Bertrand Equilibria and Sharing Rules

Steffen Hoernig*
Universidade Nova de Lisboa
CEPR, London

May 2001

Abstract

This paper analyzes the role of sharing rules in the existence and form of Bertrand equilibria for general demand and cost functions. It derives characteristics of equilibria that either do not depend on sharing rules or only on very general properties. We show that sign-preserving sharing leads to existence and uniqueness of zero-profit equilibria under fairly mild assumptions, while all other rules, including the "classical rule" of equal sharing, may either lead to non-existence of pure and mixed equilibria, or existence of a continuum of positive-profit equilibria.

JEL: C72, D43, L13

Keywords: Bertrand oligopoly, Bertrand paradox, Sharing rule, Sign-preserving sharing

*shoernig@fe.unl.pt. Travessa Estêvão Pinto, 1099-032 Lisboa, Portugal. I would like to thank Fernando Branco for helpful discussions on the non-existence example. All errors are mine.

1 Introduction

The question of existence and multiplicity of pure and mixed equilibria in Bertrand pricing games has been widely researched. The results obtained depend on properties of the demand and production cost specifications, and on three "rules" embodied in the payoffs: the sharing rule, the rationing rule, and what can be termed the "supply rule".

The *sharing rule* determines how demand is distributed between the set of firms charging the lowest price. The sharing rules has a decisive role to play, because in a large number of cases (including all classical ones) candidate equilibria will involve more than one firm charging the lowest price. Common rules are "equal sharing", where each firm receives the same share of demand, and "random sharing", where total demand is attributed to one single firm with equal probability for all firms charging the lowest price. The latter sharing rule is used in the papers by Baye and Morgan (1997 , 1999), and referred to by Vives (1999), p. 199. The reason that it is of interest is that zero-profit pure equilibria exist, and are unique, under rather weak assumptions. Thus it translates the standard intuition that Bertrand equilibria involve zero profits, which for equal sharing only holds under constant returns to scale.

The *rationing rule* determines how much demand is left to firms not charging the lowest price when the low-price firms do not serve the whole market. Finally, the *supply rule* determines whether a firm must serve total demand at the price it quotes (output is "demand-determined"), or whether it can refuse to sell more than a certain quantity ("voluntary trading"), normally defined by the supply function or a capacity constraint.

Voluntary trading gives rise to Bertrand-Edgeworth competition, while this paper deals with the traditional case of Bertrand competition, where supply is demand-determined.[1] In particular, we assume that output is demand-determined, therefore the rationing rule is irrelevant. Instead we concentrate on the sharing rule. For simplicity, we only consider the symmetric case where all firms have the same production technology. Differences in production costs give rise to additional issues which are not central to the concerns of this paper.

[1] See Vives (1999), ch. 5.1 for an overview, and ch. 5.2 for Bertrand-Edgeworth competition.

2 Bertrand Oligopoly and Sharing Rules

Assume that there are n identical firms in an industry for a homogeneous good[2], posting prices $p_1, .., p_n \in \mathbb{R}_+$. Let $\hat{p}_i = \min_{j \neq i} p_j$, and $m = |\{i = 1..n \mid p_i \leq \hat{p}_i\}|$ the number of firms charging the lowest price. The demand function $D : \mathbb{R}_+ \to \mathbb{R}_+$ is non-increasing, and the cost function $C : \mathbb{R}_+ \to \mathbb{R}_+$ non-decreasing with $C(0) = 0$. A monopolist's profit is $\pi_1(p) = pD(p) - C(D(p))$, and $\pi_m(p)$ denotes the profits of each of the m firms charging the lowest price p. Thus firm i's payoff is

$$\tilde{\Pi}_i(p_1, ..., p_n) = \begin{cases} 0 & if \quad p_i > \hat{p}_i \\ \pi_m(p_i) & if \quad p_i = \hat{p}_i \\ \pi_1(p_i) & if \quad p_i < \hat{p}_i \end{cases}. \tag{1}$$

The sharing rule is implied by $\pi_m(p)$, as discussed below. *Equal sharing* amounts to $\pi_m(p) = pD(p)/m - C(D(p)/m)$, and is the sharing rule traditionally assumed. *Random sharing* means that one of the firms charging the lowest price is randomly selected with identical probability, and yields $\pi_m(p) = \pi_1(p)/m$.

It is easy to see that under constant returns to scale, $C(q) = cq$ with some $c \geq 0$, the payoffs under the two sharing rules coincide,

$$\pi_m(p) = pD(p)/m - cD(p)/m = \pi_1(p)/m \tag{2}$$

and that they differ in general otherwise. For increasing returns to scale and equal sharing, $\pi_m(p) < \pi_1(p)/m$, while for decreasing returns to scale even $\pi_m(p) > \pi_1(p)$ may be true.

The equivalence of payoffs under constant returns may explain why the random sharing rule has not received much attention. Nevertheless, under random sharing each firm's profits simply are a multiple of a monopolist's profits, independently of the demand and cost functions. Therefore it is not surprising that existence and uniqueness of symmetric pure equilibria are trivial, as shown below. Under equal sharing, as discussed below the situation is much more complicated, with non-existence, uniqueness or existence of continua of pure and mixed symmetric equilibria depending on production cost.

The most general definition of a sharing rule would be an arbitrary mapping $(D, C, p, m) \mapsto \pi_m(p)$, on suitably defined spaces of demand and cost functions. In fact, in the more general setting of "games with endogenous

[2] We assume that entry decisions and costs are sunk, thus do not consider two-stage models of entry and production.

sharing rules" Simon and Zame (1990) show that there is always a "sharing rule" such that a mixed equilibrium exists if the strategy space is compact and demand is continuous. This sharing rule is an *a priori* arbitrary selection from the payoff correspondence, in particular in the Bertrand context it may change with any assumptions about demand, cost, and the number of firms.

As a first attack on the issue, and to impose some regularity on the problem, we will be more restrictive and let a sharing rule be given by two mappings $\phi : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ and $\chi : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}_+$, continuous and non-decreasing in their second argument[3], with

$$\pi_m(p) = \phi\left[m, \chi\left[m, D(p)\right] p - C\left(\chi\left[m, D(p)\right]\right)\right].$$

The maps ϕ and χ are independent of D and C, and the double appearance of χ assures that quantity sold equals quantity produced for each firm. Equal sharing corresponds to $\phi(m, x) = x$ and $\chi(m, x) = x/m$, while random sharing is obtained with $\phi(m, x) = x/m$ and $\chi(m, x) = x$.

Apart from being continuous in their second argument, there are other natural properties that one may want to impose on (ϕ, χ). Let us first make the following definitions:

Definition 1 *A function $f : \mathbb{R} \to \mathbb{R}$ is*

1. *zero-preserving if $f(x) = 0$ if $x = 0$;*
2. *sign-preserving if $f(x) \gtreqless 0$ if and only if $x \gtreqless 0$ for all $x \in \mathbb{R}$.*

One may impose that $\chi(m, D(p))$ is zero whenever $D(p) = 0$, i.e. when there is no demand at the price p. This is nothing else but χ being zero-preserving in its second argument. Similarly, it makes sense to have $\pi_m(p) = 0$ whenever $D(p) = 0$. Together with χ being zero-preserving, this means that ϕ should be zero-preserving as well. A stricter condition would be that either χ or ϕ are sign-preserving functions, which is implied to being zero-preserving and strictly increasing.

A class of sharing rules of special interest is the following:

Definition 2 *A sharing rule (ϕ, χ) is* sign-preserving *(SP) if $\pi_m(p)$ has the same sign as $\pi_1(p)$, for all m and p, and all demand and cost functions.*

While random sharing is sign-preserving, equal sharing is not, as will be discussed below. In fact, sign-preserving sharing (SPS) rules take on a simple form (All proofs can be found in the appendix).

[3] One may also imagine sharing rules that depend on the total number of firms in the market, $\pi_{m,n}(p)$, but even for these our arguments continue to hold.

Proposition 3 *A sharing rule (ϕ, χ) is sign-preserving if and only if ϕ is sign-preserving in its second argument and $\chi(m, x) = x$ for all m and x, i.e. $\pi_m(p) = \phi(m, \pi_1(p))$.*

In the following we will argue that the family of SPS rules has the same properties concerning existence and uniqueness of (zero-profit) Bertrand equilibria as in the traditional case of equal sharing and constant returns to scale in production. In a sense made precise below it is the only family that has this property.

Let us now define some other useful properties of sharing rules:

Definition 4 *A sharing rule (ϕ, χ) is, given sets of demand functions $\mathcal{D}$ and cost functions $\mathcal{C}$,*

1. *decreasing if $\pi_m(p) < \pi_1(p)$ for all $(D, C, p, m) \in \mathcal{D} \times \mathcal{C} \times \mathbb{R}_+ \times \mathbb{N}$ with $m > 1$ and $\pi_1(p) > 0$;*
2. *non-decreasing if there is $(D, C, p, m) \in \mathcal{D} \times \mathcal{C} \times \mathbb{R}_+ \times \mathbb{N}$ such that $\pi_m(p) \geq \pi_1(p)$;*
3. *sum-decreasing if $m\pi_m(p) \leq \pi_1(p)$ for all $(D, C, p, m) \in \mathcal{D} \times \mathcal{C} \times \mathbb{R}_+ \times \mathbb{N}$.*
4. *quantity-decreasing if $\chi(m, D(p)) \leq D(p)$ for all $(D, p, m) \in \mathcal{D} \times \mathbb{R}_+ \times \mathbb{N}$.*

Any sum-decreasing sharing is decreasing. An SPS rule is decreasing if ϕ is decreasing in m, etc. Random sharing is sum-decreasing, as are other simple sharing rules such as the SPS rule $\pi_m(p) = \lambda\pi_1(p)/m$ with $\lambda \in (0, 1)$ and $m \geq 2$. Equal sharing is sum-decreasing for constant returns, and non-decreasing in general.

The property of being quantity-decreasing is a natural one: No firm should be attributed an output higher than market demand.[4] Still, it will only be used in section 5.1 to show non-existence of equilibria under certain sharing rules.

3 Some Properties of Bertrand Equilibria

In this section we will derive some characteristics of Bertrand equilibria, and how these are related to assumptions about the sharing rule, the demand and

[4] This requirement can obviously be strengthened.

cost functions. Starting from the assumption that equilibria exist, there is surprisingly much that one say, even though we do not presuppose a certain sharing rule, nor continuity.

First we would like to raise a rather technical issue, about the difference between the "set of prices played in (mixed) equilibrium" $\mathcal{P}$, and the "support of the equilibrium (mixed) strategy" S. With continuous strategy spaces, $\mathcal{P}$ may essentially be a strict subset of S, i.e. not all prices in the equilibrium price support are equilibrium prices. This fact complicates some of our arguments below, but for brevity we relegate the discussion of this issue to appendix 7.1.

Let let $J_p = \{j|\Pi_j > 0\}$ and $J_z = \{j|\Pi_j = 0\}$ be the sets of firms with positive and zero expected equilibrium profits, respectively (There can be none with negative profits, given that $C(0) = 0$.), and $J = J_p \cup J_z$ be the set of all firms.

The following two examples show that there can be Bertrand equilibria where some firms make positive profits in equilibrium while others make zero profits. They also show that firms in J_z may play lower prices even though they make zero profits (example 1), while firms $j \in J_p$ may play one or more prices in equilibrium (example 2).

Example 1 (see Figure 1). Under ES with two firms, demand is $D(p) = \max\{0, 1-p\}$, cost $c(0) = 0$, $c(q) = F_1 \in [1/9, 2/9)$ for $q \in (0, 1/3]$, and $c(q) = F_2 = 1/4$ for $q > 1/3$. An equilibrium is firm 1 playing $p_1 = 2/3$ and firm two randomizing between $p_2 = 1/2$, played with probability $P \in (0,1)$, and other prices larger than 2/3. Expected payoffs are $\Pi_1 = (2/9 - F_1) P > 0$, and $\Pi_2 = 0$. Note that π_1 has a left-discontinuity at the equilibrium price 1/3, $\pi_2(1/3) \leq 0$, and that lower prices create losses.

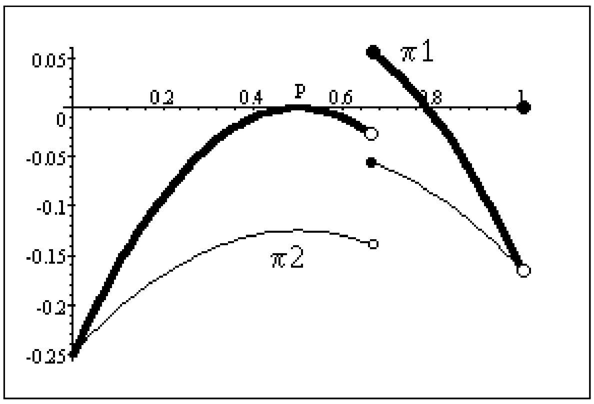

Figure 1: Only one firm makes zero profits.

Example 2. ES with three firms, demand $D(p) = \max\{0, 1-p\}$, cost: $c(0) = 0$, $c(q) = q^2/2 + 0.047$ (u-shape average cost). Let $p_1 = 0.31$ and

$p_2 = 0.32$, then for the profits before fixed cost π^g it is true that $\pi_1^g (p_i) < 0 < \pi_3^g (p_i) < 0.047 < \pi_2^g (p_i)$, $i = 1, 2$. Let firms 1 and 2 play the two-firm mixed equilibrium on p_1 and p_2 (as described in section 4), and firm 3 some $p > p_2$. This is clearly an equilibrium with positive profits for firms 1 and 2, while firm 3 cannot make positive profits playing p_1, p_2 or any other price below p_2, and has to make do with zero profits. It is important to note that this result is not driven by the discontinuity in costs (C can be made continuous without changing the result), but by the presence of increasing returns followed by decreasing returns to scale.

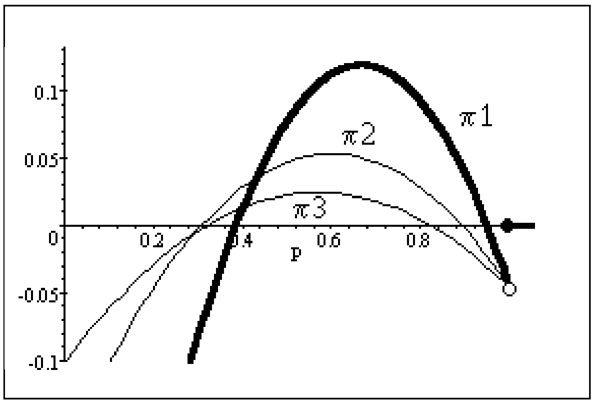

Figure 2: A mixed equilibrium where only one firm makes zero profits.

The following proposition lists some of the properties of Bertrand equilibria where some firms have positive profits. Let "maximum profits" be defined as $\pi^m = \sup_{k,p} \pi_k (p)$. In the classic Bertrand model with ES and constant marginal costs these are equal to monopoly profits $\sup_p \pi_1 (p)$.

Proposition 5 *Assume there exists a mixed or pure Bertrand equilibrium in a market for a homogeneous good with $n \geq 2$ identical firms under a given sharing rule, and at least one firm has positive expected profits. Then the following holds:*

1. *If maximum profits π^m are finite, then all firms in J_p play the same finite maximum price $t < \infty$. For all firms in J_p this price is an atom, or possibly a limit point of atoms if $|J_p| = 1$, while all firms in J_z play prices $p \geq t$ with positive probability.*

2. *If $J_z \neq \emptyset$, the sets $\mathcal{P}_j$ of all firms $j \in J_p$ are countable.*[5] *If for $j \in J_p$, $r \in \mathcal{P}_j$ is not an atom, then it is a limit point of atoms and if additionally $r < t$ then it must be an atom for all firms in J_z. Any price $p < t$ with $\pi_1 (p) > 0$ must be an atom.*

[5] As of yet, it is an open question whether the same holds when $J_z = \emptyset$.

Some as of yet unresolved questions are:

1. With $\pi^m < \infty$ and $J_z = \emptyset$, are there equilibria where firms play prices that are not (limit points of) atoms? (look at Kaplan and Wettstein 2000)

2. Do firms in J_z play atoms where firms in J_p play atoms?

3. Can we say something about limit points of atoms, in particular t for $|J_p| = 1$?

The following proposition lists some properties of the payoff functions and the sharing that must be satisfied in non-zero profit equilibria. All the proofs are in the appendix.

Proposition 6 *Assume that $\pi^m < \infty$ and there exists a Bertrand equilibrium where at least one firm makes positive profits. Let $t = \max \mathcal{P}_j$ for $j \in J_p$ be the joint highest price played by positive-profit firms.*

1. *If $M = |J_p| \geq 2$ there is some $m = M..n$ with $\pi_m(t) > 0$ and $\pi_m(t) \geq \pi_1(t)$, or $\pi_1(t) > 0$ and has a left-discontinuous upward jump at t.*

2. *If $J_z \neq \emptyset$ then there are at most countably many prices $p < t$ with $\pi_1(p) > 0$. For all prices $p \in \mathcal{P}_j$, $j \in J_p$, it is true that there is some $m = 2..n$ with $\pi_m(p) > 0$ and $\pi_m(p) \geq \pi_1(p)$, or $\pi_1(p) > 0$ and π_1 has a left-discontinuous upward jump at p with $\liminf_{p' \nearrow p} \pi_1(p') \leq 0$.*

Statement 1 of this proposition means that it is necessary for positive-profit equilibria that either the sharing rule is non-decreasing or that π_1 has a specific kind of discontinuity at the maximum price that positive-profit firms play. Statement 2, which includes the more difficult case of a single firm with positive profits, extends this statement to *all* prices played in equilibrium. Moreover, and this can be a very strong condition, it follows that there can be no continuum of positive single firm profits below the highest price positive-profit firms play in equilibrium if there are some firms that make zero profits in equilibrium.

From proposition 6 follows directly:

Corollary 7 *If 1. maximum profits are finite, 2. the sharing rule is decreasing, and 3. profits π_1 do not have left-discontinuous upward jumps at points p where $\pi_1(p) > 0$, then any pure or mixed Bertrand equilibrium must involve zero profits for every firm.*

This corollary is a vast generalization of the necessity part of Kaplan and Wettstein (2000, p. 69) to arbitrary sharing rules, demand and cost functions[6]: They show that under equal sharing, constant returns to scale and

[6] Baye and Morgan (1999) do only show the existence of positive profit equilibria under RS and infinite monopoly profits but do not demonstrate necessity.

continuous demand, positive profit equilibria exist if and only if $\lim_{p\to\infty} D(p)\,p = \infty$. Their case is contained in our proposition since with constant returns to scale equal sharing is decreasing, and π_1 is continuous, and shows that the condition on maximum profits is necessary.

The second condition, on the sharing rule, given a set of demand and cost functions, is also necessary. This is made clear by the result of Dastidar (1995) that with ES and strictly increasing returns to scale, and some continuity and differentiability assumptions, positive profit equilibria exist. In this case the sharing is non-decreasing.

Lastly, the following example shows that also the third condition is necessary:

Example 3: Under RS and two firms, let $D(p) = \max\{0, 1-p\}$, $c(q) = 0$ if $q \leq 2/3$, and $c(q) = F > 2/9$ if $q > 2/3$. Then $p = 1/3$, the point of discontinuity, for both firms is an equilibrium yielding expected profits of $1/9$ to each firm.

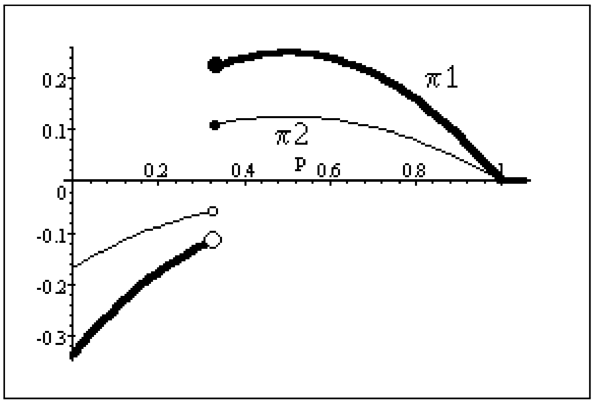

Figure 3: Positive profits because of discontinuity.

Sufficient conditions for the third condition to hold are that π_1 is left-continuous, that π_1 is lower semi-continuous (as discussed in Baye and Morgan 1997), and of course that π_1 is continuous.

Finally, it must also be stressed that our assumption of $C(0) = 0$ for all cost functions is not at all innocuous. "Zero profits" as such are only a benchmark if they correspond to the profits that make a firm indifferent between participating in the equilibrium or pricing itself out of the market. In other words, if a firm has fixed costs even if it produces nothing, then these fixed costs can be neglected for the determination of the (post-entry) equilibrium while one should remember than "zero profits" then refers to gross profits. Any fixed cost implied by C are production-related, not entry-related.[7]

[7] Sharkey and Sibley (1993) assume that fixed costs are avoidable, but occur in an entry

After this treatment of positive profit Bertrand equilibria we will now consider zero-profit equilibria. It is rather difficult to say anything about the form of the equilibrium strategies, on the contrary to the case with positive profits. For example, the argument in the proof of the previous proposition involving the highest price played in equilibrium does not work since firms with zero profits can play arbitrarily high prices without changing their payoffs: Nothing can be deduced from a maximum price (if one exists at all). We make the following definitions concerning π_1:

Definition 8 *An "initial viable price" (IVP) is a price s such that $\pi_1(s) \geq 0$ and $\pi_1(p) \leq 0$ for all $p < s$. If in addition $\pi_1(s) = 0$ then s is an "initial break-even price" (IBP).*[8]

Contrary to Theorem 1 in Baye and Morgan (1997), which is based on the implicit assumption of RS, in general an IBP (or IVP) may *not* give rise to a symmetric pure equilibrium, because $\pi_1(s) \geq 0$ does not imply $\pi_n(s) \geq 0$ (for RS it does). Together with the following proposition this means that an IBP in general is neither necessary nor sufficient for the existence of a zero-profit ("Bertrand paradox") equilibrium.

The following proposition lists some properties that payoffs must satisfy in any zero-profit Bertrand equilibrium. Given a mixed equilibrium, let $s = \min_j \min S_j$, which again exists because every S_j is closed and bounded from below. Without loss of generality order firms by the values of $\sup S_j$, $j = 1..n$, such that $t_1 \leq t_2 \leq ... \leq t_n$.

Proposition 9 *Assume there exists a mixed or pure zero-profit Bertrand equilibrium in a market for a homogeneous good with n identical firms under a given sharing rule. Then*

1. *Single firm profits are non-positive, $\pi_1(p) \leq 0$, for all $p < s$, and for almost all $p \leq t_2$. If $\pi_1(p) > 0$ for some $p \in [s, t_2]$, then it must be an atom for at least one firm, or for at least two firms if $p < t_1$, and there must be an $m = 2..n$ such that $\pi_m(p) < 0$.*

2. *For any price $p \in \mathcal{P}_j$, for some $j \in J$, and with $p \leq t_2$, it is true that either $\pi_1(p) = 0$, or there is an $m = 2..n$ such that $\pi_m(p) \geq 0 > \pi_1(p)$, or $\pi_1(p) > 0$ and π_1 is left-discontinuous at p.*

stage, while production decisions are made in a subsequent stage. Incidentally, verbally they assume ES but base their mathematics on RS (see their equation 3.1) since they do not assume constant marginal cost.

[8] The latter notion was introduced in Baye and Morgan (1997).

Comment: 1. The first statement of the proposition is very strong: It means that the supports of the two lowest-price firms (firms 1 and 2) can at most contain a countable number of prices p where single-firm profits π_1 are positive. This result establishes a tight upper limit on these supports, see e.g. the example in the following section.

2. An IVP or IBP may not be part of the equilibrium supports, since even if one knows that for a price p with $s < p < t_2$ it is true that $\pi_1(p) \geq 0$, then one cannot exclude the existence of prices $p' \in (s, p)$ with $\pi_1(p') > 0$ even if there can only be countably many such prices. It is also not necessary that $s \in \mathcal{P}_j$ for any j. On the other hand, if there is such a j, then s is an IVP if the sharing rule is decreasing; if furthermore π_1 is left-continuous then s is an IBP, leading to a generalization of Baye and Morgan's (1997) result.

3. Dastidar (1995) has shown that under ES and strictly decreasing returns to scale there may be (among many others) a pure symmetric zero-profit equilibrium with $\pi_n(s) = 0$ and $\pi_1(s) < 0$, see our section 4. In this example $\mathcal{P}_j = \{s\}$ for all firms, and the sharing rule is non-decreasing.

In the following section we will apply these propositions to show the non-existence of pure and mixed Bertrand equilibria in a particular example.

4 Equal Sharing

The classical treatments of Bertrand oligopoly, which have handed down to us the cherished intuitions of "zero profits" and "price equals marginal cost" were based on a specific cost function, that is, constant returns, and a specific sharing rule, equal sharing. While the former has been the subject of some investigation, the latter went almost unnoticed. In this section we will give an account of the most important results on equilibria under ES. For simplicity we will assume that demand and costs are continuous unless explicitly stated otherwise.

Under constant returns to scale in production, pricing at marginal cost constitutes a symmetric pure equilibrium. For two firms, Harrington (1989) has shown that this is the unique equilibrium when demand is bounded, continuous and has a finite choke-off price. With more than two firms, there is a continuum of zero-profit equilibria where at least two firms always choose price equal to marginal cost, and the other firms randomize arbitrarily between prices not lower than marginal cost. This multiplicity of equilibria is "inessential" in the sense that equilibrium production, profits and consumer welfare are the same for all these equilibria.

On the other hand, Kaplan and Wettstein (2000) show that infinite monopoly profits and price are not only sufficient but also necessary in duopoly for (mixed) equilibria other than this standard Bertrand equilibrium to obtain. In these equilibria expected profits are positive and equal to $\pi_1(s)$, where s is the lowest price played. Price supports are $[s, \infty)$, and equilibrium is based on the trade-off between ever higher profits and ever lower probability of achieving them as prices go to infinity.[9]

Let us now consider strictly decreasing returns. Dastidar (1995) has shown, subject to continuity and differentiability of demand and cost functions, that in symmetric oligopoly there is a continuum of pure symmetric equilibria. These contain the "competitive" equilibrium where price equals marginal costs but firms make positive profits, and exactly one equilibrium where all firms make zero profits. These equilibria exist on an interval $s \in \left[\underline{p}_n, \bar{p}_n\right]$ where $\pi_n(s) \geq \pi_1(s) > \pi_1(p)$ for all $p < s$, thus underbidding is not profitable. It is essential to notice that this interval exists precisely because equal sharing under strictly decreasing returns is non-decreasing.

We show now that it is straightforward to even construct a continuum of *mixed* equilibria from the above pure equilibria[10]. This means that for strictly decreasing returns to scale there is a true plethora of equilibria that are truly different from each other. For simplicity we analyze a symmetric duopoly where both firms randomize between two prices. Let $p_1 < p_2$ with $p_i \in \left[\underline{p}_2, \bar{p}_2\right]$, then $\pi_2(p_2) > \pi_2(p_1) \geq \max\{0, \pi_1(p_1)\}$. The equilibrium probability of playing the lower price $P = P(p_1)$ is defined through

$$\Pi = P\pi_2(p_1) + (1-P)\pi_1(p_1) = (1-P)\pi_2(p_2),$$

or

$$\begin{aligned} P &= \frac{\pi_2(p_2) - \pi_1(p_1)}{\pi_2(p_2) - \pi_1(p_1) + \pi_2(p_1)} \in [0,1], \\ \Pi &= \frac{\pi_2(p_1)\pi_2(p_2)}{\pi_2(p_2) - \pi_1(p_1) + \pi_2(p_1)} \geq 0 \end{aligned}$$

Note also that if firm 2 plays p_1 or p_2, firm 1 will not choose any price different from p_1 and p_2. Since π_1 is increasing on $\left[\underline{p}_2, \bar{p}_2\right]$, any price below p_1 leads to payoffs $\pi_1(p) < \Pi$, $p \in (p_1, p_2)$ leads to profits $(1-P)\pi_1(p) < \Pi$, and $p > p_2$ to zero profits. Expected equilibrium profits are zero if and only if $p_1 = \underline{p}_2$, otherwise they are positive.

[9] Baye and Morgan, in the context of random sharing, have also demonstrated the existence of these equilibria, but not shown necessity of infinite monopoly profits.

[10] See Hoernig (2001) for the corresponding existence result involving arbitrary finite numbers of firms and prices.

After showing that under decreasing returns to scale there may be "too" many equilibria, we now show that under increasing returns there may be none, not even a mixed one. Consider the following example, also discussed in Vives (1999), p. 118, in a similar form.

Example 4: Let there be n firms, ES, and $D(p) = \max\{0, 1-p\}$, $C(0) = 0$ and $C(q) = F > 0$ for $q > 0$. Let monopoly be strictly viable, $F \in (0, 1/4)$. We have $\pi_1(p) = (1-p)p - F > \pi_m(p) = (1-p)p/m - F$ whenever $\pi_1(p) > 0$ and for any $m = 2..n$, thus ES is decreasing. There is a left-discontinuous upward jump at $p = 1$ but with $\pi_1(1) = 0$. Since maximum profits are finite, by corollary 7 there are no pure or mixed equilibria involving positive profits for any firms.

Assume now there is a zero-profit equilibrium. The highest price p^* such that there are only countably many $p \leq p^*$ with $\pi_1(p) > 0$ is $p^* = \left(1 - \sqrt{1-4F}\right)/2$, and by proposition 9, $t_2 \leq p^*$, where t_2 is the second-lowest supremum of the equilibrium supports S_j. In fact, $\pi_1(p^*) = 0$ and $\pi_m(p) < 0$ for all $p < p^*$ and all $m = 1..n$. Thus no firm wants to play a lowest price in this range, i.e. the only possibility is that at least firms 1 and 2 play p^* with probability 1, and all other firms do not play prices below p^*. This leads to negative profits for firms 1 and 2 since $\pi_m(p^*) < 0$ for all $m = 2..n$, a contradiction. Thus we have shown that there do not exist pure or mixed equilibria, neither with positive nor with zero profits.

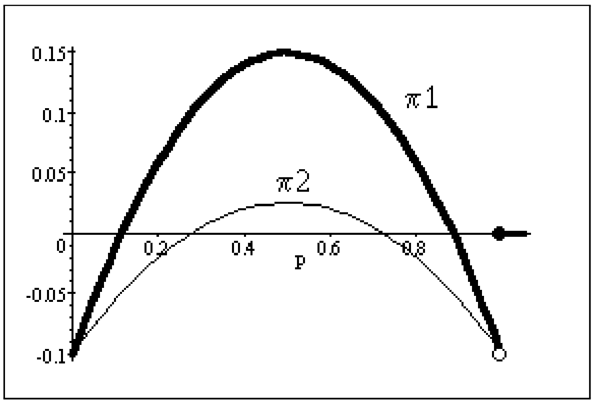

Figure 4: No pure or mixed equilibria.

Incidentally, in this example the sum of payoffs is not upper semi-continuous, violating one of the assumptions of the Dasgupta-Maskin (1986) existence theorem, while it fulfills this assumption under random sharing. Vives (1999, p.118) shows that indeed an equilibrium exists under RS, a pure symmetric equilibrium where both firms play p^* and make zero expected profits.

It remains to note that of course this list is not exhaustive of all possibilities under ES, since in general cost functions may exhibit increasing or decreasing returns over some regions. Our examples 1 and 2 are of this type.

5 Sign-preserving Sharing

In this section we will highlight some of the special properties that sign-preserving sharing has, in particular in relation to existence and uniqueness of equilibrium outcomes. We will then continue to show that other sharing rules are either plagued by non-existence or multiplicity of equilibria.

First we note that equilibria under SPS are much simpler than in the general case:

Proposition 10 *Under SPS, in any equilibrium firms either all make positive profits, or all make zero profits. Firms with positive profits only play prices p with $\pi_1(p) > 0$, and all prices $p < t_1$ played by zero-profit firms involve $\pi_1(p) = 0$.*

Comment: 1. The last proposition means in particular that we do not have to worry about atoms, since under SPS atoms cannot cancel out payoffs of opposite sign.

The following proposition is a generalization of the Baye and Morgan (1997) result on the relation between IBP's and zero-profit equilibria. Its derivation shows that it depends *decisively* on the SPS assumption.

Proposition 11 *Under SPS any IVP gives rise to a pure symmetric equilibrium. There is a zero-profit equilibrium if and only if there is an IBP.*

Comment: 1. Note the we had to invoke SPS for *both* the necessity and the sufficiency part of the last proposition.

We will now investigate when equilibria exist, and given the last proposition it is useful to concentrate on IVP's and IBP's.

Lemma 12 *Assume that monopoly is viable, i.e. there is a price $\bar{p}$ such that $\pi_1(\bar{p}) \geq 0$. Under SPS, a (smallest) IVP exists if π_1 upper semi-continuous from the right on $[0, \bar{p}]$.*

Comment: 1. A simple example of non-existence of equilibrium stemming from the failure of upper semi-continuity is the following: With RS, $D(p) = 1 - p$, $C(q) = cq$ for $0 \leq q < 1 - c$, and $C(q) = 2cq$ for $q \geq 1 - c$ (let $c \in (0,1)$). Profits π_1 are right-discontinuous at $p = c$, and by corollary 7 and proposition 9 neither pure nor mixed equilibria exist.

2. Sufficient conditions for π_1 being usc from the right are π_1 being usc, or right-continuous, or continuous. These are conditions on the profit function, but it is useful to obtain some conditions on the fundamentals:

Lemma 13 *If demand D and costs C are lower semi-continuous then π_1 is right-continuous.*

Comment: 1. It may a be a little bit surprising that the condition on D is lower- and not upper semi-continuity. In most games choosing optimal strategies involves maximization, and here upper semi-continuity of payoffs is indispensable. The reason that upper semi-continuity of demand is not relevant for the existence of equilibria in our context is that discontinuities in demand cause downward jumps in revenue which pose no problem when transmitted to the payoffs. Non-existence can only be caused by magnification of the jumps in demand through the cost function into upward jumps in profits. Right-continuity at these upward jumps then follows from D being lsc, not usc.

2. In fact, since demand D is non-increasing, left- (right-) continuity of D are equivalent to upper (lower) semi-continuity. Similarly, since costs C are non-decreasing, left- (right-) continuity of C are equivalent to lower (upper) semi-continuity. Therefore we can use these terms interchangeably in the statements of our propositions. The proofs are elementary and similar to the ones in the previous lemmas.

Let us now turn to IBP's and zero-profit equilibria.

Lemma 14 *If demand is bounded and π_1 lower semi-continuous from the left then any IVP p_0 is an IBP. For the latter it is sufficient that D and C are upper semi-continuous.*

Comments: 1. The assumption of bounded demand is necessary: With $D(p) = 1/p$ and $C(q) = 0$ one can define a continuous profit function $\pi_1(p) = 1$ on $\mathbb{R}_+$, with IVP $p_0 = 0$ and positive equilibrium profits equal to $1/n$.

2. The assumption of left-continuity is also essential: In Example 3 $p = 1/3$ is an IVP but not an IBP. In fact, no IBP exists.

Finally, we tackle the question of uniqueness of equilibrium, which can be a thorny question in general. There are two aspects to this question. The first aspect is the uniqueness of the "zero-profit outcome", as discussed in Baye and Morgan (1997), while the second one is the uniqueness of the equilibrium price as such. As concerns the first aspect, corollary 7 shows that if π_1 has no right-continuous upward jumps where it is positive, and if the sharing rule is decreasing then zero profits is indeed the unique equilibrium outcome. SPS being decreasing means simply that $\phi(m, x) < x$ for all x and $m \geq 2$, but apart from this SPS does not add anything fundamentally new to this aspect.

It is not possible to establish uniqueness of equilibrium prices without imposing continuity conditions, but uniqueness of zero-profit equilibria is straightforward, even if at the price of a rather strict condition:

Lemma 15 *Under SPS, if π_1 is strictly quasi-concave there is at most one symmetric zero-profit Bertrand equilibrium.*

Comments: 1. As in the case of ES and constant marginal cost, with more than two firms zero-profit equilibria are generically non-unique. All firms but two can play any prices above t_2, and even randomize between them.

We summarize the preceding discussion in the following theorem:

Theorem 16 *Assume monopoly is viable. Under SPS, if demand and cost are lower semi-continuous then (pure symmetric) equilibria exist. If demand and cost are continuous, then zero-profit Bertrand equilibria exist .If in addition single-firm profits π_1 are also strictly quasiconcave there is at most one zero-profit equilibrium, or if $\phi(m, x) < x$ for all $x \in \mathbb{R}$ and $m \geq 2$ then there are no positive-profit equilibria.*

Therefore under SPS the classical "Bertrand paradox" of zero profits is restored, with the new interpretation that "price is equal to average cost". The "competitive outcome" of price equal to marginal cost is the exception rather than the rule, and generically occurs only under constant returns to scale. It is also worth noting that uniqueness only applies to symmetric equilibria, and when there are more than two firms then there exists a continuum of pure (and even mixed) equilibria.

5.1 Comparison to other sharing rules

In the previous section we have seen that under the SPS rule Bertrand equilibria exist under fairly general conditions, and that only the zero-profit outcome arises if SPS is decreasing. We will now show that either sharing rules that are not sign-preserving can be problematic in that for each of them there are examples where either no (pure or mixed) equilibria exist[11], or there is a continuum of equilibria. For simplicity we will deal with a market of only two firms.

Proposition 17 *Consider a market for a homogeneous good with two firms, and a given sharing rule (ϕ, χ). Assume that ϕ and χ are zero-preserving, that*

[11] We will use the techniques of example 4 to show this.

χ is strictly increasing in its second argument and quantity-decreasing. If this sharing rule is not sign-preserving then either no (pure or mixed) Bertrand equilibria exists, or there is a continuum of pure and mixed equilibria.

This result means that there may be little hope in getting rid of the problems of non-existence on the one hand, or "excessive" multiplicity of equilibria on the other hand, without recurring to sign-preserving sharing rules.

6 Conclusions

In this paper, unlike most of the literature, we took a different point of view in analyzing Bertrand oligopoly. We did not take a sharing rule as given, in particular "equal sharing", but derived very general properties of pure and mixed Bertrand equilibria which are either independent of the sharing rule, or depend on a very small set of their properties.

We then set out classic and recent results about Bertrand equilibria under the classic "equal sharing" rule, proving that in extreme cases there may be continua of (non-trivial) mixed equilibria, or no pure or mixed equilibrium at all. "Sign-preserving sharing", a generalization of "random sharing" was shown to not suffer from the these problems, with existence and uniqueness of (zero-profit) equilibria occurring under fairly general conditions.

Lastly, if some reasonable assumptions are imposed (such as "no positive sales if demand is zero")), we showed that a sharing rule that is not sign-preserving gives rise either to non-existence of equilibria or existence of a continuum of positive-profit equilibria.

Topics for further research are the following: Determine the relation between our treatment and conditions in papers such as Dasgupta and Maskin (1986), Simon (1987) and Reny (1999), and to possible treatments of ties in auctions (have not found one yet...). An analysis of Bertrand-Edgeworth oligopoly using our techniques may also yield interesting results.

7 Appendix

7.1 Support vs Equilibrium Prices

The set $\mathcal{P}$ contains all prices between which a firm will randomize in equilibrium, and a fundamental requirement is that all prices in $\mathcal{P}$ yield the *same* expected payoff. On the other hand, the support S is defined as the smallest closed subset of the strategy space with probability measure one (Kirman

1981, p. 198).[12] Thus S contains exactly the prices all of whose neighborhoods contain positive probability weight, either because they themselves are atoms or limit points of atoms, or because they are part of a continuum of prices played in equilibrium. This means that $\mathcal{P} \cap S$ contains all prices that are payoff-relevant,[13] therefore we can without loss of generality assume that $\mathcal{P} \subset S$. Then $\mathcal{P}$ is dense in S, but may be a strict subset of S. This fact creates some problems.

An illustrative example may be helpful: Let $U : [0, 1] \to \{0, 1\}$ be defined as $U(0) = 0$, and $U(x) = 1$ for $x > 0$ (This could be the payoff given opponents's strategies). Then any $x \in (0, 1]$ maximizes payoffs, but $x = 0$ does not. Consider the uniform distribution on $(0, 1]$, which also leads to expected profits of 1. The point $x = 0$ is part of the support as defined above, but leads to profits of zero and is a dominated strategy.[14]

Working with the set S is rather efficient, since it is closed and at least bounded from below. Still, following the above argument the best we can say is the following: Assume that a property A holds for any price in $\mathcal{P}$. Then in each neighborhood of any price $p \in S$ there is a price p' for which A holds (because $p' \in \mathcal{P}$). This observation explains some of the complications in the arguments below which are necessary since we do not presume continuity from the outset.

7.2 Proofs

Proof of Proposition 3:
Proof. Clearly for $\chi(m, x) = x$ we have that $\pi_m(p) = \phi(m, \pi_1(p))$, and this sharing rule is sign-preserving if and only if ϕ is sign-preserving.

Now assume that $\pi_m(p)$ is sign-preserving. We will choose a price p and a cost function C to produce a contradiction[15]. First, for each m, ϕ cannot be always positive or always negative as $\pi_1(p)$ can be positive or negative for arbitrary demand and cost functions. We can conclude that there are y_1 and y_m such that $\phi(1, y_1) > 0$ and $\phi(m, y_m) < 0$.

[12] Or, given the price distribution F, the support S of F is defined as the set of all prices p such that $F(p + \varepsilon) - F(p - \varepsilon) > 0$ for all ε, as in Kaplan and Wettstein (2000).

[13] The points in $\mathcal{P} \backslash S$ are of total weight zero, which are irrelevant in stochastic and payoff terms.

[14] Vives (1999), p. 45, writes that "..in a mixed strategy equilibrium all actions in the support of the equilibrium distribution of a player are best responses to the distributions used by the rivals." Does he use a different notion of support, or simply have in mind the finite case?

[15] These are *a priori* arbitrary, and the demand function is implicit. An interesting extension of this work would be to limit the classes of demand and cost functions considered.

Assume now that there are x_1 and m such that $\chi(m, x_1) = x_m < x_1$. Let $c_1 = C(x_1)$ and $c_m = C(x_m)$ for some still to be determined cost function C, for which we will only make use of these two values. We will now choose a price $p \geq 0$ and costs $c_1 \geq c_m \geq 0$ (because costs are non-decreasing) such that

$$y_1 = x_1 p - c_1, \; y_m = x_m p - c_m.$$

In other words, we need to find a price $p \geq 0$ such that

$$c_1 = x_1 p - y_1 \geq c_m = x_m p - y_m \geq 0,$$

which is obtained for any price

$$p \geq \left\{ 0, \frac{y_1}{x_1}, \frac{y_m}{x_m}, \frac{y_m - y_1}{x_m - x_1} \right\}. \tag{3}$$

The same condition results for $\chi(m, x_1) = x_m > x_1$ and $c_m \leq c_1$. In both cases, by construction $\pi_1(p) = \phi(1, x_1 p - c_1) > 0$ and $\pi_m(p) = \phi(m, x_m p - c_m) < 0$, i.e. the sharing rule is not sign-preserving. ■

Proof of Proposition 5:

Proof. . 1. Let S_j, $j = 1..n$, be the support of firm j's equilibrium strategy, i.e. the smallest closed set that contains probability mass 1 (see appendix 7.1), $F_j(p) = \Pr(p_j \leq p)$ its distribution function, $G_j(p) = \Pr(p_j \geq p)$, and note that $G_j(p)$ is left-continuous in p. $\mathcal{P}_j$ is the set of prices played in equilibrium, and we can without loss of generality assume that $\mathcal{P}_j \subset S_j$. Furthermore, $\mathcal{P}_j$ is dense in S_j, and the expected profits in equilibrium are $\bar{\Pi}_j = \Pi_j(p)$ for all p in $\mathcal{P}_j$.

Every S_j, $j \in J_p$, then has a finite maximum $t_j = \sup \mathcal{P}_j$ since the maximum profits π^m are finite: For each firm, the probability that a price p is the lowest price converges to zero as p goes to infinity since $\lim_{p \to \infty} F_i(p) = 1$ for all firms $i = 1..n$. As $\pi_1(p)$ remains bounded by π^m, expected profits converge to zero as well. Yet, expected profits are positive by assumption, therefore each support must have a finite supremum, and since it is closed it has a finite maximum $t_j \in S_j$.

Let there be more than one firm making positive profits, $M = |J_p| > 1$. The maxima t_j for $j \in J_p$ must all be equal to the same $0 \leq t < \infty$, because any price above the lowest t_j would never be the lowest price and yield zero profits instead. By the same reasoning, $G_j(t) > 0$ for all firms in J_z. Moreover, the price t must be an atom for all $j \in J_p$, and therefore $t \in \mathcal{P}_j$: Assume to the contrary that for firm $i \in J_p$ the price t is not an atom, $\lim_{p \nearrow t} F_i(p) = 1$. Consider firm $j \neq i$ with equilibrium payoff $\bar{\Pi}_j > 0$

and note that there is a sequence $\{p_k\} \subset \mathcal{P}_j$ with $\Pi_j(p_k) = \bar{\Pi}_j$ for all k, and $\lim_{k\to\infty} p_k = t$, because $\mathcal{P}_j$ is dense in S_j. Then we arrive at a contradiction because

$$\bar{\Pi}_j = \lim_{k\to\infty} \Pi_j(p_k) \leq \pi^m \lim_{k\to\infty} (1 - F_i(p_k)) = 0.$$

The statement for $M = 1$, say $J_p = \{1\}$, follows from the proof in the next paragraph applied to $t = \max S_1 = \sup \mathcal{P}_1$. Note that we cannot conclude $t \in \mathcal{P}_1$.

2. Assume now that $J_z \neq \emptyset$, and that $r \in S_j$ for some firm $j \in J_p$ is neither an atom nor a limit point of atoms. Since every neighborhood of r must contain positive probability mass, $\mathcal{P}_j$ contains (uncountably many) prices arbitrarily close to r which are not atoms. Thus there is a price $p \in \mathcal{P}_j$, with $F_j(p) < 1$, that is not an atom for firm 1 and for some other firm $i \in J_z$. Expected profits of playing p are $\Pi_j(p) = \bar{\Pi}_j > 0$ for firm j, and $\Pi_i(p) \leq 0$ for firm i. But since both firms play p with zero probability, we have

$$\Pi_i(p)(1 - F_i(p)) = \Pi_j(p)(1 - F_j(p)) > 0,$$

a contradiction. Therefore S_j and $\mathcal{P}_j$ both contain only atoms or limit points of atoms.

Assume that $\mathcal{P}_j$ for a firm $j \in J_p$ contains some price $p < t$ which is not an atom, and that there is some firm $i \in J_z$ for which p is not an atom, either. By the argument of the last paragraph firm i could make positive profits playing p, a contradiction to $i \in J_z$. Thus p must be an atom for all firms in J_z. Additionally, since there can only be countably many atoms for each player, there can only be countably many limit points of atoms in $\mathcal{P}_j$, i.e. $\mathcal{P}_j$ is a countable set.

As for the last sentence, note simply that any price $p < t$ which is not an atom for any firm gives rise to positive profits if $\pi_1(p) > 0$. In fact, there must either be an atom by at least one firm in J_p or at least two firms in J_z. ■

Proof of Proposition 6:

Proof. . 1. Let $M = |J_p| \geq 2$, and let $t = \max \mathcal{P}_j$, $j \in J_p$, as defined above. The payoff of playing t for a firm $i \in J_p$ is

$$\bar{\Pi}_i = \left(\prod_{j\neq i} G_j(t)\right) \sum_{k=M}^{n} R_{ik}(t)\, \pi_k(t) > 0,$$

where $R_{ik}(t) \geq 0$, $k = M..n$, is the probability that $k-1$ firms other than i play t, given that all firms $j \neq i$ play at least t, and $\sum_{k=M}^{n} R_{ik}(t) = 1$. The sum term arises because firms $j \in J_z$ may also be playing t with positive probability, otherwise it would simply be equal to $\pi_M(t)$. Underbidding t slightly, in the limit firm i can make profits $\left(\prod_{j \neq i} G_j(t)\right) \limsup_{p \nearrow t} \pi_1(p)$, therefore it is necessary that

$$\limsup_{p \nearrow t} \pi_1(p) \leq \sum_{k=M}^{n} R_{ik}(t) \pi_k(t).$$

Since the term on the right-hand side is a weighted average of $\pi_k(t)$, $k = M..n$, there is an $m = M..n$ such that $\pi_m(t) > 0$ and $\pi_m(t) \geq \limsup_{p \nearrow t} \pi_1(p)$. Then either $\pi_m(t) \geq \pi_1(t)$ or $\pi_1(t) > \pi_m(t) \geq \max\left\{0, \limsup_{p \nearrow t} \pi_1(p)\right\}$.

2. Let $J_z \neq \emptyset$. By proposition 6, any price $p < t$ with $\pi_1(p) > 0$ must be an atom for some firm, but there can only be countably many atoms. Therefore $\pi_1(p) \leq 0$ for almost all prices $p < t$. Any of the at most countably many prices $p \in \mathcal{P}_j$ for any $j \in J_p$ must fulfill $\sum_{k=1}^{n} R_{jk}(p) \pi_k(p) > 0$, thus $\pi_1(p) > 0$ and π_1 has a left-discontinuous upward jump at p, or there is some $m = 2..n$ such that $\pi_m(p) > 0$ and $\pi_m(p) \geq \pi_1(p)$. ■

Proof of Proposition 9

Proof. Assume that all firms have zero expected profits in equilibrium. It is clear that $\pi_1(p) \leq 0$ for all $p < s$ because otherwise firms would undercut each other. We can also immediately conclude that $\pi_1(p) \leq 0$ for almost all $p < t_2$, following the argument in the proof of proposition 6, observing that it is enough to consider deviations of the firm with the smallest upper limit of its support. Any price $p \leq t_2$ with $\pi_1(p) > 0$ must be an atom for at least one firm i, because otherwise some firm $j \neq i$ could make positive profits playing this price. On the other hand, for $p < t_1$ it must be an atom for at least two firms because otherwise the single firm playing this price makes positive profits.

Consider a price $p \in \mathcal{P}_j$, $p \leq t_1$, for some $j \in J$, with $G_i(p) > 0$ for all $i \neq j$. This means that either $p < t_1$, or $p = t_1 < \infty$ and is an atom for all firms with $\sup S_i = t_1$. The expected profits for firm j of playing p are $\left(\prod_{i \neq j} G_i(p)\right) \sum_{k=1}^{n} R_{jk}(p) \pi_k(p) = 0$, thus the weighted average of $\pi_k(p)$, $k = 1..n$, is zero. Thus either $\pi_1(p) = 0$, or $\pi_1(p) > 0$ and π_1 has is left-discontinuous at p, or there is an $m = 2..n$ such that $\pi_m(p) > 0$ and $\pi_m(p) \geq 0 > \pi_1(p)$. The same argument holds for all prices $p \in \mathcal{P}_1$, $t_1 \leq p \leq t_2$, with $G_i(p) > 0$ for $i > 1$. ■

Proof of Proposition 10

Proof. Assume $i \in J_p \neq \emptyset$ and $j \in J_z \neq \emptyset$. Then since any price p played by i leads to positive expected profits and since all $\pi_m(p)$, $m = 1..n$, have the same sign, they must all be positive. Thus firm j could also play p and make positive profits even if p was an atom, a contradiction. Finally, any price $p' \leq t_1$ played by j in equilibrium makes zero expected profits but has positive probability of being the lowest price, therefore $\pi_m(p') = 0$, $m = 1..n$. ■

Proof of Proposition 11

Proof. Let p be an IVP, i.e. $\pi_1(p) \geq 0$ and $\pi_1(p') \leq 0$ for all $p' < p$. By SPS, $\pi_m(p) \geq 0$ for all $m = 2..n$, therefore playing p with probability 1 is a pure symmetric equilibrium with non-negative profits. An IBP p is an IVP with $\pi_1(p) = 0$, i.e. leads to a zero-profit equilibrium. For the converse, assume there is a zero-profit equilibrium. By proposition 9 any price $p \leq t_2$ such that $\pi_1(p) > 0$ must be an atom for at least one firm. By SPS the expected profits of playing p for this firm are positive, a contradiction to zero profits. Therefore we must have $\pi_1(p) \leq 0$ for all $p \leq t_2$. There are equilibrium prices $p \leq t_2$, $p \in \mathcal{P}_j$ for some j, and by proposition 10 they fulfill $\pi_1(p) = 0$. Thus all such p are IBP's. ■

Proof of Lemma 12

Proof. Define $p^* \leq \bar{p}$ as $p^* = \inf\{p \geq 0 | \pi_1(p) \geq 0\}$. By upper semi-continuity from the right, we must have $\pi_1(p^*) \geq \lim_{p \searrow p^*} \pi_1(p) \geq 0$. By definition of p^*, $\pi_1(p) < 0$ for all $p < p^*$, thus p^* is an IVP, and it is the smallest one. ■

Proof of Lemma 13

Proof. Fix a price $p_0 \geq 0$. Since D is lsc and non-increasing, we have $\lim_{p \searrow p_0} D(p) = D(p_0)$,and since C is lsc and non-decreasing, we have $\lim_{q \nearrow D(p_0)} C(q) = C(D(p_0))$. Therefore $\lim_{p \searrow p_0} \pi_1(p) = \pi_1(p_0)$ and π_1 is right-continuous at p_0. ■

Proof of Lemma 14

Proof. Let $p_0 > 0$ be an IVP. Then by lower semi-continuity from the left of π_1 at p_0:

$$0 \geq \liminf_{p \nearrow p_0} \pi_1(p) \geq \pi_1(p_0) \geq 0,$$

i.e. $\pi_1(p_0) = 0$. If $p_0 = 0$ is an IVP then $\pi_1(0) \geq 0$ by definition. On the other hand, $\pi_1(0) = 0 \times D(0) - C(D(0)) \leq 0$ since D is bounded, and again it follows that $\pi_1(p_0) = 0$.

Fix a price $p_0 \geq 0$. Since D is usc and non-increasing, we have $\lim_{p \nearrow p_0} D(p) = D(p_0)$,and since C is usc and non-decreasing, we have $\lim_{q \searrow D(p_0)} C(q) = C(D(p_0))$. Therefore $\lim_{p \nearrow p_0} \pi_1(p) = \pi_1(p_0)$ and π_1 is left-continuous at p_0. ■

Proof of Lemma 15

Proof. Let $p_1 = \inf \{p \geq 0 | \pi_1 (p) \geq 0\}$ and $p_2 = \sup \{p \geq 0 | \pi_1 (p) \geq 0\}$. If $p_1 < p_2$, then since π_1 is strictly quasi-concave on the continuum of prices $p \in (p_1, p_2)$ profits π_1 are positive, thus by proposition 9 $t_2 \leq p_1$, where t_2 is the second-smallest upper limit of the supports. The same is true for $p_1 = p_2$. Since by definition $\pi_1 (p) < 0$ for any $p < p_1$, the two firms with lowest upper limit of support play p_1 with probability 1. Thus the unique symmetric zero-profit equilibrium is the pure equilibrium where all firms play p_1. ■

Proof of Proposition 17

Proof. For clarity, we omit the first argument from ϕ and χ. If (ϕ, χ) is not sign-preserving there are either situations where $\pi_1 \geq 0$ and $\pi_2 \leq 0$, or $\pi_1 \leq 0$ and $\pi_2 \geq 0$ (but without equality for both). We will show that in the first case there may be no equilibria, whereas in the second case there may exist a continuum of them.

1. First assume that there is a price p_0, with demand $d_0 = D (p_0)$ and production cost $C (d_0)$ such that (let $\chi_0 = \chi (d_0)$)

$$\begin{aligned} \pi_{10} &= p_0 d_0 - C (d_0) > 0, \\ \pi_{20} &= \phi [p_0 \chi_0 - C (\chi_0)] < 0. \end{aligned}$$

If one the above two inequalities is an equality raise or lower p_0 slightly without changing d_0 to achieve strict inequalities.

We must have $d_0 > 0$ since $\pi_{10} > 0$. Let $\zeta_0 = p_0 \chi_0 - C (\chi_0)$, then $\phi (x) < 0$ for all $x \leq \zeta_0$. Since ϕ and χ are zero-preserving we have that $\chi_0 > 0$, and $\zeta_0 < 0$, thus $C (\chi_0) > 0$. Since χ is quantity-decreasing we have $\chi_0 \leq d_0$ and $C (\chi_0) \leq C (d_0)$.

Define the cost function $C^* (0) = 0$ and $C^* (x) = C^* = C (\chi_0) > 0$ for all x, and the demand function $D^* (p) = \max \left\{ 0, \chi^{-1} \left(\frac{\chi_0}{p_0} (2p_0 - p) \right) \right\}$, which is well-defined since χ is strictly increasing. It is continuous, strictly decreasing and positive on $[0, 2p_0]$, with $D (0) = \chi^{-1} (2\chi_0) < \infty$. Furthermore, for $p < 2p_0$,

$$\begin{aligned} \pi_2 (p) &= \phi [p \chi (D^* (p)) - C^* (\chi (D^* (p)))] \\ &= \phi \left[p \frac{\chi_0}{p_0} (2p_0 - p) - C^* \right], \end{aligned}$$

while $\pi_2 (p) = 0$ for $p \geq 2p_0$ since ϕ is zero-preserving. By construction, the π_2 takes on a local maximum at $p = p_0$, with $\pi_2 (p_0) = \phi (p_0 \chi_0 - C^*) < 0$, we have $\pi_2 (p) < 0$ for all $p \in [0, 2p_0)$.

On the other hand, for $p < 2p_0$

$$\pi_1(p) = p_0 \chi^{-1}\left(\frac{\chi_0}{p_0}(2p_0 - p)\right) - C^*$$

and $\pi_1(p) = 0$ for $p \geq 2p_0$, thus π_1 is continuous apart from the jump to $\pi_1(2p_0) = 0$. We have that $\pi_1(p_0) = p_0\chi^{-1}(\chi_0) - C^* \geq p_0 d_0 - C(d_0) > 0$, therefore monopoly is strictly viable.

The sharing rule (ϕ, χ) is decreasing since $\pi_1(p) > \pi_2(p)$ whenever $\pi_1(p) > 0$ since there $\pi_2(p) < 0$, and by corollary 7 there are no equilibria involving positive profits for any firm. Let $0 < p^* < p_0$ be the unique price such that $\pi_1(p^*) = 0$, then $\pi_1(p) < 0$ (and $\pi_2(p) < 0$) for all $p < p^*$. By proposition 9 the only possible pure or mixed equilibrium is both firms playing p^* with probability 1, which leads to negative payoffs, a contradiction. Thus for this example there are no equilibria.

2. Now assume that there is a price p_0, with demand $d_0 = D(p_0)$ and production cost $C(d_0)$ such that (let $\chi_0 = \chi(d_0)$)

$$\begin{aligned}
\pi_{10} &= p_0 d_0 - C(d_0) < 0, \\
\pi_{20} &= \phi\left[p_0\chi_0 - C(\chi_0)\right] > 0.
\end{aligned}$$

If one the above two inequalities is an equality raise or lower p_0 slightly without changing d_0 to achieve strict inequalities.

Let $D^*(p) = (4 - p/p_0)\, d_0/3$ and keep the cost function. We have $D^*(p_0) = d_0$, and revenue takes on its unique local maximum at $p = 2p_0$. Then revenue is increasing and costs are non-increasing on $[0, 2p_0]$, thus $\pi_1(p) = pD^*(p) - C(D^*(p))$ is increasing. Therefore there is an $\varepsilon > 0$ such that π_1 is negative on $[0, p_0 + \varepsilon)$, and $\pi_2(p) = \phi\left[p\chi\left[D^*(p)\right] - C(\chi\left[D^*(p)\right])\right]$ is positive on $E = (p_0 - \varepsilon, p_0 + \varepsilon)$. Then every price $p \in E$ gives raise to a pure symmetric equilibrium with positive profits, thus there is a continuum of pure equilibria. Furthermore, from these non-trivial mixed equilibria involving any finite number of prices can be constructed as shown in Hoernig (2001). ■

References

[1] Baye, Michael R. and Morgan, John (1997). "Necessary and Sufficient Conditions for Existence and Uniqueness of Bertrand Paradox Outcomes". Princeton, Woodrow Wilson School, Discussion Paper in Economics 186.

[2] Baye, Michael R. and Morgan, John (1999). "A Folk Theorem for One-Shot Bertrand Games". *Economics Letters*, 65:59–65.

[3] Dasgupta, Partha and Maskin, Eric (1986). "The Existence of Equilibrium in Discontinuous Economic Games, I:Theory". *Review of Economic Studies*, 53:1–26.

[4] Dastidar, Krishnendu Ghosh (1995). "On the Existence of Pure Strategy Bertrand Equilibrium". *Economic Theory*, 5(1):19–32.

[5] Harrington, Joseph E. (1989). "A Re-Evaluation of Perfect Competition as the Solution to the Bertrand Price Game". *Mathematical Social Sciences*, 17(3):315–328.

[6] Hoernig, Steffen H. (2001). "Mixed Bertrand Equilibria under Decreasing Returns to Scale: An Embarassment of Riches". Mimeo, Universidade Nova de Lisboa.

[7] Kaplan, Todd R. and Wettstein, David (2000). "The Possibility of Mixed-Strategy Equilibria with Constant-Returns-to-Scale Technology under Bertrand Competition". *Spanish Economic Review*, 2(1):65–71.

[8] Kirman, Alan P. (1981). "Measure Theory with Applications to Economics". In Arrow, Kenneth J. and Intriligator, Michael D., editors, *Handbook of Mathematical Economics, Volume I*, chapter 5, pages 159–209. Amsterdam: North Holland.

[9] Reny, Philip J. (1999). "On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games". *Econometrica*, 67(5):1029–56.

[10] Sharkey, William W. and Sibley, David S. (1993). "A Bertrand Model of Pricing and Entry". *Economics Letters*, 41(2):199–206.

[11] Simon, Leo K. (1987). "Games with Discontinuous Payoffs". *Review of Economic Studies*, 54:569–97.

[12] Simon, Leo K. and Zame, William R. (1990). "Discontinuous Games and Endogenous Sharing Rules". *Econometrica*, 58(4):861–72.