Horizontal Monopolization via Alliances, or Why a Conspiracy Admits No More Than Four

By

Jong-Say YONG

Department of Economics

National University of Singapore

Send correspondence to: Department of Economics
National University of Singapore
10 Kent Ridge Crescent
Singapore 119260
email: ecsyjs@@nus.edu.sg
Tel: (65)874-6834
Fax: (65)775-2646

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Abstract

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This paper asks the question: Is it possible for firms in an oligopolistic industry to conspire to monopolize the market by forming an alliance? We develop a modified concept of the core to take into account the possibility of outside players too may form alliances, and show that for horizontal monopolization to occur in a Cournot industry with linear demand and constant cost, there must be fewer than four firms in the industry. The corresponding result for a differentiated Bertrand industry is that the core always exists if there are fewer than three firms, whereas if there are four or more firms in the industry, the existence of the core depends on the degree of substitutability between products. It bears emphasizing that these results are not driven by the transaction costs of forming large alliances, nor the costs of coordinating and enforcing prices or production quota. Rather, it is the defection of sub-groups of firms that prevent firms from cooperating in an industry with five or more firms.

Our results contrast sharply with previous literature on horizontal alliances (in the form of horizontal mergers or cartels) which focuses on two issues: stability (e.g., Dáspremont et al. [1983]) and profitability (e.g., Salant, Switzer and Reynolds [1983]) of horizontal alliances. Both strands of literature rely almost exclusively on noncooperative game-theoretic solution concepts, and the central result being that stable and profitable alliances of varying sizes exist.

Keywords: Horizontal alliances, Core, Cooperative games, Cartel.

JEL Classification: C71, L12, L42

1 Introduction

This paper asks the question: Is it possible for firms to conspire to monopolize a market by forming an alliance? It turns out that, in the case of linear demand and constant cost, for this to happen there must be fewer than four firms in a Cournot industry. With five or more firms in the industry, any conspiracy to monopolize will not be stable in the sense of the core. The corresponding result for a Bertrand industry with differentiated products and a linear demand system is that the core is always nonempty if there are few than three firms in the industry. With four or more firms in the industry, the stability of any conspiracy will depend on the degree of substitution between products. It is worth pointing out that these results are not driven by the transaction costs of forming large alliances, nor the costs of coordinating and enforcing prices or production quota. Rather, it is the defection of sub-groups of firms that prevent firms from cooperating in an industry with large number of firms.

A horizontal alliance is a binding agreement between participating firms to coordinate strategies, e.g., to restrict output or raise prices.[1] It can take the form of a horizontal merger or a cartel agreement. Previous studies on the stability of horizontal alliances (in the form of mergers or cartels) rely almost exclusively on the notion of Nash equilibrium and its extensions as a stability concept. The seminal work is Dáspremont et al. [1983], see also Shaffer [1995], and Nocke [1999]. Their focus is on the incentive of an individual firm to deviate from the proposed equilibrium. A cartel is stable if and only if no single firm inside the cartel wants to leave, and no single firm outside the cartel wants to join. Among the well-known results is that stable cartels of varying sizes exist.

Related is the literature on the incentive for horizontal mergers, which focuses on the conditions under which a merger can be profitable for participating firms. Well-known examples are Salant, Switzer, and Reynolds [1983], Perry and Porter [1985], Deneckere and Davidson [1985], and Farrell and Shapiro [1990], among others. As in the cartel stability literature, it is shown that profitable mergers involving varying number of firms exist.

[1] The agreement may also specify transfer payments between firms. We do not concern ourselves here with the problems of how the payoffs are divided among members of an alliance.

The prediction of both strands of literature that there always exists profitable merger opportunities and stable cartels in any oligopolistic industry seems rather unappealing. If it were true, highly concentrated industries would have been the norm. In reference to their result, Deneckere and Davidson [1985] remarked that: " ... short of antitrust policy, the industry would concentrate almost completely towards monopoly." However, this clearly did not happen in the real world, even for countries with no effective antitrust policy.

We can think of at least two reasons why profitable and stable horizontal alliances may not form. Within the noncooperative game framework, Selten [1973], and more recently Prokop [1999], and Morasch [2000] model the cartel formation process and show that not all profitable and stable cartels will form in a subgame perfect equilibrium. We note, however, that due to the nature of noncooperative games, the timing and form of bargaining can lead to vastly different and even contradictory predictions about which cartels will form.

This paper offers another reason. We argue that the Nash notion of stability is too weak. It implicitly imposes an asymmetry on the ability of firms to form alliances. Specifically, firms are able to get together to form an alliance, but are not able to leave the alliance in groups. Further, firms who are outside the alliance are not able to join the alliance as a group, nor form separate alliances among themselves. In short, the Nash notion only permits deviations by individual firms. No group deviations are allowed.

To highlight the limitations of the Nash stability assumption, we offer the following parable. Imagine that there is a trading exchange where n players are engaging in some trading activities. Initially each player trades as a Nash player, and in an equilibrium each player earns the Nash equilibrium profit. Suppose now $m < n$ players enter a smoked-filled room to negotiate an alliance agreement. The important question is: how much can this alliance earn? But this clearly depends on what outside players do. Suppose outside players remain as Nash players even though they are aware that m players have entered into the smoke-filled room. The payoff for the alliance can then be determined as a Nash equilibrium between the alliance as a single entity and $(n-m)$ independent Nash players. If the alliance's payoff so determined is higher than what its members can earn collectively in a Nash equilibrium, then the alliance is deemed profitable. This is in essence the literature on the incentive for horizontal merger.

To continue with the same story, suppose a chairperson is elected in the smoked-filled room. He or she announces the payoff to all potential members, and asks individual traders in the room whether anyone wants to leave. Suppose no one leaves. Next, the chairperson opens the door, and asks individual outside players whether anyone wants to join the alliance. Suppose nobody does. The alliance is thus deemed stable. And this is in essence the focus of the cartel stability literature.

How plausible is this notion of stability? What if when the chairperson opens the door, no one is left in the trading hall? Could it not be possible that outsiders too are in another smoke-filled room negotiating their own alliance agreement? Therefore, for the Nash notion to be plausible, one must rule out this possibility. That is, one must establish that there is no incentive for outside players to get into smoke-filled rooms to negotiate their own alliance agreements. We show that this is generally *not* the case for both Cournot and Bertrand firms. Given that an alliance has formed, there is always an incentive for outside players to also form alliances when there are three or more outside players. We refer to such an equilibrium the *efficient outside alliance* (EOA) equilibrium. We then define a characteristic-function form game where each coalition computes its worth using the EOA equilibrium. The stability of the grand alliance, hence the conspiracy to monopolize, is investigated by examining whether the core of this characteristic-function game is empty.

To put our approach into context, we note that the standard notion of the core assumes that a coalition's payoff is not affected by actions of outside players. This is clearly not the case in the present context. Whether outside players form alliances or not surely has an impact on the coalition's payoff. Several approaches in the cooperative game literature has emerged to deal with this issue. One may, following Scarf [1971], assume that outside firms act to minimize the payoff of the coalition. This leads to the notions of α- and β-core, which Zhao [1997] and Zhao [2000] make use of. However, because these two notions of the core make defection difficult if at all possible, the stability of the grand alliance tends to be "exaggerated." As a result, Zhao [2000] finds that the core is always nonempty in the linear demand case. On the other hand, Chander and Tulkens [1995, 1997] propose the notion of γ-core, which is based on a partial agreement equilibrium in which outside firms act as independent Nash players. In essence, this prohibits outside players from forming alliances among themselves. As such, this can lead a coalition to over estimate its own worth. Hence defection becomes more likely, which means that the stability of the grand alliance tends to be "understated."

This paper proposes an alternative that highlights the consideration an alliance ought to make when it forms, namely that outside firms too will form alliances to protect and further their own interests. Specifically, EOA equilibrium assumes that outside firms behave in a rational and efficient manner, and are able to form alliances among themselves such that their joint payoff is maximized. Using linear demand and constant cost, we derive a characteristic function game based on the notion of EOA equilibrium, and show that for a Cournot industry, the core is always nonempty if and only if there are fewer than four firms. The corresponding result for a Bertrand industry with differentiated products is that the core is always nonempty only if there are fewer than three firms in the industry. For the case of four or more firms, the existence of the core depends on the degree of substitution between products. We thus conclude that the stability and profitability of alliances are greatly exaggerated in the literature.

We organize the paper as follows. Section 2 outlines the model. Section 3 discusses the efficient outside alliance equilibrium, and analyze the resulting characteristic-function game. Section 4 considers the case of a Bertrand industry with differentiated products. Section 5 concludes the paper.

2 The Model

Consider a Cournot industry with n firms selling homogenous goods. Let $N = \{1, \dots, n\}$ be the set of firms. Firms use identical constant return to scale production technology represented by a cost function of the form: $C(q) = cq$, where $c > 0$ is a constant. Let $q_i \in R_+$ be the output of firm i, and $Q = (q_1, \dots, q_n)$ be a strategy profile.

We assume a linear (inverse) demand function:[2]

$$D(X) = a - X,$$

[2]The slope is set to unity to simplify algebraic expressions that follow. This specification can accommodate any general linear functions by suitably rescaling the unit of measurement.

where $X = \sum_{i \in N} q_i$ is the industry output.

Define a monopoly outcome as the joint profit maximization outcome, where all firms collectively choose $Q \in R^n_+$ to maximize $\Pi(X) = XD(X) - C(X)$. We refer to this as the cooperative outcome, in which all firms conspire to monopolize the industry by forming a grand alliance. Given the particular forms of demand and cost functions, the monopolist choosing Q is no different from it choosing X, hence we shall define the maximization problem in terms of X. The profit-maximizing total output is $X^M = (a - c)/2$, and the monopoly profit is $\Pi^M = (a - c)^2/4$.

We refer to the Cournot outcome as the Nash equilibrium outcome in which $n \geq 2$ independent firms competing in quantities. Each firm chooses q_i to maximize $\pi_i = q_i D(X) - C(q_i)$. Given the assumptions on cost and demand, the Cournot outcome is unique, and is given by $q_i^c = a - c/(n+1)$, in which each firm earns

$$\pi^c = \frac{(a-c)^2}{(n+1)^2}.$$

It is obvious that in the Cournot outcome firms collectively earn strictly less than that under monopoly for all $n \geq 2$. On the surface this seems to suggest that firms ought to form a single alliance. The question is whether such an alliance is stable. We make two remarks, which explain why we are unable to extend the Nash notion of stability by making use of solution concepts such as strong Nash equilibrium, and coalition-proof Nash equilibrium (CPNE) in our present context.

(1). The game has no strong Nash equilibrium. The reason is that a strong Nash equilibrium is a Nash equilibrium that is also Pareto efficient. Given that the Cournot outcome does not maximize the joint profit for all firms, it is not Pareto efficient (from the firms' perspective), hence it is not a strong Nash equilibrium. Since there is no other Nash equilibrium, we conclude that a strong Nash equilibrium fails to exist.

(2). Any CPNE, if it exists, cannot be different from the Cournot outcome. The reason is that a CPNE is always a Nash equilibrium, and since the Nash equilibrium is unique, the conclusion follows immediately.

We therefore turn to the idea of the core as a stability requirement for the alliance. We define an *alliance* as a nonempty subset $S \subset N$, and an *alliance structure*, denoted B, as a partition of N. We refer to the *size* of an alliance $S \subset N$ as the number of members that S has, and we refer to N as the *grand alliance*.

The joint strategy of an alliance $S \subset N$ of size m is a vector $Q_S \equiv (q_1, \dots, q_m) \in R_+^m$. Let $X_S = \sum_{i \in S} q_i \in R_+$ be the total output produced by S. Define $Q_{N \setminus S} \equiv (q_i)_{i \in N \setminus S}$ as the strategy profile of all outside firms, and $X_{N \setminus S} \equiv \sum_{i \in N \setminus S} q_i$ as the total output of all outside firms. The payoff of the alliance S is thus $\Pi_S : R_+^n \to R$, where

$$\Pi_S(Q_S, Q_{N \setminus S}) = X_S D(X_S + X_{N \setminus S}) - C(X_S).$$

Since all firms have identical and constant average cost, it involves no loss of generality treating the alliance's strategy as X_S. Define a characteristic function form game $\Gamma = \langle N, w(\cdot) \rangle$, where $w(S)$, called the *worth* of alliance S, is given by

$$w(S) = \max_{X_S \in R_+} \Pi_S(X_S, Q_{N \setminus S}).$$

Clearly, the worth of S depends on the strategies chosen by players not in S. As mentioned earlier, the notions of α- and β-core assume that the alliance S computes its worth under the worst-case scenario, in that outside players act to minimize the worth of S. See Scarf [1971], and also Zhao [2000]. However, these notions are of little use in the present context since they require outside firms to produce unrealistically large quantities of output and suffer large losses. Recognizing this problem, Chander and Tulken [1997] propose a partial agreement equilibrium in which outside players act as independent Nash players. This implies, however, that outside players are not allowed to form alliances among themselves even if it is in their interests to do so. This asymmetry in the firms' abilities to form alliances seems artificial in the present context, since it is generally not in the outside players' best interest to *not* form any alliances. Given that all firms are rational profit-maximizing agents, and free to negotiate and renegotiate alliance agreements either individually or in groups, they ought to be able to arrive at the best possible alliance arrangement. With this in mind, we propose that outside players should be forming alliances and choosing output in such a way that their joint payoff is maximized. We call these efficient outside alliances.

3 Efficient Outside Alliances

Given an alliance structure B consisting of k alliances, a Nash equilibrium between these alliances is a strategy profile $(X_S^*)_{S\in B}$ such that

$$X_S^* = \underset{X_S \in R_+}{\text{argmax}}\ \Pi_S(X_S, \text{-}^*_{N\setminus S}),$$

where $\text{-}^*_{N\setminus S} = (X_J^*)_{J\in B, J\neq S}$ is a $(k-1)$ vector, and $X_J^* = \sum_{i\in J} q_i^*$ is the total output of alliance J. For the linear demand and constant cost functions considered above, a Nash equilibrium with k alliances is no different from a Nash equilibrium with k firms. The equilibrium quantities and payoffs are, for all $S \in B$,

$$X_S^* = \frac{a-c}{k+1}, \tag{1}$$

$$\Pi_S^* = \frac{(a-c)^2}{(k+1)^2}. \tag{2}$$

Note that these alliances do not necessarily consist of the same number of firms, and some alliances may have only a single member. We next define an efficient outside alliance equilibrium.

Definition: Efficient outside alliance (EOA) equilibrium
Given an alliance $S \subset N$, an *efficient outside alliance (EOA) equilibrium* is a Nash equilibrium between S and h^* outside alliances where h^* is such that the joint payoff of all players not in S is maximized.

Let $B\setminus S$ be the set of $(k-1)$ outside alliances. For convenience, we define $h = (k-1)$. To find the EOA equilibrium, we look for X_J^* that maximizes Π_J for all $J \in B$, and a h^* that maximizes $\sum_{J\in B\setminus S} \Pi_J^*$. For any h, given the linear demand and constant cost functions, equilibrium quantities and payoffs are given by (1) and (2). The combined payoffs of the h outside alliances are

$$\sum_{J\in B\setminus S} \Pi_J^* = \frac{h(a-c)^2}{(h+2)^2}. \tag{3}$$

In an EOA equilibrium, S computes its payoff based on the presumption that outside firms too are able to form alliances among themselves to further their own interests. We assume that they form alliances among themselves in such a way that the number of outside alliances h^* maximizes (3).

Note that if h is to be a continuous variable, (3) would be strictly concave in h, and the first-order condition would give rise to $h^* = 2$. Since the solution is an integer, this result simply carries over to the integer case. Hence $h^* = 2$ gives the maximum joint payoff for outside firms. That is, for $(n - m) \geq 2$, outside firms should form exactly two alliances to maximize their joint payoff. We state Proposition 1

Proposition 1 *Suppose an alliance $S \subset N$ of size $m < n$ forms. Given the linear demand and constant cost specified above, there exists a unique EOA equilibrium in which S earns*

$$\Pi_S(m) = \begin{cases} (a-c)^2/16 & \text{if } (n-m) \geq 2, \\ (a-c)^2/9 & \text{if } (n-m) = 1. \end{cases} \tag{4}$$

Proof

For $(n - m) = 1$, there are only two alliances in the industry, and their payoffs can be computed easily from (2) by setting $k = 2$. For $(n - m) \geq 2$, we have $h^* = 2$ in an EOA equilibrium, thus the payoff of each alliance can be computed from (2) by setting $k = 3$. ■

For an alliance $S \subset N$ of size $m \geq 1$, define the per-member payoff as $\pi(m) \equiv \Pi_S(m)/m$. Then, it can be readily verified that

$$\pi(1) \geq \pi(m) \quad \text{for all} \quad 2 \cdot \quad m < n. \tag{5}$$

This observation will prove useful later in establishing the existence of the core.

We define a characteristic function game $\Gamma_\lambda = \langle N, w^\lambda(\cdot)\rangle$, where for an alliance $S \subset N$ of size $m \geq 1$, $w^\lambda(S) = \Pi_S(m)$ is given by (4), and $w^\lambda(N) = \Pi^M$. Note that Γ_λ is symmetric and superadditive. It is sometimes convenient to use the notation $w(m)$ to denote the worth of a size-m alliance.

To find the core of Γ_λ, we let $Y = \{y_1, \ldots, y_n\}$ be a payoff allocation. Then, it is well-known that Y is in the core of Γ_λ if and only if the following two conditions hold:

$$\sum_{i \in S} y_i \geq w(S), \quad \text{for all } S \subset N, \tag{6}$$

$$\sum_{i \in N} y_i = w(N). \tag{7}$$

Making use of (5), we have the following Proposition:

Proposition 2 *Let $Y = (y_1, \ldots, y_n)$ be a payoff allocation. The core of the game Γ_λ is non-empty if and only if (7) and the following condition hold:*

$$y_i \geq w(1) \quad \textit{for all} \quad i \in N. \tag{8}$$

Proof

The "only if" part of the statement is obvious from (6). We prove the "if" part of the statement by noting that if $y_i \geq w(1)$ for all $i \in N$, then for any $S \subset N$ of size $m \geq 2$, summing both sides of (8) over all $i \in S$ yields

$$\sum_{i \in S} y_i \geq mw(1).$$

It follows from the observation in (5) that

$$\sum_{i \in S} y_i \geq m\pi(1) \geq m\pi(m) = \Pi_S(m) = w(S). \blacksquare$$

The next Proposition shows that for the game Γ_λ, the core is nonempty if and only if there are fewer than four firms in the industry.

Proposition 3 *For the game Γ_λ, the core is nonempty if and only if $n \cdot\ 4$.*

Proof

We prove the "if" part of the proposition by construction. Define an equal-share allocation $\bar{Y} = (y, \ldots, y)$, a n-vector where the payoffs of all players are identical and equal y. For $\bar{Y}$ to be in the core, it must satisfy (7) and (8), which after substituting (4), becomes

$$y = \frac{(a-c)^2}{4n} \quad \text{for all} \quad n \geq 2, \tag{9}$$

$$y \geq \begin{cases} (a-c)^2/16 & \text{if} \quad n \geq 3 \\ (a-c)^2/9 & \text{if} \quad n = 2. \end{cases} \tag{10}$$

It is straightforward to verify that (9)–(10) hold for $n \cdot\ 4$.

We show the "only if" part of the proposition by showing that the core is empty for $n \geq 5$. Suppose not. Let $Y = (y_1, \ldots, y_n)$ be a core allocation. Then, it must be true that

$$\sum_{i=1}^{n} y_i = (a-c)^2/4. \tag{11}$$

But individual rationality requires that $y_i \geq (a-c)^2/16$. Summing over all i gives

$$\sum_{i=1}^{n} y_i \geq n(a-c)^2/16,$$

which clearly contradicts (11) for all $n \geq 5$. ■

We say that an alliance structure B is core-stable if there exists a payoff allocation Y such that for all $S \in B$, $\sum_{i \in S} y_i = w(S)$, and that Y is in the core. Intuitively, an alliance structure is core-stable if no group of firms, whether from the same alliance or from different alliances, can get together and form a new alliance in such a way that all members of the new alliance can be made better off.[3] Obviously, if the core of a game is empty, there cannot be any alliance structure that is core stable. We refer to the grand alliance as the alliance which consists of all firms in the industry. We next show that the grand alliance is the only core-stable alliance structure, if one exists.

Proposition 4 *For n · 4, the grand alliance is the only alliance structure that is core stable.*

Proof

The proof is by contradiction. Suppose there exists an alliance structure B^* which consists of $L \geq 2$ alliances that is also core stable. Let $Y = (y_1, \ldots, y_n)$ be the corresponding core allocation.

Let Π_S be the total payoff received by members of an alliance $S \in B^*$ in an EOA equilibrium. Given the linear demand and constant cost,

$$\Pi_S = \frac{(a-c)^2}{(L+1)^2}.$$

Given that Y is a core allocation, the sum of all firms' payoffs must equal the total industry profit:

$$\sum_{j \in N} y_j = \sum_{S \in B^*} \Pi_S = \frac{L(a-c)^2}{(L+1)^2}.$$

[3]See Greenberg [1994], Definition 6.1, p.1316. This notion of core stability is used, for example, in the study of partnership by Farrell and Scotchmer [1988].

But suppose now all L alliances merge into a single grand alliance, then collectively they earn the monopoly profit, which is $\Pi^M = (a-c)^2/4$. Taking the difference, we have

$$\Pi^M - \sum_{S \in B^*} \Pi_S = \frac{(L+1)^2 - 4L}{4(L+1)^2} \cdot (a-c)^2 > 0,$$

since by supposition $L \geq 2$. Therefore it is possible to strictly improve the payoff of every member of every alliance by merging all alliances into a single grand alliance. For example, if the gain is to be distributed equally among all players, then each player can obtain an additional amount of

$$\frac{1}{n}[\Pi^M - \sum_{S \in B^*} \Pi_S] > 0.$$

This means that every member of the new alliance can be made better off, a contradiction. ■

We have therefore shown that for a homogenous Cournot industry with linear demand and identical firms, a conspiracy to monopolize the industry is unlikely to succeed unless there are fewer than four firms in the industry. One may interpret the result by treating core stability as a necessary condition for alliance formation, as in Zhao [2000]. Thus a non-empty core merely means that it is *possible* for firms to collude. Whether they will indeed do so or not then hinge on other factors such as the compatibility of corporate cultures, personalities of CEOs, and so on.

4 Differentiated Bertrand Competition

We next consider the case of Bertrand competition with differentiated products. Let there be n firms and n products where each firm produces one product at constant marginal cost c. Let $N = \{1, \ldots, n\}$ be the set of firms.[4] We specify a linear demand system as in Deneckere and Davidson [1985], where the demand for product i is:[5]

$$q_i(P) = a - p_i - b(p_i - \overline{p}), \quad i = 1, \ldots, n,$$

where $P \equiv (p_1, \ldots, p_n)$, $\overline{p} = \sum_{j \in N} p_j$ is the industry average price, and $a, b > 0$ are parameters. Of particular importance is the parameter b, which captures the degree of

[4]We also use the same notation N below for the set of products. This will be clear from the context and should cause no confusion.

[5]This demand system is also found in Shubik [1980].

substitution between products. As b approaches 0, products become non-substitutable, and each firm is a monopoly in its own market. On the other hand, as b approaches ∞, products become perfect substitutes, and competition becomes standard Bertrand.

Without loss of generality, set $c = 0$. We refer to the cooperative outcome as one in which all firms merge to become a single entity, denoted as M. It is straightforward to show that, to maximize profit, M sells each product $i \in N$ at price $p_i^M = a/2$, and it earns the monopoly profit:

$$\Pi^M = na^2/4.$$

Next, consider a general alliance structure, B, which is a partition of N into k nonempty subsets. The joint strategy of an alliance $S \in B$ is a vector $P_S \equiv (p_j)_{j\in S} \in R_+^{|S|}$, and the payoff of the alliance is a function $\Pi_S : R_+^{|S|} \to R$, where

$$\Pi_S = \max_{P_S} \sum_{j\in S} p_j q_j(P).$$

Define the Nash equilibrium between the k alliances as a strategy profile $(P_S^*)_{S\in B}$ such that

$$P_S^* = \operatorname*{argmax}_{P_S \in R_+^{|S|}} \Pi_S(P_S, P_{N\backslash S}^*) = \sum_{j\in S} \operatorname*{argmax}_{p_j} p_j\, q_j(P),$$

where $P_{N\backslash S}^* \equiv (p_i^*)_{i\notin S}$.

Proposition 5 *Suppose an alliance $S \subset N$ forms. Then in an EOA equilibrium all firms not in S form a single alliance.*

Proof

Recall that an EOA equilibrium is a Nash equilibrium between S and h^* outside alliances where h^* is such that the joint payoff of all players not in S is maximized. We prove $h^* = 1$ by showing that for any given alliance structure B with $k \geq 2$ alliances, the merger of any two alliances, B_i and B_j, increase the joint profit of the merged entity $(B_i \cup B_j)$, i.e., $\Pi(B_i \cup B_j) > \Pi(B_i) + \Pi(B_j)$. To show this, we apply the argument of Deneckere and Davidson [1985].[6] Because of the negative externalities in setting prices, a merger allows

[6]Specifically, their Theorem 3 and the Corollary to Theorem 4.

the merged entity to internalize the externalities by raising prices of their products. This must increase their joint profit. Furthermore, since reaction functions are upward sloping, higher prices charged by the merged firms will allow all other firms to raise their prices. This will in turn let the merged entity to raise their prices even further, and hence their profitability will further improve. ■

Therefore, for any given $S \subset N$, the EOA equilibrium is simply a Nash equilibrium between two alliances: S and $T \equiv N \backslash S$. Let (P_S^*, P_T^*) denote the vector of equilibrium prices. For an alliance S of size m, we define the equilibrium per-member profit as

$$\pi_S = \Pi_S / m.$$

Using the result of Deneckere and Davidson [1985], we can easily show that the smaller of the two alliances earns more per member in an EOA equilibrium.

Proposition 6 *Let S and T be two alliances with respective sizes m and $(n-m)$ in an EOA equilibrium. Then, $\pi_S > \pi_T$ if and only if $m < (n-m)$.*

Proof
The proof follows immediately from Theorem 3 of Deneckere and Davidson [1985]. ■

Given the EOA equilibrium for any $S \subset N$ is well-defined, we can define a characteristic-function game:

$$\Gamma_\lambda = \langle N, w^\lambda(\cdot) \rangle,$$

where $w^\lambda(S) = \Pi_S$. Since all firms are identical, the game Γ_λ is symmetric, and we can write the worth of an alliance S as a function of its size. Of particular relevance is the worth of single-member alliances, which is given by:

$$\begin{aligned} w(1) &= \max_{p_j \in R_+} \; p_j \, q_j(p_j, P_{-j}^*) \\ &= \frac{na^2(2n + b(n+1))[2n^2 + nb(3n-1) + b^2(n^2-1)]}{[4n^2(1+b) + 3b^2(n-1)]^2}, \end{aligned}$$

where $P_{-j}^* = (p_1^*, \ldots, p_{j-1}^*, p_{j+1}^*, \ldots, p_n^*)$. The next proposition shows that the existence of the core depends on the parameter b, which as mentioned above, is a measure of the degree of substitutability between products.

Proposition 7 *For the game Γ_λ, the core is non-empty if and only if $b \geq \max\{0, b^0\}$, where*

$$b^0 = \frac{2n[(n-2)^2 - 3 + (n-2)\sqrt{(n-2)^2+3}]}{9(n-1)}.$$

Proof

Let m be the size of the alliance $S \subset N$, $1 \cdot \; m \cdot \; n$. There are thus $\binom{n}{m}$ different alliances of size m. Let $\Xi(m)$ be the set of all size-m alliances.

Pick any player $j \in N$. We count the number of alliances of size m that player j belongs, i.e., those $S \in \Xi(m)$ which contain player j. For such alliances, since player j is a member, the other $(m-1)$ members must come from the set $N \backslash \{j\}$. Thus, there are $\binom{n-1}{m-1}$ different ways of constituting a size-m alliance with firm j as a member.

Let $Y = (y_1, \ldots, y_n)$ be a core allocation. Recall that for the core to be non-empty, both (6) and (7) must hold. We fix the size of S at m, then (6) defines $\binom{n}{m}$ inequalities for all S of size m. Summing over all such inequalities, we have the right-hand side as

$$\binom{n}{m} w(S).$$

The left-hand side becomes:

$$\sum_{S \in \Xi(m)} \sum_{i \in S} y_i = \sum_{i \in N} \sum_{S \in \Xi(m), S \ni i} y_i = \binom{n-1}{m-1} \sum_{i \in N} y_i,$$

where use is made of the fact that $N = \cup_{S \in \Xi(m)} S$. Equating the left- and right-hand sides, and making use of (7), we then have

$$\binom{n-1}{m-1} w(N) \geq \binom{n}{m} w(S),$$

which can be simplified to

$$mw(N) \geq nw(S), \tag{12}$$

for all S of size m. For the core to be nonempty, (12) must hold for S of all sizes $m = 1, \ldots, n$. By Proposition 6,

$$w(1) \geq w(S)/m$$

for all S with $m \geq 2$ members. Therefore, we can express (12) as

$$mw(N) \geq nw(1). \tag{13}$$

Substituting the values of $w(N)$ and $w(1)$, we have $b \geq b^0$, but since $b \geq 0$ by assumption, we have the "if" part of the result.

To show the "only if" part of the proposition, we show that the core is nonempty if $b \geq \max\{0, b^0\}$, i.e., if (13) holds. Let $\bar{Y} = (y, \dots, y)$ be an equal-share allocation in which all firms receive the identical payoff y, where $y = w(N)/n$. Then for any $S \subset N$ of size m,

$$\sum_{i \in S} y = mw(N)/n \geq w(1),$$

where the inequality follows from the supposition. By Proposition 6, we have

$$\sum_{i \in S} y \geq w(S)$$

for all $S \subset N$. ■

The condition $b \geq b^0$ is depicted in Figure 1, from which we can draw an immediate corollary.

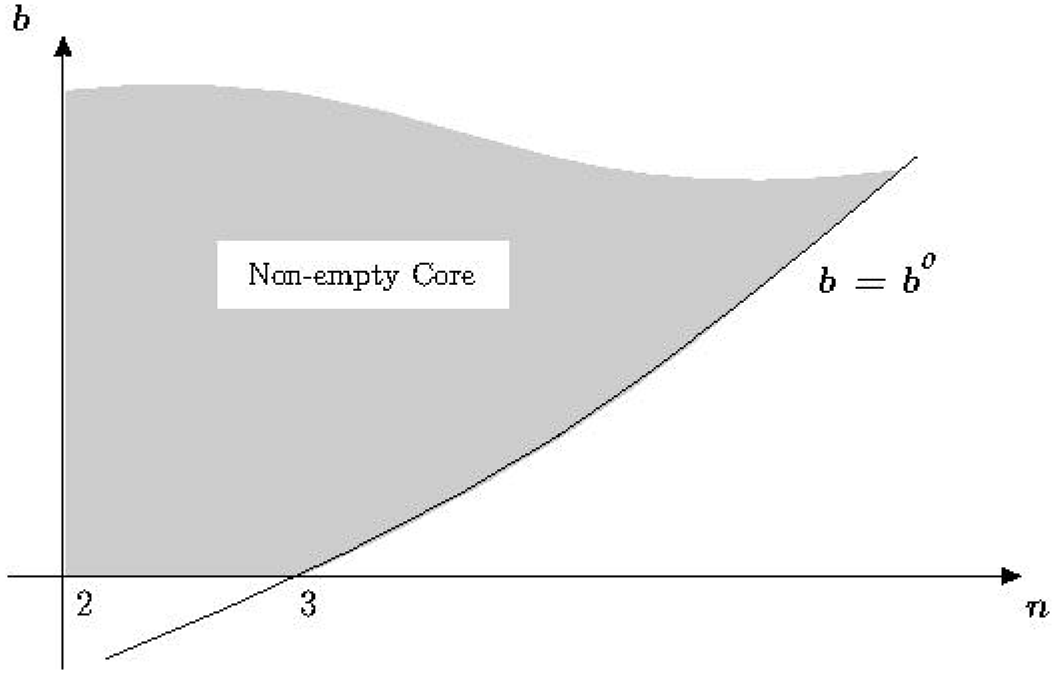

Figure 1: Condition for Non-empty Core

Corollary *For $n \cdot$ 3, the game Γ_λ always has a nonempty core.*

Proof

It can be easily verified that $b^0 < 0$ for $n = 2, 3$. Thus, the condition for nonempty core is satisfied for all nonnegative values of b. ■

Intuitively, a Bertrand firm that remains independent free rides on the benefit of higher prices charged by other firms who choose to form alliances. As n increases, the free-riding benefit goes up. Although as n goes up the monopoly profit increases as well, it is not enough to offset the increase in free-riding benefit. We illustrate this with a numerical example where we let $a = 10$. If all firms merge to become a monopoly, the equal-share allocation is $\pi^M/n = 25$ for all n. Thus, the core is empty if a firm that chooses to remain independent can earn more than this amount. Table 1 shows that, in an EOA equilibrium, the profit of an independent firm is increasing in n, but the same profit at first increases then decreases in b. In particular, at low values of b, as n increases from 3 to 4, the independent firm's payoff begins to cross over the threshold of 25. However, at higher values of b (i.e., when products become close substitutes), this crossing over occurs at a larger n.

	$b = 0.1$	$b = 0.5$	$b = 1.0$	$b = 2.0$	$b = 5.0$
$n = 2$	24.985	24.691	24.000	22.222	17.284
$n = 3$	24.999	24.933	24.655	23.623	19.681
$n = 4$	25.010	25.127	25.212	24.931	22.316
$n = 5$	25.017	25.268	25.632	25.978	24.691
$n = 10$	25.035	25.611	26.697	28.870	32.719
$n = 20$	25.046	25.815	27.360	30.853	39.784

Table 1: EOA Equilibrium Payoff of an Independent Firm

We make two remarks:

(1). As $b \to \infty$, the competition approaches symmetric Bertrand with no differentiation, and the core is always non-empty. This is obvious as any firm which refuses to join the grand alliance can only earn zero payoff in a standard Bertrand.

(2). At $b = 0$, each firm is an isolated monopoly. There is no interaction between products, hence there is no free-riding benefit. A firm's payoff between joining and not joining the grand alliance is the same. Thus the core consists of all possible partition of N. Note that a firm that chooses to remain independent does not enjoy any advantage.

It is instructive to compare the results here with those of the previous section. Under Cournot competition, when $m < n$ firms form an alliance, they internalize the negative externalities arising from their quantity choices by restricting their combined output. If other firms were to produce at the same output levels, the firms in the alliance would clearly have been better off (and so would the outside firms). However, the outside firms can benefit even more by raising their output levels, and since quantities are strategic substitutes, this hurts the alliance. In the linear case, firms who choose to form the alliance are always worse off. Thus, in an EOA equilibrium, of $n \geq 5$ Cournot firms, a firm is always better off remaining independent, and since every firm is identical, no alliances will form. In particular, joining the grand alliance, thereby getting a share of the monopoly profit, is never an attractive proposition unless $n \cdot$ 4.

5 Conclusion

The paper proposes a new equilibrium notion between alliances, which we refer to as the efficient outside alliance (EOA) equilibrium. In this equilibrium, we allow outside firms to respond to the formation of an alliance by forming their own alliances. We assume that outside firms respond in such a way that their joint profit is maximized. Based on this equilibrium notion, we propose a modified version of the gamma-core solution concept of Chander and Tulkens [1997]. This core solution is applied to study the stability of horizontal alliances in both Cournot and differentiated Bertrand industries. The results suggest that the stability and profitability of horizontal mergers and alliances have been grossly over-exaggerated in the literature. In the linear demand case, the core under both Cournot and Bertrand (with differentiated products) competition is empty unless the industry consists of only four or fewer firms. Our results also contradict sharply with those of Zhao [2000], who shows that the core is nonempty in the linear case using the notions of α- and β-core.

It should be emphasized that we have deliberately avoided considering cost synergies between firms for the sake of simplicity. However, the framework can be readily extended to allow for such cases by following, for example, Farrell and Shapiro [1990]. We conjecture that, with cost synergies, the possibility of profitable and stable horizontal alliances should expand, hence a conspiracy would be able to admit more firms.

A more serious limitation of the Cournot model is the complete symmetry between firms and alliances. As such, an alliance that comprises several firms is no different from a single firm or any other alliances in the industry. Such is clearly not the case in the real world, where an alliance of several firms frequently enjoys a size advantage over its non-allied competitors. A simple way to introduce more realism into the model is perhaps to specify capacity constraints for firms. Further research will be needed to determine whether our results hold if the model is extended in this direction.

Bibliography

[1] Chander, P. and H. Tulkens [1995], "A Core-Theoretic Solution for the Design of Cooperative Agreements on Transfrontier Pollution," *International Tax and Public Finance*, 2(2), 279–93.

[2] Chander, P. and H. Tulkens [1997], "The Core of an Economy with Multilateral Environmental Externalities," *International Journal of Game Theory*, 26, 379–401.

[3] Dáspremont, C., A. Jacquemin, J.J. Gabszewicz, and J.A. Weymark [1983], "On the Stability of Collusive Price Leadership," *Canadian Journal of Economics*, 16(1), 17–25.

[4] Deneckere, R. and C. Davidson [1985], "Incentives to Form Coalitions with Bertrand Competition," *Rand Journal of Economics*, 16, 473–86.

[5] Farrell, J. and C. Shapiro [1990], "Horizontal Mergers: An Equilibrium Analysis," *American Economic Review*, 80(1), 107–26.

[6] Nocke, V. [1999], "Cartel Stability Under Capacity Constraints: The Traditional View Restored," *Working Paper*, Nuffield College, Oxford.

[7] Morasch, K. [2000], "Strategic Alliances as Stackerbelg Cartels–Concept and Equilibrium Alliance Structure," *International Journal of Industrial Organization*, 18, 257–82.

[8] Perry, M.K. and R.H. Porter [1985], "Oligopoly and the Incentive for Horizontal Merger," *American Economic Review*, 75(1), 219–27.

[9] Prokop, J. [1999], "Process of Dominant-Cartel Formation," *International Journal of Industrial Organization*, 17, 241–57.

[10] Salant, S.W., S. Switzer, and R.J. Reynolds [1983], "Losses from Horizontal Merger: The Effects of an Exogenous Change in Industry Structure on Cournot-Nash Equilibrium," *Quarterly Journal of Economics*, 98, 185–99.

[11] Scarf, H. [1971], "On the Existence of a Cooperative Solution for a General Class of N-person games," *Journal of Economic Theory*, 3, 169–81.

[12] Selten, R. [1973], "A Simple Model of Imperfect Competition, Where 4 are Few and 6 are Many," *International Journal of Game Theory*, 3, 141–201.

[13] Shaffer, S. [1995], "Stable Cartels with a Cournot Fringe," *Southern Economic Journal*, 61, 744–54.

[14] Shubik, M. [1980], *Market Structure and Behavior*, Camb.: Harvard Univ. Press.

[15] Zhao, J. [1997], "A Cooperative Analysis of Covert Collusion in Oligopolistic Industries," *International Journal of Game Theory*, 26, 249–66.

[16] Zhao, J. [2000], "Non-empty Core as a Precondition for Horizontal Mergers," *Working Paper*, Department of Economics, Ohio State University.