right)$, then i's expected payoff is:

$$E\left[u_i\left(x,F_j\left(y\right)\right)\right]=v_i\left(x\right),$$

because $x>y$ in this case. From assumption B2, it follows that $v_i\left(x\right)\leq v_i\left(t^{**}\right)$ for any $x\in\left(t^{**},+\infty\right)$. Hence any $x\in\left(-\infty,t^{**}\right]$ is a best reply to the probability distribution function $F_j\left(t\right)$, from (4).

The same reasoning is valid for player j. Hence the distribution functions from (4) do really constitute a mixed equilibrium. End of proof.

3. Applications

There are many examples of games Γ_A and Γ_B, which fit assumptions A1-A4 and B1-B4 correspondingly. Let us look at some of them.

3.1. The Bertrand Model. The standard two-firm Bertrand model is game Γ_A, where

$$\left(u_i\left(x,y\right),u_j\left(x,y\right)\right)_A=\begin{cases}\left(0,y\right), & \text{if } x>y\\ \left(x,0\right), & \text{if } x<y\\ \left(0.5x,0.5x\right) & \text{if } x=y\end{cases},$$

or $v_i\left(t\right)=v_j\left(t\right)\equiv 0$, $w_i\left(t\right)=w_j\left(t\right)=t$, $\underline{t}=0$, and $\alpha=0.5$. It is easy to check that all assumptions A1-A4 are fulfilled. As a corollary of Theorem 1, we have

Proposition 1. There exists continuum of equilibria in mixed strategies in two-firm Bertrand model. Moreover, for any $\underline{p} > 0$, the following probability distribution constitutes a mixed equilibrium:

$$F(p) = \begin{cases} 0, & \text{if } p < \underline{p} \\ 1 - \frac{\underline{p}}{p} & \text{if } p \geq \underline{p} \end{cases}.$$

Note that the expected profit of each firm in the mixed equilibrium is $\underline{p} > 0$.

3.2. Auctions with toeholds. Another class of games that fits into the theoretical model analyzed in Section 2 is a two-bidder auctions with toeholds. Two risk neutral individuals are interested in acquiring an object. Bidder $i\,(j)$ has a valuation $v_i\,(v_j)$ and owns a share $\theta_i\,(\theta_j = 1 - \theta_i) > 0$ of the object. Bidders' values and shares are common knowledge. Bidders submit bids simultaneously. The higher bidder gets the object and pays either her bid in the first price auction, or the opponent's bid in the second price auction. If bids are equal, then the object is allocated to bidder i with probability $\alpha \in [0, 1]$ and to bidder j with probability $(1 - \alpha) \in [0, 1]$. Thus, bidder i's payoff is $v_i - (1 - \theta_i)p$, if he wins, and $\theta_i p$, if he loses, where p is the selling price. We consider below two possible mechanisms: first- and second-price auctions.

The Second-Price Auction. Suppose that the two bidders compete in a second-price auction. It is easy to see that the auction game with toeholds is exactly game Γ_A, where

$$(u_i(x, y), u_j(x, y))_A =$$
$$= \begin{cases} (v_i - (1-\theta_i)y, (1-\theta_i)y), & \text{if } x > y \\ (\theta_i x, v_j - \theta_i x), & \text{if } x < y \\ (\alpha\,[v_i - (1-\theta_i)x] + (1-\alpha)\,\theta_i x, \alpha(1-\theta_i)x + (1-\alpha)\,(v_j - \theta_i x)) & \text{if } x = y \end{cases},$$

or $v_i(t) = v_i - (1-\theta_i)t$, $v_j(t) = v_j - \theta_i t$, $w_i(t) = \theta_i t$, $w_j(t) = (1-\theta_i)t$, and $\underline{t} > \max\{v_i, v_j\}$. It is easy to check that all assumptions A1-A4 are fulfilled. As a corollary of Theorem 1, we have

Proposition 2. There exists a continuum of mixed strategy equilibria in the sealed bid second-price auction in which player i randomizes over bids in the interval $[\underline{b}, +\infty)$ according to the distribution function

$$F_i(b) = 1 - \left(\frac{\underline{b} - v_j}{b - v_j}\right)^{1-\theta_i}, \; i \neq j,$$

where $\underline{b}$ is any number greater than $\max\{v_i, v_j\}$.

The interesting feature of the mixed equilibria is that the expected payoff of each bidder is $\underline{b} > \max\{v_i, v_j\}$ and this payoff is independent from bidder's valuation!

The First-Price Auction. Suppose that the selling mechanism is the first-price auction. Then, the auction game with toeholds coincides with game Γ_B, where

$$(u_i(x,y), u_j(x,y))_B =$$
$$= \begin{cases} (v_i - (1-\theta_i)x, (1-\theta_i)x), & \text{if } x > y \\ (\theta_i y, v_j - \theta_i y), & \text{if } x < y \\ (\alpha [v_i - (1-\theta_i)x] + (1-\alpha)\theta_i x, \alpha(1-\theta_i)x + (1-\alpha)(v_j - \theta_i x)) & \text{if } x = y \end{cases},$$

or $v_i(t) = v_i - (1-\theta_i)t,\ v_j(t) = v_j - \theta_i t,\ w_i(t) = \theta_i t,\ w_j(t) = (1-\theta_i)t$, and $\bar{t} < \min\{v_i, v_j\}$. It is easy to check that all assumptions B1-B4 are fulfilled. As a corollary of Theorem 2, we have

Proposition 3. There exists a continuum of mixed strategy equilibria in the sealed bid first-price auction in which player i randomizes over bids in the interval $(-\infty, \bar{b}]$ according to the distribution function

$$F_i(b) = \left(\frac{v_j - \bar{b}}{v_j - b}\right)^{\theta_i},\ i \neq j,$$

where $\bar{b}$ is any number lower than $\min\{v_i, v_j\}$.

The expected payoff of bidder i in the mixed strategy equilibrium is $v_i - (1-\theta_i)\bar{b}$, where $\bar{b} < \min\{v_i, v_j\}$. Note that this payoff dependents on bidder i's valuation, which was not the case in the second price auction.

4. Conclusion

The main feature of the class of games studied in this paper is the presence of externalities between players. We have pointed out that the use of a random tie-breaking rule makes this game discontinuous. We have shown that, if the players' strategy space coincides with the set of real numbers, a continuum of Nash equilibria in mixed strategies do exist.

One might wonder what would happen if we modified the game in such a way to allow for a deterministic tie-breaking rule. In our toeholds example, since valuations are common knowledge, one could think of breaking a tie in favor of the bidder with the higher valuation for the object. This formulation has been analyzed by Ettinger (2001). The author shows that the first- and second-price auctions admit a unique equilibrium in undominated strategies. It is easy to prove that, if bidders can play weakly dominated strategies, the set of equilibrium outcomes both in first- and second-price auction will coincide with the interval between buyers' valuations. However, whenever a random tie-breaking rule is introduced, the set of equilibrium outcomes will be found outside the interval between bidders' valuations.

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