ON THE JOINT DENSITY OF THE SUM AND SUM OF SQUARES OF NONNEGATIVE RANDOM VARIABLES

by

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Abstract

If either or both of the sum and sum of squares of n variates is(are) minimal sufficient, inferential procedures are based on their joint density. In cases where the variates are non-negative the derivation of this joint density is non-trivial, and no closed-form expression for it seems to be known. Using a differential-geometric approach, we derive this joint density for the class of exponential models in which either or both of these statistics is(are) minimal sufficient. The results have numerous applications; one of these, the censored normal model, is considered briefly.

Key Words and Phrases: Censored Normal, Exponential family, Non-negative variates, Regular Simplex, Sum, Sum of Squares.

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1 INTRODUCTION

In statistical models for which either or both of the statistics

$$S_{n1} = \Sigma_{i=1}^{n} x_i, \text{ and } S_{n2} = \Sigma_{i=1}^{n} x_i^2,$$

are minimal sufficient, the joint density of (S_{n1}, S_{n2}) becomes the basis for inference. If the sample space for $x = (x_1, x_2, ..., x_n)'$ is all of R^n, this joint density is easily obtained, but if, as is frequently the case, the sample space consists of just non-negative values of x, *i.e.*, is R_+^n, the problem is far from trivial, and there appears to be no known closed form expression for the joint density in the literature. This paper provides an expression for that joint density. In doing so we shall see that the density cannot be characterised by a single function, but has a different functional form on each of $n-1$ disjoint intervals. This situation arises elsewhere, and has been discussed by, among others, Mulholland (1965), (1970). Examples with similar characteristics to the problem studied here include the statistic $Q_1 = y'Ay/y'y$, with $y \sim N(0, I_n)$ (studied by many authors, but see in particular von Neumann (1941), Koopmans (1942), Anderson (1971), Saldanha and Tomei (1996), and Hillier (2001)), and the more general form $Q_2 = y'Ay/y'By$, again with $y \sim N(0, I_n)$ and B positive definite (also studied by many, but see in particular Forchini (2001)).

Our approach to the problem is differential-geometric in character. In particular, our starting point will be Theorem 8.3.1 from Tjur (1981), which gives an expression for the density of a suitably behaved statistic as a surface integral over the manifold in the sample space on which the statistic is constant (see also Hillier and Armstrong (1999)). For a general, continuously differentiable, $p-$dimensional statistic $S = S(x)$ defined on an open subset X of R^n, and having the property that the $p \times n$ matrix $DS(x) = \{\partial S_i(x)/\partial x_j\}_{i=1,\ldots,p}^{j=1,\ldots,n}$ has rank p whenever $S(x) = s$, the $s-$level set of S,

$$M(s) = \{x; x \in X, S(x) = s\}, \tag{1}$$

is an $(n-p)-$dimensional manifold embedded in R^n (Spivak (1965), Theorem 5-1, p. 111). The density of S at the point s is then given by:

$$pdf_S(s) = \int_{M(s)} |DS(x)DS(x)'|^{-\frac{1}{2}} pdf(x)(dM(s)), \tag{2}$$

where $pdf(x)$ is the density of the underlying random vector x, $|\cdot|$ denotes the determinant of the indicated matrix, and $(dM(s))$ denotes the (canonical) volume element on $M(s)$ (defined more precisely below).

For $S(x) = (S_{n1}, S_{n2})'$ it is easy to see that the hypotheses above are satisfied, and that $|DS(x)DS(x)'| = (ns_2 - s_1^2)$ is constant on $M(s)$, which in this case is an $(n-2)-$dimensional manifold. Here, of course, we must have $s_1^2 < ns_2$ (Cauchy-Schwarz). And if, as we shall assume, $pdf(x)$ is a member of the exponential family with minimal sufficient statistic S, the density of x (with respect to Lebesgue measure) has the form

$$pdf(x;\theta) = \exp\{\theta_1 S_{n1} + \theta_2 S_{n2} + \kappa(\theta)\},\ x \in X \subseteq R^n, \tag{3}$$

and is therefore also constant on $M(s)$. In this case (2) gives at once

$$pdf_S(s;\theta) = (ns_2 - s_1^2)^{-\frac{1}{2}} \exp\{\theta_1 s_1 + \theta_2 s_2 + \kappa(\theta)\} \int_{M(s)} (dM(s)). \tag{4}$$

The integral in (4) is simply the surface content of the $(n-2)-$dimensional manifold

$$M(s) = \{x : x \in X, \Sigma_{i=1}^n x_i = s_1, \Sigma_{i=1}^n x_i^2 = s_2\}, \tag{5}$$

i.e., that part of the surface formed by the intersection of the hyperplane $\Sigma_{i=1}^n x_i = s_1$ with the hypersphere $\Sigma_{i=1}^n x_i^2 = s_2$ that lies in X. In case X is all of R^n this is easily evaluated because the hypersphere $\Sigma_{i=1}^n x_i^2 = s_2$ intersects the hyperplane $\Sigma_{i=1}^n x_i = s_1$ in an $(n-1)-$dimensional hypersphere of radius $\sqrt{(ns_2 - s_1^2)/n}$, and the content of this surface is simply the surface content of an $(n-1)-$sphere with this radius, namely,

$$C_{n-1}[(ns_2 - s_1^2)/n]^{(n-2)/2}, \tag{6}$$

where $C_k = 2\pi^{\frac{k}{2}}/\Gamma(\frac{k}{2})$ denotes the surface content of the unit sphere in k dimensions. Thus, equation (4) immediately yields, for instance, familiar results for the case where the x_i are *i.i.d.* $N(\mu, \sigma^2)$ (which, of course, are usually expressed in terms of the joint density of S_{n1} and $S_{n2}^* = (nS_{n2} - S_{n1}^2)/n$).

However, in the case where the x_i are restricted to be non-negative (as, for instance, in the censored or truncated normal model, or when the x_i are *i.i.d.* with an exponential distribution, see below), the manifold $M(s)$ is much more complicated:

it consists of that part of the surface of the hypersphere $\Sigma_{i=1}^{n} x_i^2 = s_2$ that intersects the hyperplane $\Sigma_{i=1}^{n} x_i = s_1$ *in the non-negative orthant.* Thus, our problem will be to evaluate the integral in (4) for this case. Naturally, the marginal densities of S_{n1} and S_{n2} can be obtained from the joint density, but may also be obtainable by direct application of (2) with $S(x)$ one-dimensional. Whether or not this latter approach is straightforward depends on the context.

Before proceeding, we note an important implication of (4). For any two members of the exponential family (3), indexed by parameter vectors θ_a and θ_b respectively, (4) holds for both. Eliminating the surface integral from this pair of equations we obtain the result:

$$pdf_S(s;\theta_b) = \left[\frac{pdf(x;\theta_b)}{pdf(x;\theta_a)}\right] pdf_S(s;\theta_a) \tag{7}$$

(*cf.*, Durbin (1980)). That is, the density of S induced by any member of this family can be (trivially) obtained from that induced by any other. Thus, for instance, the density of S under censored normal sampling, when the conditional density for $n \geq 2$ uncensored observations, given n, is reasonably complicated (and both S_{n1} and S_{n2} are minimal sufficient) can be obtained from that under the much simpler independent exponential sampling (when S_{n1} alone is sufficient).

The plan of the paper is as follows. In Section 2 we first simplify the manifold over which the integral is to be evaluated slightly, and mention some of its more obvious properties. We then provide some background information on, first, the regular simplex in k dimensions (because this turns out to be the key to the result), and, second, integration on manifolds. The main result, and its derivation, are given in Section 3, and Section 4 gives two applications. The more routine and tedious aspects of the derivation of the main result are relegated to the Appendix.

2 PRELIMINARIES

2.1 Simplifications

The manifold $M(s)$ may be simplified slightly by replacing the x_i by x_i/s_1, $i = 1, ..., n$. $M(s)$ is thereby transformed into the manifold $M(u) = \{x > 0,\ \Sigma_{i=1}^n x_i = 1,\ \Sigma_{i=1}^n x_i^2 = u\}$, where $u = s_2/s_1^2$, and it is easy to check that under this rescaling the volume elements $(dM(u))$ and $(dM(s))$ are related by $(dM(s)) = s_1^{n-2}(dM(u))$. We may therefore confine attention to the integral:

$$V_n^*(u) = \int_{M(u)} (dM(u)), \tag{8}$$

where

$$M(u) = \{x > 0, \Sigma_{i=1}^n x_i = 1, \Sigma_{i=1}^n x_i^2 = u\}. \tag{9}$$

Note that if, in (4), $\theta_2 = 0$ (so that the x_i are independent exponential variates), and we transform from (S_{n1}, S_{n2}) to (S_{n1}, U_n), with $U_n = S_{n2}/S_{n1}^2$, we have at once from (4) that S_{n1} and U_n are independent (because the integral does not involve s_1), that S_{n1} is exponentially distributed, and that $pdf_{U_n}(u) = (nu-1)^{-\frac{1}{2}} V_n^*(u)$. This result is useful in its own right, and also for checking the results to follow. Also, in view of (2), $V_n^*(u)$ is proportional to the density of the sum of squares $U_n = \Sigma_{i=1}^n x_i^2$, at the point $U_n = u$, when x is uniformly distributed on the set $X = \{x; x > 0, \Sigma_{i=1}^n x_i = 1\}$.

In the case $n = 2$ it is easy to see that, for $1/2 < u < 1$, the circle $x_1^2 + x_2^2 = u$ cuts the line $x_1 + x_2 = 1$ in two points in the non-negative quadrant, and in this case we take $V_2^*(u) = 2$. We henceforth assume that $n \geq 3$. We denote the *surface* of a hypersphere in $k-$dimensions with radius ρ by $S_k(\rho)$. $S_k(\rho)$ is itself a $(k-1)-$dimensional manifold, and, as above, we denote the content of the surface $S_k(1)$ by:

$$C_k = 2\pi^{\frac{k}{2}}/\Gamma(\tfrac{k}{2}). \tag{10}$$

We shall also need to integrate over (parts of) the *interior* of various hyperspheres. With a slight abuse of terminology, we refer to the interior of the hypersphere with the same radius and centre as $S_k(\rho)$ as the interior of $S_k(\rho)$.

The squared distance from the origin to the hyperplane $\Sigma_{i=1}^n x_i = 1$ is n^{-1}, and the hyperplane meets the coordinate axes where each $x_i = 1$, so the hypersphere $S_n(\sqrt{u}) = \{x \in R^n;\ x'x = u\}$ can only intersect the hyperplane $\Sigma_{i=1}^n x_i = 1$ in the non-negative orthant if $n^{-1} < u < 1$. That is,

Proposition 1 $V_n^*(u) = 0$ *for* $u \leq n^{-1}$ *or* $u \geq 1$.

(At $u = n^{-1}$ and $u = 1$, $S_n(\sqrt{u})$ intersects the hyperplane in one point, and in n

isolated points, respectively. Thus, in these cases $M(u)$ is a 0−dimensional manifold, and has content zero.)

Now, the hyperplane $\Sigma_{i=1}^n x_i = 1$ intersects the non-negative orthant in a *regular simplex* of dimension $n-1$, Σ_{n-1} say, with sides of length $\sqrt{2}$. The vertices of this simplex lie on the surface of an $(n-1)-$dimensional hypersphere whose centre, c_{n-1} say, is at the point on the hyperplane nearest the origin (c_{n-1} is called the *centroid* of Σ_{n-1}). Since the hyperplane is orthogonal to the line joining the origin to c_{n-1}, for any point x in the simplex, $x'x = n^{-1} + \rho_{n-1}^2$, where ρ_{n-1}^2 is the squared distance from x to c_{n-1}. Thus, the intersection of $S_n(\sqrt{u})$ with the hyperplane $\Sigma_{i=1}^n x_i = 1$ consists of that part of the $S_{n-1}(\rho_{n-1})$ (with centre at c_{n-1}) that lies *inside* the simplex Σ_{n-1}, where

$$\rho_{n-1} = \sqrt{u - n^{-1}}. \tag{11}$$

That is:

Proposition 2 *For $n^{-1} < u < 1$, $V_n^*(u)$ is the content of that part of the surface of an $(n-1)$-dimensional hypersphere with radius ρ_{n-1} and centre at the centroid of Σ_{n-1} that lies inside Σ_{n-1}. Denoting this quantity by $V_{n-1}(\rho_{n-1})$, $V_n^*(u) = V_{n-1}(\rho_{n-1})$.*

Before seeking to evaluate $V_{n-1}(\rho_{n-1})$ for $n^{-1} < u < 1$ we briefly describe those properties of the regular simplex that bear on the calculation.

2.2 The Regular Simplex

A $k-$dimensional regular simplex, Σ_k, is determined by $k+1$ points (its *vertices),* each equidistant from the remainder. Thus, the vertices lie on the surface of a $k-$dimensional hypersphere, the centre of which is the *centroid* of the simplex, c_k say. In the case of a simplex of side length s, the lines joining the centroid to the vertices are of length $r_k = s\sqrt{k/2(k+1)}$, the radius of the hypersphere containing the simplex. In our case, $s = \sqrt{2}$, so in future we take $r_k = \sqrt{k/(k+1)}$, and Σ_k will always denote a regular simplex of side-length $\sqrt{2}$.

Each choice of k vertices from the original $(k+1)$ determines a *face* of Σ_k, itself a ($(k-1)$-dimensional) regular simplex with the same side length as the original Σ_k. The distance from the centroid c_k of Σ_k to any of its faces is:

$$f_k = 1/\sqrt{k(k+1)}. \tag{12}$$

Thus, f_k is the radius of the largest hypersphere (with centre at the centroid of Σ_k) that lies entirely inside Σ_k Setting $k = n-1$, the entire surface $S_{n-1}(\rho_{n-1})$ lies inside Σ_{n-1} if $0 < \rho_{n-1} < f_{n-1}$, or $n^{-1} < u < (n-1)^{-1}$, so we can state our first result for $V_{n-1}(\rho_{n-1})$ immediately:

Case 1 *For* $n^{-1} < u < (n-1)^{-1}$,

$$V_n^*(u) = V_{n-1}(\rho_{n-1}) = C_{n-1}\rho_{n-1}^{n-2} \tag{13}$$

These results for a face determined by k points generalise as follows. Any $r+1 \geq 1$ of the original $k+1$ vertices of Σ_k determine a regular simplex, Σ_r, of dimension r, with the same side length as Σ_k. We call this an $r-face$ of Σ_k; there are obviously $\binom{k+1}{r+1}$ distinct such $r-$faces. The $k+1$ $0-$faces are single points, the vertices of Σ_k, while the $k(k+1)/2$ $1-$faces are line segments, its sides. The line joining the origin (the centroid of the original Σ_k) to the centroid of any $r-$face is orthogonal to all points in the $r-$face, and has length

$$f_{k,r} = \sqrt{\frac{k}{k+1} - \frac{r}{r+1}}. \tag{14}$$

Note that the f_k defined above are, in this notation, the $f_{k,k-1}$. We omit the extra subscript for the special case $r = k-1$.

Since $f_{k,r}$ increases as r decreases, *faces of lower dimension are further from the centroid of the original simplex.* Thus, an $S_k(\rho)$ with centre at the centroid of Σ_k intersects just those $r-$faces for which $\rho > f_{k,r}$. In particular, for $f_{k,r} < \rho < f_{k,r-1}$, $S_k(\rho)$ intersects all $r-$faces, but no $r'-$face with $r' < r$. That is, $S_k(\rho)$ lies "inside" the set of $(r-1)-$faces of Σ_k, but partly "outside" the set of its $r-$faces.

Setting $k = n-1$ and $\rho = \rho_{n-1} = \sqrt{u - n^{-1}}$, the interval $f_{n-1,r} < \rho_{n-1} < f_{n-1,r-1}$ corresponds to the interval $(r+1)^{-1} < u < r^{-1}$. As u passes through a point seperating two of these intervals, the surface $S_{n-1}(\rho_{n-1})$ passes "outside" a new set of $r-$faces of Σ_{n-1} (but remains "inside" the set of lower dimensional faces), and $V_{n-1}(\rho_{n-1})$ thus becomes a different function of u. Therefore:

Proposition 3 $V_n^*(u)$, *thought of as a function of* u, *has a different functional form on each of the* $n-1$ *intervals:*

$$(r+1)^{-1} < u < r^{-1} \ , \ r = 1, ..., n-1. \tag{15}$$

Our approach to the evaluation of the content of $M(u)$ will be based on the observation that both the surface of an $S_k(\rho)$, and its interior, can be partitioned into disjoint "pieces" corresponding to the faces of Σ_k, and that points in R^k can be assigned local coordinates which reflect that fact. Before introducing this partitioning we briefly describe the key ideas relating to integration on manifolds that will be needed later.

2.3 Surface Integration

The surfaces we are concerned with are *differentiable manifolds* embedded in R^k (where, initially, $k = n-1$). In the case of a manifold M of dimension $p < k$, this means that, in the neighbourhood of each point $x \in M$, there is an open set $A \subset R^k$ containing x, an open set $B \subset R^p$, and a one-to-one differentiable function $f : B \mapsto R^k$ such that (a) $f(B) = M \cap A$, (b) $f^{-1} : f(B) \mapsto B$ is continuous, and (c) the matrix $Df(y) = \{\partial f_i(y)/\partial y_j\}_{i=1,k}^{j=1,p}$ has rank p for each $y \in B$. Such a function is called a *local coordinate chart* for M near x. Given a local coordinate chart f near a point $x \in M$, the (canonical) volume element on M, (dM), is defined locally by:

$$(dM) = |Df(y)'Df(y)|^{\frac{1}{2}} (dy), \tag{16}$$

where (dy) denotes ordinary Lebesgue measure on R^p. M can be equipped with a system of such (overlapping) local coordinate charts (an *atlas)* that cover it, so that (dM) is well-defined everywhere on M, and one can therefore integrate functions defined on M in a natural way. It is straightforward to show that the integral of a function defined on the manifold is independent of the system of local coordinate charts used.

Now, the surface of a simplex Σ_k (*i.e.*, the set of its $(k-1)-$faces) is evidently in one-to-one correspondence with a surface $S_k(\rho)$ centered at the centroid of Σ_k. In particular, the $(k+1)$ faces of Σ_k partition $S_k(\rho)$ into $(k+1)$ disjoint pieces, each piece being in the "direction" (from the centroid of Σ_k) of a single face. (We say that points in R^k are *in the direction of* a particular face of Σ_k if the line joining the centroid of Σ_k to the point in question intersects that face, or would do so if extended positively). Thus, we can use the faces of Σ_k to construct (local) coordinates for points on $S_k(\rho)$, as follows.

Choose one $(k-1)-$face of Σ_k, a Σ_{k-1}. Coordinate axes in R^k, with c_k as origin, can be chosen so that this face is parallel to one axis, say the first, so that the first coordinate of each point on the Σ_{k-1} is constant, and this is obviously f_k, the length of the line joining c_k to the centroid of the face, c_{k-1}. Let $y_{k-1} \in R^{k-1}$ denote the coordinates of a point in the R^{k-1} containing Σ_{k-1}, with its centroid, c_{k-1}, as origin. Then points in R^k in the direction of Σ_{k-1} can be written in the form

$$x = \alpha \begin{pmatrix} f_k \\ \\ y_{k-1} \end{pmatrix}, \ \alpha > 0, \ y_{k-1} \in \Sigma_{k-1}, \tag{17}$$

and it is easy to see that for points of this form on $S_k(\rho)$, $\alpha = \rho(f_k^2 + y'_{k-1}y_{k-1})^{-\frac{1}{2}}$. This defines, locally (for points in the direction of one $(k-1)-$face of Σ_k), a coordinate chart for $S_k(\rho)$ given by:

$$x = f(y_{k-1}) = \rho(f_k^2 + y'_{k-1}y_{k-1})^{-\frac{1}{2}} \begin{pmatrix} f_k \\ \\ y_{k-1} \end{pmatrix}, \ y_{k-1} \in \Sigma_{k-1}, \tag{18}$$

and it is straightforward to check that, with these coordinates, the volume element defined in (16) is given by:

$$(dS_k(\rho)) = f_k \rho^{k-1}(f_k^2 + y'_{k-1}y_{k-1})^{-\frac{k}{2}}(dy_{k-1}). \tag{19}$$

Clearly, the union of such coordinate charts over all $(k+1)$ faces of Σ_k completely covers $S_k(\rho)$, and, by construction, they overlap only in spaces of dimension $(k-2)$ (the Σ_{k-2} where two $(k-1)-$faces intersect), which have $((k-1)-$dimensional Lebesgue) measure zero.

In fact, if we replace ρ by $r > 0$, in (18), and allow r to vary, *all* points in $x \in R^k$ in the direction of one face of Σ_k can be represented uniquely in the form:

$$x = r(f_k^2 + y'_{k-1}y_{k-1})^{-\frac{1}{2}} \begin{pmatrix} f_k \\ \\ y_{k-1} \end{pmatrix}, \ r > 0, \ y_{k-1} \in \Sigma_{k-1}, \tag{20}$$

and it is easy to see that the Euclidean volume element (Lebesgue measure), (dx), on R^k factors (locally) as:

$$\begin{aligned}(dx) &= f_k r^{k-1}(f_k^2 + y'_{k-1}y_{k-1})^{-\frac{k}{2}} dr(dy_{k-1}) \\ &= (dS_k(r))dr, \ r > 0, \ y_{k-1} \in \Sigma_{k-1}. \qquad (21)\end{aligned}$$

That is, locally, Lebesgue measure on R^k factors into the product of a measure on part of the surface of the hypersphere $S_k(r)$ and Lebesgue measure on R^+. This, of course, is well-known from the polar coordinate representation of points in R^k. In the present context, though, the coordinates (20) prove more useful, and will enable us to integrate over the interior of an S_k, as well as over its surface.

3 MAIN RESULTS

Consider an $S_k(\rho)$ with centre at the centroid of a Σ_k, which we take as the origin, and with $f_k < \rho < r_k$, so that part, but not all of, $S_k(\rho)$ lies inside Σ_k. We denote the surface content of $S_k(\rho)$ that is inside Σ_k by $V_k(\rho)$, and its *complement*, the surface content *outside* Σ_k, by:

$$\bar{V}_k(\rho) = C_k \rho^{k-1} - V_k(\rho). \qquad (22)$$

Clearly, because of the symmetry of both Σ_k and $S_k(\rho)$, to evaluate $\bar{V}_k(\rho)$, and hence $V_k(\rho)$, we need only consider the content of $S_k(\rho)$ that is in the direction of, but outside, *one* face of Σ_k.

The surface, $S_k(\rho)$ intersects each face of Σ_k in (at least part of) a hyperspherical surface $S_{k-1}(\rho_{k-1})$ centered at the centroid of that face, and having radius $\rho_{k-1} = \sqrt{\rho^2 - f_k^2}$. If $\rho_{k-1} < f_{k-1}$ each such $S_{k-1}(\rho_{k-1})$ lies entirely inside its respective Σ_{k-1}. But if $\rho_{k-1} > f_{k-1}$ each $S_{k-1}(\rho_{k-1})$, in turn, intersects each of the faces of the Σ_{k-1} in (at least part of) yet another (lower dimensional) hypersphere $S_{k-2}(\rho_{k-2})$, with $\rho_{k-2} = \sqrt{\rho_{k-1}^2 - f_{k-1}^2}$, and so on. The radii of the successively lower-dimensional hyperspheres ocurring in this process are given by the recursive relation:

$$\rho_{k-r-1}^2 = \rho_{k-r}^2 - f_{k-r}^2, \ r = 0, ..., k-1, \ \rho_k \equiv \rho. \tag{23}$$

For ρ in the interval $f_{k,k-r} < \rho < f_{k,k-r-1}$, $\rho_{k-s} > f_{k-s}$ for $s = 1, ..., r-1$, but $\rho_{k-r} < f_{k-r}$. That is, in the above process it is not until we arrive at a face of dimension $k-r$, a Σ_{k-r}, that the entire $S_{k-r}(\rho_{k-r})$ lies inside the Σ_{k-r}. This is the basis of the calculations to follow.

Suppose first that $f_k < \rho < f_{k,k-2}$, so that $\rho > f_k$ but $\rho_{k-1} < f_{k-1}$. Then $S_k(\rho)$ intersects each $(k-1)-$face of the original Σ_k, but none of the $r-$faces with $r < k-1$. Thus, in the direction of a single face of Σ_k, the part of $S_k(\rho)$ outside that face is a (complete) "cap" on $S_k(\rho)$. Using the coordinates (18), we therefore have, for the content *outside* a single face,

$$\begin{aligned} & f_k \rho^{k-1} \int_{y'_{k-1} y_{k-1} \leq \rho_{k-1}^2} (f_k^2 + y'_{k-1} y_{k-1})^{-\frac{k}{2}} (dy_{k-1}) \\ = \ & f_k C_{k-1} \rho^{k-1} \int_0^{\rho_{k-1}} r_1^{k-2} (f_k^2 + r_1^2)^{-\frac{k}{2}} dr_1, \end{aligned} \tag{24}$$

on converting y_{k-1} to polar coordinates and integrating over the surface $S_{k-1}(1)$. Note that the integral here is over the *interior* of $S_{k-1}(\rho_{k-1})$. Adding these $(k+1)$ equal components, and subtracting the result from the original surface content of $S_k(\rho)$, we obtain:

$for\ f_k < \rho < f_{k,k-2}$,

$$V_k(\rho) = C_k\rho^{k-1} - (k+1)f_k C_{k-1}\rho^{k-1}\int_0^{\rho_{k-1}} r_1^{k-2}(f_k^2 + r_1^2)^{-\frac{k}{2}}dr_1. \quad (25)$$

The integral here can, of course, be evaluated explicitly, but we defer the evaluation of all integrals until later.

Applying this result for the case $k = n-1,\ \rho = \rho_{n-1}$, with $f_{n-1} < \rho_{n-1} < f_{n-1,n-3}$, we obtain:

Case 2 *For* $(n-1)^{-1} < u < (n-2)^{-1}$,

$$V_{n-1}(\rho_{n-1}) = C_{n-1}\rho_{n-1}^{n-2} - nf_{n-1}C_{n-2}\rho_{n-1}^{n-2}\int_0^{\rho_{n-2}} r_1^{n-3}(f_{n-1}^2 + r_1^2)^{-\frac{n-1}{2}}dr_1. \quad (26)$$

where $\rho_{n-2}^2 = \rho_{n-1}^2 - f_{n-1}^2 = u - (n-1)^{-1}$.

Suppose next that $f_{k,k-2} < \rho < f_{k,k-3}$, so that $\rho_{k-1} > f_{k-1}$, but $0 < \rho_{k-2} < f_{k-2}$. In this case equation (24) *overstates* the content of $S_k(\rho)$ outside Σ_k but in the direction of one of its faces, because only part of the (interior of) $S_{k-1}(\rho_{k-1})$ over which the integral in the first line of (24) is evaluated actually lies inside the Σ_{k-1} :

we need to restrict the integral in the first line of (24) to the part of the interior of $S_{k-1}(\rho_{k-1})$ that is inside the face.

To calculate the content to be *excluded* from (24) we can proceed much as above: we now partition the interior of $S_{k-1}(\rho_{k-1})$ into k pieces, each piece corresponding to one face of the Σ_{k-1}, a Σ_{k-2}. In one of these, we use the analogue of the coordinates (20) for y_{k-1} itself. That is, we put, for points y_{k-1} in the direction of one of the faces of Σ_{k-1},

$$y_{k-1} = r_1(f_{k-1}^2 + y_{k-2}'y_{k-2})^{-\frac{1}{2}} \begin{pmatrix} f_{k-1} \\ y_{k-2} \end{pmatrix}, \ r_1 > 0, \ y_{k-2} \in \Sigma_{k-2}, \tag{27}$$

and, from (21), the volume element becomes:

$$(dy_{k-1}) = f_{k-1} r_1^{k-2} (f_{k-1}^2 + y_{k-2}'y_{k-2})^{-\frac{k-1}{2}} dr_1 (dy_{k-2}). \tag{28}$$

In these coordinates, points y_{k-1} are in the interior of $S_{k-1}(\rho_{k-1})$ if $r_1 < \rho_{k-1}$. And, since the $S_{k-1}(\rho_{k-1})$ intersects a face of Σ_{k-1}, a Σ_{k-2}, in an entire $S_{k-2}(\rho_{k-2})$ (under the hypothesis $\rho_{k-2} < f_{k-2}$), such points are outside Σ_{k-1} in the direction of one of its faces if $y_{k-2}'y_{k-2} < \rho_{k-2}^2$ and $r_1 > \sqrt{f_{k-1}^2 + y_{k-2}'y_{k-2}}$. The latter inequality arises because the radius of the hyperspherical surface with centre at the centroid c_k of the original Σ_k and containing the point $y_{k-2} \in \Sigma_{k-2}$ is $\sqrt{f_{k-1}^2 + y_{k-2}'y_{k-2}}$, while the point (27) lies on a hypersphere of radius r_1, and is therefore outside Σ_{k-1} (in the direction of this face) only when $r_1 > \sqrt{f_{k-1}^2 + y_{k-2}'y_{k-2}}$. Hence, in these coordinates, the portion of the interior of $S_{k-1}(\rho_{k-1})$ in (24) to be excluded is, for each of the k faces of Σ_{k-1}, of the form:

$$\{(r_1, y_{k-2}); \sqrt{f_{k-1}^2 + y_{k-2}' y_{k-2}} < r_1 < \rho_{k-1},\ y_{k-2}' y_{k-2} < \rho_{k-2}^2\},$$

so that, for each $(k-2)-$face of the Σ_{k-1}, the content to be excluded from (24) is:

$$\begin{aligned} & f_k f_{k-1} \rho^{k-1} \int_{y_{k-2}' y_{k-2} \le \rho_{k-2}^2} \int_{\sqrt{f_{k-1}^2 + y_{k-2}' y_{k-2}}}^{\rho_{k-1}} r_1^{k-2} (f_k^2 + r_1^2)^{-\frac{k}{2}} \\ & \times (f_{k-1}^2 + y_{k-2}' y_{k-2})^{-\frac{k-1}{2}} dr_1 (dy_{k-2}) \\ = \ & f_k f_{k-1} C_{k-2} \rho^{k-1} \int_0^{\rho_{k-2}} \int_{\sqrt{f_{k-1}^2 + r_2^2}}^{\rho_{k-1}} r_1^{k-2} r_2^{k-3} \\ & \times (f_k^2 + r_1^2)^{-\frac{k}{2}} (f_{k-1}^2 + r_2^2)^{-\frac{k-1}{2}} dr_1 dr_2 \end{aligned} \tag{29}$$

on again converting y_{k-2} to polar coordinates and integrating over $S_{k-2}(1)$. Since, for each $(k-1)-$face of Σ_k, there are k such $(k-2)-$faces, we have

$for\ f_{k,k-2} < \rho < f_{k,k-3}$,

$$\begin{aligned} V_k(\rho) \ = \ & C_k \rho^{k-1} - (k+1) f_k C_{k-1} \rho^{k-1} \int_0^{\rho_{k-1}} r_1^{k-2} (f_k^2 + r_1^2)^{-\frac{k}{2}} dr_1 \\ & + k(k+1) f_k f_{k-1} C_{k-2} \rho^{k-1} \int_0^{\rho_{k-2}} \int_{\sqrt{f_{k-1}^2 + r_2^2}}^{\rho_{k-1}} r_1^{k-2} r_2^{k-3} \\ & \times (f_k^2 + r_1^2)^{-\frac{k}{2}} (f_{k-1}^2 + r_2^2)^{-\frac{k-1}{2}} dr_1 dr_2. \end{aligned} \tag{30}$$

Applying this result for the case $k = n-1$, $\rho_{n-1} = \sqrt{u - n^{-1}}$, we have

Case 3 *For* $(n-2)^{-1} < u < (n-3)^{-1}$,

$$V_{n-1}(\rho_{n-1}) \ = \ C_{n-1} \rho_{n-1}^{n-2} - n f_{n-1} C_{n-2} \rho_{n-1}^{n-2} \int_0^{\rho_{n-2}} r_1^{n-3} (f_{n-1}^2 + r_1^2)^{-\frac{n-1}{2}} dr_1$$

$$+n(n-1)f_{n-1}f_{n-2}C_{n-3}\rho_{n-1}^{n-2}\int_0^{\rho_{n-3}}\int_{\sqrt{f_{n-2}^2+r_2^2}}^{\rho_{n-2}} r_1^{n-3}r_2^{n-4}$$
$$\times(f_{n-1}^2+r_1^2)^{-\frac{n-1}{2}}(f_{n-2}^2+r_2^2)^{-\frac{n-2}{2}}dr_1dr_2. \quad (31)$$

Example 1 $k=3$.

In the case $k=3$, Σ_3 is a regular tetrahedron with four equilateral triangles as faces. The three possible cases (a) $0<\rho<f_3$, (b) $f_3<\rho<f_{3,1}$, and (c) $f_{3,1}<\rho<f_{3,0}=r_3$ are depicted in Figure 1.

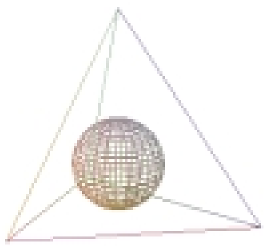

Figure 1(a)

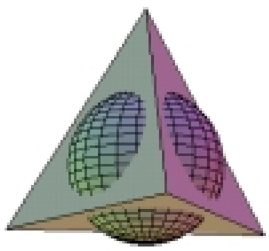

Figure 1(b)

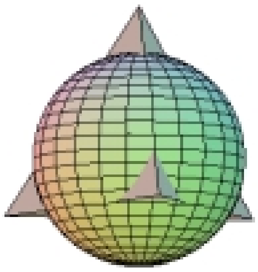

Figure 1(c)

The results so far obtained yield:

$for\ 0 < \rho < 1/2\sqrt{3}$ (Case 1):

$$V_3(\rho) = 4\pi\rho^2, \tag{32}$$

$for\ 1/2\sqrt{3} < \rho < 1/2$ (Case 2):

$$\begin{aligned} V_3(\rho) &= 4\pi\rho^2 - (4\pi/\sqrt{3})\rho^2 \int_0^{\rho_2} r_1(f_3^2 + r_1^2)^{-\frac{3}{2}} dr_1 \\ &= 4\pi\rho/\sqrt{3} - 4\pi\rho^2, \end{aligned} \tag{33}$$

and

$for\ 1/2 < \rho < \sqrt{3}/2$, (Case 3):

$$\begin{aligned} V_3(\rho) &= 4\pi\rho^2 - (4\pi/\sqrt{3})\rho^2 \int_0^{\rho_2} r_1(f_3^2 + r_1^2)^{-\frac{3}{2}} dr_1 \\ &\quad + (2\sqrt{2})\rho^2 \int_0^{\rho_1} \int_{\sqrt{f_2^2+r_2^2}}^{\rho_2} r_1(f_3^2 + r_1^2)^{-\frac{3}{2}} (f_2^2 + r_2^2)^{-1} dr_1 dr_2 \\ &= 4\pi\rho/\sqrt{3} - 4\pi\rho^2 + (2\sqrt{2})\rho^2 \int_0^{\rho_1} (f_2^2 + r_2^2)^{-1} \left\{ (f_2^2 + f_3^2 + r_2^2)^{-\frac{1}{2}} - \frac{1}{\rho} \right\} dr_2 \\ &= 4\pi\rho/\sqrt{3} - 4\pi\rho^2 - 4\sqrt{3}\rho \arctan(\rho_1/f_2) + 24\rho^2 \arctan(\rho_1/\rho\sqrt{2}), \end{aligned} \tag{34}$$

with $\rho_1^2 = \rho^2 - f_2^2 - f_3^2$. ■

One further case will illustrate the structure of the general result. Thus, consider the case $f_{k,k-3} < \rho < f_{k,k-4}$, so that both $\rho_{k-1} > f_{k-1}$ and $\rho_{k-2} > f_{k-2}$, but $\rho_{k-3} < f_{k-3}$. In this case the integral in the second term in (29) is now too large, because not all points y_{k-2} in the region $y_{k-2}' y_{k-2} \leq \rho_{k-2}^2$, over which that integral is evaluated, lie in the $(k-2)-$face Σ_{k-2}. Again, though, we can partition the Σ_{k-2} into its $(k-3)-$faces ($(k-1)$ of them), use the analogue of the coordinates (20) for y_{k-2}, and evaluate the contribution to the integral for one such face as before. To simplify the notation, let us define the sequence of constants

$$a_{k,s} = \Pi_{i=1}^{s}(k-i+2) f_{k-i+1} = \sqrt{\tfrac{k+1}{k-s+1}},\ s = 1, ..., k, \tag{35}$$

and the functions

$$\begin{aligned} H_{k,s}(\rho) &= \int_0^{\rho_{k-s}} \int_{\sqrt{f_{k-s+1}^2 + r_s^2}}^{\rho_{k-s+1}} \cdots \int_{\sqrt{f_{k-1}^2 + r_2^2}}^{\rho_{k-1}} [\Pi_{i=1}^{s} r_i^{k-i-1}] \\ &\quad \times [\Pi_{i=1}^{s} (f_{k-i+1}^2 + r_i^2)^{-\frac{k-s+1}{2}}][dr_1 dr_2 ... dr_s], \end{aligned} \tag{36}$$

where the ρ_{k-i} are as defined in equation (22) above. Series expansions for the functions $H_{k,s}(\rho)$, as well as some of their properties, are given in the Appendix.

In this notation, (25) and (30) become:

$$\begin{aligned} V_k(\rho) &= C_k \rho^{k-1} - a_{k,1} C_{k-1} \rho^{k-1} H_{k,1}(\rho), \\ f_k &< \rho < f_{k,k-2}, \end{aligned} \tag{37}$$

and

$$V_k(\rho) \;=\; C_k\rho^{k-1} - a_{k,1}C_{k-1}\rho^{k-1}H_{k,1}(\rho) + a_{k,2}C_{k-2}\rho^{k-1}H_{k,2}(\rho), \qquad (38)$$

$$f_{k,k-2} \;<\; \rho < f_{k,k-3}.$$

The content to be excluded from (29), for each of the $(k-3)-$faces of the Σ_{k-2}, is:

$$f_kf_{k-1}f_{k-2}C_{k-3}\rho^{k-1}\int_0^{\rho_{k-3}}\int_{\sqrt{f_{k-2}^2+r_3^2}}^{\rho_{k-2}}\int_{\sqrt{f_{k-1}^2+r_2^2}}^{\rho_{k-1}} r_1^{k-2}r_2^{k-3}r_3^{k-4}$$
$$\times(f_k^2+r_1^2)^{-\frac{k}{2}}(f_{k-1}^2+r_2^2)^{-\frac{k-1}{2}}(f_{k-2}^2+r_3^2)^{-\frac{k-2}{2}}dr_1dr_2dr_3.$$
$$= \; f_kf_{k-1}f_{k-2}C_{k-3}\rho^{k-1}H_{k,3}(\rho). \qquad (39)$$

There are $(k-1)$ such terms, so that, subtracting these from the second term in (29), for $f_{k,k-3} < \rho < f_{k,k-4}$,

$$\bar{V}_k(\rho) = a_{k,1}C_{k-1}\rho^{k-1}H_{k,1}(\rho) - a_{k,2}C_{k-2}\rho^{k-1}H_{k,2}(\rho) + a_{k,3}C_{k-3}\rho^{k-1}H_{k,3}(\rho), \qquad (40)$$

and so

$for\ f_{k,k-3} < \rho < f_{k,k-4},$

$$V_k(\rho) = C_k\rho^{k-1} - a_{k,1}C_{k-1}\rho^{k-1}H_{k,1}(\rho) + a_{k,2}C_{k-2}\rho^{k-1}H_{k,2}(\rho) - a_{k,3}C_{k-3}\rho^{k-1}H_{k,3}(\rho). \qquad (41)$$

In general, for the case $f_{k,k-r} < \rho < f_{k,k-r-1}$,we need to continue this process of iteratively modifying the calculation at the previous step – the contribution to $\bar{V}_k(\rho)$ from faces of dimension higher than $k-r$ – and only at the last stage (for the

$(k-r)-$faces of each $\Sigma_{k-r+1})$ integrating over the complete interior of an $S_{k-r}(\rho_{k-r})$. We state the result in:

Theorem 1 *Let $S_k(\rho)$ denote the surface of a hypersphere in k dimensions with center at the centroid of the $k-$dimensional simplex Σ_k. Let $V_k(\rho)$ denote the surface content of the part of $S_k(\rho)$ that lies inside Σ_k, and let $f_{k,r}$ and ρ_{k-r} be as defined in equations (14) and (22) respectively. Then:*

(a) $V_k(\rho)$ is a different function of ρ on each interval

$$f_{k,k-r} < \rho < f_{k,k-r-1},\ r = 0, 1, ..., k-1,\ f_{k,k} = 0; \tag{42}$$

(b) In the interval $f_{k,k-r} < \rho < f_{k,k-r-1}$, $V_k(\rho)$ is given by

$$V_k(\rho) = C_k\rho^{k-1} + \rho^{k-1}\sum_{s=1}^{r}(-1)^s a_{k,s}C_{k-s}H_{k,s}(\rho), \tag{43}$$

where $a_{k,s}$, C_{k-s} , and $H_{k,s}(\rho)$ are as defined in equations (10), (36), and (37) respectively.

Applying the theorem to the case $k = n-1$, $\rho = \rho_{n-1} = \sqrt{u - n^{-1}}$, and noting that $\rho^2_{n-r-1} = u - (n-r)^{-1}$ we have:

Corollary 1 *For each $r = 0, 1, ..., n-2$, for u in the interval*

$$(n-r)^{-1} < u < (n-r-1)^{-1}, \tag{44}$$

$$
\begin{aligned}
V_n^*(u) &= \int_{M(u)} (dM(u)) \\
&= C_{n-1}\rho_{n-1}^{n-2} + \rho_{n-1}^{n-2} \sum_{s=1}^{r} (-1)^s a_{n-1,s} C_{n-s-1} H_{n-1,s}(\rho_{n-1}) \qquad (45)
\end{aligned}
$$

where $M(u) = \{x > 0, \Sigma_{i=1}^n x_i = 1, \Sigma_{i=1}^n x_i^2 = u\}$ *and* $\rho_{n-1} = \sqrt{u - n^{-1}}$.

4 EXAMPLE: Censored Normal Model

Let z_i, $i = 1, ..., n$ be N independent $N(0, 1)$ variates, and let $Y_i^* = (\gamma + z_i)/\theta$, with $-\infty < \gamma < \infty$, $\theta > 0$. Assume that we observe not the latent variates Y_i^* but the censored variates

$$
\begin{aligned}
Y_i &= Y_i^* \text{ if } Y_i^* > 0 \\
&= 0 \text{ if } Y_i^* \leq 0. \qquad (46)
\end{aligned}
$$

The data then consists of the number, n, of uncensored observations, and their (necessarily positive) values, together with the configuration of the uncensored values in the sample. Since, in this simple case, the configuration is unimportant we may, for each n, sum over those configurations, and it is easy to see that the conditional density of the uncensored observations is given by

$$
pdf(y_+|n) = (2\pi)^{-\frac{n}{2}} (\theta/F(\gamma))^n exp\{-\frac{1}{2}\Sigma_+(\theta y_i - \gamma)^2\}, \qquad (47)
$$

where $F(\cdot)$ denotes the *cdf* of the standard normal distribution, y_+ denotes the $n-$vector of positive observations, and Σ_+ denotes the sum over the indices for which

$y_i > 0$. The marginal density of n is Binomial(N, $F(\gamma)$). Of interest, at least initially, is the conditional joint density, given n, of the (conditionally) sufficient statistics in (47), $S_{n1} = \Sigma_+ y_i$ and $S_{n2} = \Sigma_+ y_i^2$.

For $n = 0, 1, 2$ there is nothing to prove, but for $n > 2$ the conditional density (47) is a member of the exponential family (3). The earlier results thus give the conditional joint density of $S_{n1} = \Sigma_+ y_i$ and $S_{n2} = \Sigma_+ y_i^2$, given n :

$$pdf_{S|n}(s|n;\gamma,\theta)) = (ns_2 - s_1^2)^{-\frac{1}{2}} s_1^{n-2} \exp\{\theta\gamma s_1 - \frac{1}{2}\theta_2^2 s_2 - \frac{1}{2}n\gamma^2\} V_n^*(u),\ n > 2, \quad (48)$$

with $V_n^*(u)$ as given in Corollary 1.

Because the number, n, of uncensored observations is not ancillary in this model, it would usually be the unconditional density of S_n, and of functions of S, such as the maxumum likelihood estimator for (γ, θ), that would be of interest. Unfortunately, converting this conditional result into its unconditional counterpart is not at all straightforward because, as we have seen, $V_n^*(u)$ has a different functional form on intervals that depend on the conditioning variate, n. This problem, among others, is addressed in a seperate paper dealing with the properties of the maximum likelihood estimators for γ and θ in this model.

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APPENDIX
The integrals $H_{k,s}(\rho)$

Let

$$H_{k,s} = \int \cdots \int_{R_s} [\prod_{i=1}^{s} r_i^{k-i-1} (f_{k-i+1}^2 + r_i^2)^{-\frac{k-i+1}{2}}](dr_1 dr_2 ... dr_s), \tag{A.1}$$

where R_s is the region:

$$\sqrt{f_{k-i}^2 + r_{i+1}^2} \ < \ r_i < \rho_{k-i}, \ i = 1, ..., s-1 \tag{A.2}$$

$$0 \ < \ r_s < \rho_{k-s}. \tag{A.3}$$

Transform to the new variables $q_i = r_i^2$, $i = 1, ..., s$, so that $r_i = q_i^{\frac{1}{2}}$, $i = 1, ..., s$. The Jacobian is $2^{-s} \prod_{i=1}^{s} q_i^{-\frac{1}{2}}$, the integrand becomes:

$$2^{-s}[\prod_{i=1}^{s} q_i^{\frac{k-i}{2}-1} (f_{k-i+1}^2 + q_i)^{-\frac{k-i+1}{2}}], \tag{A.4}$$

and the region of integration becomes:

$$f_{k-i}^2 + q_{i+1} \ < \ q_i < \rho_{k-i}^2, \ i = 1, ..., s-1, \tag{A.5}$$

$$0 \ < \ q_s < \rho_{k-s}^2. \tag{A.6}$$

Now define

$$b_i \ = \ (\rho_{k-i}^2 - q_i)/(\rho_{k-i-1}^2 - q_{i+1}), \ i = 1, ..., s-1, \tag{A.7}$$

$$b_s \ = \ (\rho_{k-s}^2 - q_s)/\rho_{k-s}^2. \tag{A.8}$$

Then $0 < b_i < 1$ for $i = 1, ..., s$, the Jacobian of the transformation is $\rho_{k-s}^{2s} \prod_{i=1}^{s} b_i^{i-1}$, and

$$q_i = \rho_{k-i}^2 - \rho_{k-s}^2 \prod_{j=i}^{s} b_j, \tag{A.9}$$

$$= \rho_{k-i}^2 (1 - \psi_s(i) \prod_{j=i}^{s} b_j) \; i = 1, ..., s, \tag{A.10}$$

where

$$\psi_s(i) = \rho_{k-s}^2 / \rho_{k-i}^2, \; i = 0, ..., s; \; \psi_s(s) = 1, \tag{A.11}$$

Y_iAlso,

$$f_{k-i+1}^2 + q_i = \rho_{k-i+1}^2 - \rho_{k-s}^2 \prod_{j=i}^{s} b_j$$

$$= \rho_{k-i+1}^2 (1 - \psi_s(i-1) \prod_{j=i}^{s} b_j), \; i = 1, ..., s. \tag{A12}$$

Note that

$$0 < \psi_s(0) < \psi_s(1) < .... < \psi_s(s-1) < \psi_s(s) = 1. \tag{A.13}$$

In terms of the variables $b_1, ..., b_s$,

$$H_{k,s} = 2^{-s} \rho_k^{-k} \rho_{k-s}^{k+s} [\prod_{i=1}^{s} \rho_{k-i}^2]^{-1} \int \cdots \int_{C_s} [\prod_{i=1}^{s} b_i^{i-1}]$$
$$\times [\prod_{i=1}^{s} (1 - \psi_s(i) p_i)^{\frac{k-i}{2} - 1}] [\prod_{i=1}^{s} (1 - \psi_s(i-1) p_i)^{-\frac{k-i+1}{2}}] (db_1 ... db_s), \tag{A.14}$$

where C_s denotes the unit $s-$cube, $C_s = \{b_i; i = 1, ..., s; 0 < b_i < 1\}$, and we have put

$$p_i = \prod_{j=i}^{s} b_j, \; i = 1, ..., s. \tag{A.15}$$

Now, the integral of the term involving b_1 is, from standard results,

$$\begin{aligned}
& \sum_{j_1=0}^{\infty} \frac{(\frac{k}{2})_{j_1}(1)_{j_1}}{j_1!(2)_{j_1}} (\psi_s(0)p_2)^{j_1} {}_2F_1(j_1+1, 1-\frac{k-1}{2}; j_1+2; \psi_s(1)p_2) \\
= \; & (1-\psi_s(1)p_2)^{\frac{k-1}{2}} \sum_{j_1,j_2=0}^{\infty} \frac{(\frac{k}{2})_{j_1}(j_1+\frac{k+1}{2})_{j_2}}{j_1!j_2!\Gamma(j_1+j_2+2)}, \\
& \times (1)_{j_1}(1)_{j_2}\psi_s(0)^{j_1}\psi_s(1)^{j_2}p_2^{j_1+j_2} \qquad \text{(A.16)}
\end{aligned}$$

where $(c)_j = c(c+1)....(c+j-1)$ is the usual Pocchammer symbol, and we have used the Gaussian transformation of the hypergeometric function:

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z). \qquad \text{(A.17)}$$

Multiplying (A.16) by $b_2(1-\psi_s(2)p_2)^{\frac{k-2}{2}-1}(1-\psi_s(1)p_2)^{-\frac{k-1}{2}}$ and integrating out b_2 gives:

$$\begin{aligned}
& (1-\psi_s(2)p_3)^{\frac{k-2}{2}} \sum_{j[3]=0}^{\infty} \frac{(\frac{k}{2})_{j_1}(j_1+\frac{k+1}{2})_{j_2}(j_1+j_2+\frac{k+2}{2})_{j_3}}{j_1!j_2!j_3!\Gamma(j_1+j_2+j_3+3)} \\
& \times (1)_{j_1}(1)_{j_2}(1)_{j_3}\psi_s(0)^{j_1}\psi_s(1)^{j_2}\psi_s(2)^{j_3}p_3^{j_1+j_2+j_3} \qquad \text{(A.18)}
\end{aligned}$$

Let

$$\begin{aligned}
\alpha_i &= \sum_{l=1}^{i} j_l, \; i=1,...,s, \; \alpha_0 \equiv 0, && \text{(A.19)} \\
\nu(j[s]) &= \prod_{l=1}^{s} j_l!, && \text{(A.20)} \\
(c)_{j[s]} &= \prod_{l=1}^{s} (c)_{j_l}, && \text{(A.21)} \\
\psi_s^{j[s]} &= \prod_{l=1}^{s} \psi_s(l-1)^{j_l} && \text{(A.22)}
\end{aligned}$$

Continuing as above to integrate out $b_3, ..., b_{s-1}$ we obtain:

$$(1-\psi_s(s-1)b_s)^{\frac{k-s+1}{2}} \sum_{j[s]=0}^{\infty} \frac{[\prod_{i=1}^{s}(\alpha_{i-1}+\frac{k+i-1}{2})_{j_i}]}{\nu(j[s])]\Gamma(\alpha_s+s)}(1)_{j[s]}\psi_s^{j[s]}b_s^{\alpha_s}. \tag{A.23}$$

Multiplying this by $b_s^{s-1}(1-\psi_s(s)b_s)^{\frac{k-s}{2}-1}(1-\psi_s(s-1)b_s)^{-\frac{k-s+1}{2}}$ and integrating out b_s we thus obtain:

$$H_{k,s} = \frac{\rho_{k-s}^{k+s}}{(\frac{k-s}{2})_s 2^s \rho^k [\prod_{i=1}^{s}\rho_{k-i}^2]} \sum_{j[s]=0}^{\infty} \frac{[\prod_{i=1}^{s}(\alpha_{i-1}+\frac{k+i-1}{2})_{j_i}]}{\nu(j[s])(\frac{k+s}{2})_{\alpha_s}}(1)_{j[s]}\psi_s^{j[s]}. \tag{A.24}$$

Setting $k = n-1$ and $\rho = \rho_{n-1} = \sqrt{(nu-1)/n}$ in (A.36), and referring to the result in Equation (*) in the text, we have that, if the $y_i's$ are independent $\chi^2(2)$ random variables,

$$
\begin{aligned}
pdf_U(u) &= \tfrac{1}{2}\Gamma(n)(nu-1)^{-\frac{1}{2}}V_n^*(u) \\
&= \tfrac{1}{2}\Gamma(n)(nu-1)^{-\frac{1}{2}}V_{n-1}(\rho_n) \\
&= \tfrac{1}{2}\Gamma(n)(nu-1)^{-\frac{1}{2}}\{C_{n-1}\rho_n^{n-2} + \rho_n^{n-2}\sum_{i=1}^{r}(-1)^i A_{n-1,i}C_{n-1-i}H_i(\rho_n;\phi_i)\} \\
for\ each\ r &= 0,....,n-2,\ and\ (n-r)^{-1} < u < (n-r-1)^{-1} \qquad (49)
\end{aligned}
$$