Closed Form Solution of Non Linear Rational Expectations Model

On the Role of Interest Rate on Investment

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Abstract

In this paper we discuss an attractive solution of a standard non linear dynamic structural model, and apply it both to model agregate investment behaviour and to assess the effects of an interest rate policy on this variable.

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1. Introduction

The famous criticism of Lucas stated 25 years ago on the way to build up clean models (see Lucas, 1976), has forced a generation of researchers to work very carefully when doing their jobs. One of the more crucial implications of this warning is the recognition that a good model should emphasize only structural parameters and not expectation parameters. Indeed, by definition, the former ones are invariant with respect to economic policy changes, while the latter ones escape from this property since economic agents are rational. Although this review has effectively strived with the drift of excess on the reliability put on much economic models, some new difficulties arisen from it. A pure structural approach allows to stay as closed as possible to the *conditions* ruling out the optimal behavior, but this may be at the expense of the characterization of *the solution itself.* As a consequence, policy lessons a model is instructed to provide may be severely limited. This issue is especially true for investment modelling, where the presence of both high dynamics and non linearity does not often lay out an easy road towards a closed form where policy exercises or forecasting can be conducted.

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In this paper we derive a solution to a standard non linear dynamic structural model with rational expectations. This solution is obtained by assuming stochastic movements for exogenous variables, entailing a particular and attractive behavior for the variable causing non linearity. The result is exploited as a policy exercise to assess the role of interest rate on fixed investment decisions in a neoclassical context with convex adjustment costs technology.

2. A solution to non linear model

Let us consider the following non linear dynamic structural equation

$$y_t = \alpha E\left[b_{t+1}y_{t+1} \mid \Omega_t\right] + \beta E\left[b_{t+1}x_{t+1} \mid \Omega_t\right] \tag{2.1}$$

with y_t the endogenous variable; b_t and x_t two independent exogenous variables, b_t causing non linearity in (2.1), $0 < b_t < 1$; α and β two structural parameters, $0 < \alpha < 1$; $E\left[. \mid \Omega_t\right]$ the expectation of . conditional to the set of available information denoted $\Omega_t \equiv \{y_{t-1}, ..., y_1, b_t, b_{t-1}, ..., b_1, x_t, x_{t-1}, ..., x_1\}$ or $\Omega_t \equiv \left\{\underline{y_{t-1}}, \underline{b_t}, \underline{x_t}\right\}$.

A reduced form equation for (2.1) is easily obtained by conducting a standard looking-forward procedure. Indeed, writing (2.1) at period $t+1$, pre-multiplying each sides by b_{t+1}, taking expectations with respect to Ω_t and using the law of iterated expectations, leads to

$$E[b_{t+1}y_{t+1} \mid \Omega_t] = \alpha E[b_{t+1}b_{t+2}y_{t+2} \mid \Omega_t] + \beta E[b_{t+1}b_{t+2}x_{t+2} \mid \Omega_t] \tag{2.2}$$

so that (2.1) becomes

$$y_t = \alpha^2 E[b_{t+1}b_{t+2}y_{t+2} \mid \Omega_t] + \alpha\beta E[b_{t+1}b_{t+2}x_{t+2} \mid \Omega_t] + \beta E[b_{t+1}x_{t+1} \mid \Omega_t] \tag{2.3}$$

Dealing similarly with all successive future endogenous variable terms, and using conditions on α and b_t, we end up with

$$y_t = \frac{\beta}{\alpha}\sum_{j=1}^{\infty}\alpha^j E\left[\left(\prod_{i=1}^{j} b_{t+i}\right) x_{t+j} \mid \Omega_t\right] \tag{2.4}$$

Next, to get an observable solution of (2.4), we need to make some assumptions on the stochastic behavior of the exogenous variables. For x_t we opt for the usual autoregressive representation, retaining for simplicity the $AR(1)$ process, so that

$$x_{t+j} = \varphi x_{t+j-1} + a_{t+j} \tag{2.5}$$

for all $j > 0$, with $0 < \varphi \leq 1$ and a_t a white noise with variance σ_a^2.

>From (2.5) the forecasting function of x_{t+j} based on x_t is given by

$$\begin{aligned} x_{t+j} &= \varphi^j x_t + \sum_{k=1}^{j}\varphi^{j-k}a_{t+k} \\ &= g_j\left[x_t, a_{t+1}, ..., a_{t+j}; \varphi\right] \quad \text{(say)} \end{aligned} \tag{2.6}$$

for all $j > 0$.

Now the choice for a stochastic process for variable b_t is crucial. The problem is find a process which allows an exact solution for (2.4). Remaining in a representation like (2.6), we propose to investigate the following specific generating mechanism for the future of b conditional on b_t:

(i) for $j = 1$

$$\begin{aligned} b_{t+1} &= \phi b_t + e_{t+1} \\ &= f_1 \left[b_t, e_{t+1}; \phi \right] \qquad \text{(say)} \end{aligned} \tag{2.7}$$

and (ii) for $j > 1$

$$\begin{aligned} b_{t+j} &= \frac{(\phi b_t)^j + \sum\limits_{k=1}^{j} \phi^{j-k} e_{t+k}}{(\phi b_t)^{j-1} + \sum\limits_{k=1}^{j-1} \phi^{j-1-k} e_{t+k}} \\ &= f_j \left[b_t, e_{t+1}, ..., e_{t+j}; \phi \right] \qquad \text{(say)} \end{aligned} \tag{2.8}$$

with $0 < \phi \leq 1$, and again e_t a white noise with variance σ_e^2 *but having the property to be uncorrelated with* a_t to characterize the independence assumption between b and x.

Clearly a constant behavior of b_t is checked when $\phi = 1$ and all error terms $e = 0$. More importantly combining (2.7) and (2.8) leads to the following compact process

$$\prod_{i=1}^{j} b_{t+i} = (\phi b_t)^j + \sum_{k=1}^{j} \phi^{j-k} e_{t+k} \tag{2.9}$$

for all $j > 0$.

Under stochastic movements (2.5) and (2.9), equation (2.4) may then be written as

$$y_t = \frac{\beta}{\alpha} \sum_{j=1}^{\infty} \alpha^j E \left[\left\{ (\phi b_t)^j + \sum_{k=1}^{j} \phi^{j-k} e_{t+k} \right\} \left\{ \varphi^j x_t + \sum_{k=1}^{j} \varphi^{j-k} a_{t+k} \right\} \mid \Omega_t \right] \tag{2.10}$$

and, solving all conditional expectations due to independence properties, we have

$$y_t = \frac{\beta}{\alpha} \left\{ \sum_{j=1}^{\infty} (\alpha \phi \varphi b_t)^j \right\} x_t \tag{2.11}$$

A solution of (2.1) is hence the simple non linear relation

$$y_t = \left\{ \frac{\beta}{(\phi \varphi b_t)^{-1} - \alpha} \right\} x_t \tag{2.12}$$

3. Application to investment

Result (2.12) can be attractively exploited when turning to the aggregate investment issue. Indeed this resolution procedure may help to emphasize the impact of a stochastically-treated interest rate on investment *within a reduced form allowing an economic policy-like exercise.* This closed form will be the solution of a well-known non linear dynamic structural specification with rational expectations.

>From a neoclassic point of view, the firm value is based on an optimization behavior of the total amount of (rationally) expected future discounted flows of profits

$$V_t = \max \sum_{j=0}^{\infty} E\left[\left(\prod_{i=0}^{j} b_{t+i}\right)\left\{p'_{t+j}Y[K_{t+j-1}] - I_{t+j} - G[I_{t+j}]\right\} \mid \Omega_t\right] \tag{3.1}$$

with $b_t = (1+r_t)^{-1}$, r_t the interest rate, p'_t the price of output relative to the price of investment, i.e. p_t/p^I_t, $Y[K_{t-1}]$ the output produced with beginning-period capital stock, I_t investment expenses, $G[I_t]$ a convex function reflecting the installation costs of the I_t units of investment (see Eisner and Strotz, 1963) and $\Omega_t \equiv \left\{\underline{r_t}, \underline{p'_t}, \underline{K_{t-1}}\right\}$.

The objective function is maximized under the usual dynamic accumulation equation

$$K_t = (1-\delta)K_{t-1} + I_t \tag{3.2}$$

and, for simplicity, by assuming the following explicit form for both technologies

$$Y[K_{t-1}] = \theta K_{t-1}, \qquad G[I_t] = \frac{\gamma}{2} I_t^2 \tag{3.3}$$

Optimal conditions provides Euler equations

$$-1 - \frac{\partial G[I_t]}{\partial I_t} + \lambda_t = 0 \tag{3.4}$$

and

$$-\lambda_t + E\left[b_{t+1}\left\{p'_{t+1}\frac{\partial Y[K_t]}{\partial K_t} + (1-\delta)\lambda_{t+1}\right\} \mid \Omega_t\right] = 0 \tag{3.5}$$

Substituting the definition of λ_t from (3.4) in (3.5) and using (3.3), we get the non linear structural equation

$$1 + \gamma I_t = (1-\delta)E\left[b_{t+1}\left(1+\gamma I_{t+1}\right) \mid \Omega_t\right] + \theta E\left[b_{t+1}p'_{t+1} \mid \Omega_t\right] \tag{3.6}$$

Generally interest rate is supposed to be constant so that (3.6) reduced to a linear rational expectations model whose solution is well-known (see Blanchard and Khan, 1980). Now the modelling strategy issued in previous section circumvents this approximation and allows a richer treatment for r_t. Clearly equivalence between equation (3.6) and benchmark equation (2.1) is found by letting $\alpha = (1-\delta)$, $\beta = \theta$, $y_t = 1 + \gamma I_t$ and $x_t = p'_t$. Retaining the set of stochastic processes (2.6) and (2.9), an observable solution for investment is then easily derived as

$$I_t = -\frac{1}{\gamma} + \frac{\theta}{\gamma}\left\{\frac{1}{\eta + (\phi\varphi)^{-1}r_t}\right\} p'_t \tag{3.7}$$

with $\eta = (\phi\varphi)^{-1} - (1-\delta) > 0$

A convex effect of interest rate on investment (assuming fixed prices) is verified since

$$\frac{\partial I_t}{\partial r_t} = -\frac{\theta}{\gamma}\left\{\frac{(\phi\varphi)^{-1}}{\left(\eta + (\phi\varphi)^{-1} r_t\right)^2}\right\} p_t' < 0 \tag{3.8}$$

and

$$\frac{\partial^2 I_t}{\partial r_t^2} = \frac{\theta}{\gamma}\left\{\frac{2(\phi\varphi)^{-2}}{\left(\eta + (\phi\varphi)^{-1} r_t\right)^3}\right\} p_t' > 0 \tag{3.9}$$

Equation (3.8) is thus crucial to adequately assess the effect on investment of a marginal change in interest rate. Inversely, for a policy goal (again with fixed prices $\overline{p}$) aiming at increasing investment by Δ_I units, i.e. raising the actual level of investment from I_t to $\Delta_I + I_t$, some algebra manipulation shows that interest rate has to be modified by the amount

$$\Delta_r = -\Delta_I \left\{\frac{\left(\phi\varphi\eta + r_t\right)^2}{\phi\varphi\overline{p}\dfrac{\theta}{\gamma} + \Delta_I\left(\phi\varphi\eta + r_t\right)}\right\} \tag{3.10}$$

Some additional properties arises from (3.7). Denoting $\overline{I_t}$ the optimal level of investment obtained with this solution but in a constant interest rate regime, i.e. $r_t = \overline{r}$, the stochastic treatment of r_t implies the adjustment correction

$$I_t = \overline{I_t} + \frac{\theta}{\gamma}\left\{\frac{\overline{r} - \phi^{-1} r_t}{\left(\phi\varphi\eta + r_t\right)\left(\varphi\eta + \overline{r}\right)}\right\} p_t' \tag{3.11}$$

Equation (3.11) states that when interest rate is lower (or higher) than its constant approximation value, investment demand is revised upwards (or downwards) with respect to its linear case demand level.

Solution (3.7) is also attractively transformed by introducing the user cost of capital c_t (see Jorgenson, 1963) defined as $c_t = (r_t + \delta)\, p_t^I$, so that

$$I_t = -\frac{1}{\gamma} + \frac{\theta}{\gamma}\left\{\frac{p_t}{c_t + (\eta - \delta)(1 + r_t) p_t^I}\right\} \tag{3.12}$$

An extreme case is picked when $\eta = \delta$, which can only be verified with the unit root hypothesis for both interest rate and relative prices ($\phi = \varphi = 1$). Then equation (3.12) becomes

$$I_t = -\frac{1}{\gamma} + \frac{\theta}{\gamma}\left(\frac{p_t}{c_t}\right) \tag{3.13}$$

This very simple neoclassical structural form relates investment demand to the inverse of real user cost of capital, and clearly identifies the role of production and installation technologies through their respective structural parameters. Also equation (3.10) confines now to

$$\Delta_r = -\Delta_I \left\{ \frac{(\delta + r_t)^2}{\overline{p}\frac{\theta}{\gamma} + \Delta_I (\delta + r_t)} \right\} \tag{3.14}$$

In the alternative case $\eta > \delta$, i.e. $(\phi\varphi)^{-1} > 1$, then (3.12) states that investment is lower since the term $(\eta - \delta)(1 + r_t)p_t^I > 0$ is added to the user cost of capital effect. Therefore a lower degree of autocorrelation in exogenous variables discourages investment.

Returning to the policy variable movements, interest rate forecasting function is deduced from (2.7) and (2.8) and is represented by

$$r_{t+j} = \begin{cases} \dfrac{(1 + r_t) - \phi - (1 + r_t)e_{t+j}}{\phi + (1 + r_t)e_{t+j}} & [j = 1] \\ \dfrac{\phi^{j-1}(1 + r_t) - \phi^j + (1 + r_t)^j \left\{ \sum_{k=1}^{j-1} \phi^{j-k} \left(\phi^{-1} - 1\right) e_{t+k} - e_{t+j} \right\}}{\phi^j + (1 + r_t)^j \sum_{k=1}^{j} \phi^{j-k} e_{t+k}} & [j > 1] \end{cases} \tag{3.15}$$

Again hypothesis $\phi = 1$ allows to simplify the forecasting process of r since, for all $j > 0$, interest rate is now modelled by the single equation

$$r_{t+j} = \frac{r_t - (1 + r_t)^j e_{t+j}}{1 + (1 + r_t)^j \sum_{k=1}^{j} e_{t+k}} \tag{3.16}$$

Particularly, for $j = 1$, we get

$$r_{t+1} = \frac{r_t - (1 + r_t)e_{t+1}}{1 + (1 + r_t)e_{t+1}} \tag{3.17}$$

Error term e_{t+1} is under the control of the policy-maker but is unknown by the manager of the firm. A zero value for e_{t+1} leaves interest rate unchanged. On the other hand, when the stochastic shock affecting r_{t+1} is positive (or negative), interest rate will be forced to diminish (or to rise).

It is also worth noticing that, from equation (3.13), one can investigate investment variability with respect to exogenous uncertainty parameters since

$$\begin{aligned} Var\left[I_t\right] &= \left(\frac{\theta}{\gamma}\right)^2 Var\left[\frac{p_t}{c_t}\right] \\ &= \psi[\sigma_e^2, \sigma_a^2, t] \qquad \text{(say)} \end{aligned} \tag{3.18}$$

Finally, figure 1 illustrates the main result of the paper and presents a simulation of the response surface curve of investment (vertical axis) with respect to interest rate (axis 2, scaled from 2% up to 20%) and relative prices (axis 3, scaled from 0.5 up to 1.5), obtained from reduced form (3.12). Structural parameters are calibrated to the following values : $\theta = 1$, $\gamma = 0.001$ and $\delta = 13\%$ (remember that $\phi = \varphi = 1$). Clearly investment is more sensitive to r in a low-level interest rate regime than in a high-level interest rate regime.

Figure 1 : Non linear response surface of investment
to interest rate and relative prices

4. Conclusions

This paper has proposed an attractive path to get an observable solution of a standard non linear dynamic condition with rational expectations, making use of stochastic assumptions for exogenous variables. The result has been emphasized to judge, in an more subtle way than usual, the effect of an interest rate policy strategy on investment within a standard optimization problem. Two future research lines are obvious to improve the study. The first theoretical one is (i) to extend the non linear model to contain more policy variables, allowing for instance to investigate the effect of taxation policy on investment, and (ii) the use of richer dynamics to explain exogenous behaviors, particularly concerning the variable causing non linearity. The second direction is the empirical performance of both the solution and the stochastic representations with actual data. Based on recent results, we expect to significantly move forward in this agenda in the very near future.

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