# Regression-Based Unit Root Tests with Recursive Mean Adjustment for Seasonal and Non-Seasonal Time Series

A.M.Robert Taylor*
Department of Economics
University of Birmingham
Edgbaston
Birmingham, B15 2TT
United Kingdom.
E-mail: R.Taylor@bham.ac.uk

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Abstract

This paper considers tests for (seasonal) unit roots in a univariate time series process. We construct test statistics which are similar with respect to both the initial values of the process and the possibility of (differential seasonal) drift under the (seasonal) unit root null. In contrast to existing approaches, where similar inference is obtained by (seasonally) de-meaning and (seasonally) de-trending the process, utilizing all available sample data, we adopt the technique of recursive (seasonal) de-meaning and (seasonal) de-trending of the process. Representations are derived for the limiting distributions of the proposed test statistics under both the (seasonal) unit root null and under near (seasonal) integration. In the non-seasonal case the asymptotic local power of the proposed statistic is compared with the QD de-trended DF tests of Elliott et al. (1996) and Elliott (1999), and found to outperform both of these approaches in the case where the initial observation is drawn from the stationary distribution of the process. We also find in the case of quarterly data that our proposed statistics display superior finite sample size and power properties to the conventional seasonal unit root statistics of Hylleberg et al. (1990) and variants of such tests constructed using simple symmetric least squares and weighted symmetric least squares estimation.

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# 1 Introduction

It is well known that if the data generating process (DGP) is a random walk with potentially non-zero drift, both the limiting and finite-sample distributions of the Dickey and Fuller (1979) [DF] unit root test depend on the drift parameter. Exact similar inference may be obtained by including an intercept and linear time-trend in the DF test regression; see Hamilton (1994,Section 17.4, pp.486-504). This is numerically identical to de-meaning and de-trending, over the full sample, the variables involved in the DF regression prior to computing the DF test. Similar results apply to the regression-based seasonal unit root tests of, *inter alia*, Hylleberg *et al.* (1990) [HEGY], Ghysels *et al.* (1994) [GLN] and Smith and Taylor (1998) [ST1] for quarterly data, Beaulieu and Miron (1993) [BM] and Taylor (1998) for monthly data, and Smith and Taylor (1999) [ST2] and Smith and Taylor (2000) for an arbitrary seasonal aspect. For example, including seasonal intercepts and seasonal time trends in their test regressions delivers exact similar inference where the DGP is a seasonal random walk with potentially differential seasonal drift, and this is identical to seasonally de-meaning and de-trending the variables in the test regression; see ST1 for full details.

The above approach is known to have a detrimental effect on the power properties of the DF test; see, *inter alia*, Elliott *et al.* (1996) [ERS]. To understand why consider the simple $AR(1)$ time-series process $\{x_t\}_{t=1}^{T}$ generated according to $x_t = \rho x_{t-1} + \epsilon_t$, $t = 1, \ldots, T$, where $0 \leq \rho \leq 1$, $\epsilon_t \sim IN(0, \sigma^2)$ and $x_0 = 0$. Denote by $\hat{\rho}$ and $\hat{\rho}_\mu$ the Ordinary Least Squares (OLS) estimator of $\rho$, obtained by the regression of $x_t$ on $x_{t-1}$, and of $x_t$ on $x_{t-1}$ and $\mathsf{D}_t = 1$, $t = 1, \ldots, T$, respectively. Both $\hat{\rho}$ and $\hat{\rho}_\mu$ suffer a finite-sample bias which contains a leading negative signed term in $T^{-1}$; see, *inter alia*, Tanaka (1984), Sharman and Stine (1988), and Kiviet and Phillips (1993). For fixed $T$, the bias is greater when $\rho$ lies on, or in the neighbourhood of, the complex unit circle. Moreover, for given $\rho$ and $T$, the bias is greater for $\hat{\rho}_\mu$ than it is for $\hat{\rho}$. As discussed in Shin and So (1999) this is because there is an induced correlation between the lagged OLS de-trended series, $x_{t-1} - \mathsf{D}_{t-1}' \left( \sum_{j=1}^{T} \mathsf{D}_{j-1} \mathsf{D}_{j-1}' \right)^{-1} \sum_{j=1}^{T} \mathsf{D}_{j-1} x_{j-1}$, and the disturbance term $\epsilon_t$, which increases as $\rho$ tends towards one. Where $\mathsf{D}_t = (1, t)'$, this correlation, and hence the bias in $\hat{\rho}_\tau$, the resulting estimator of $\rho$, increases still further. Indeed, the bias will tend to increase with the dimension of $\mathsf{D}_t$.

The limiting distributions of $\hat{\rho}$, $\hat{\rho}_\mu$ and $\hat{\rho}_\tau$ are clearly unaffected by the above biases where $\rho < 1$. However, for $\rho = 1$, the above estimators all converge on unity at rate $T$. The difference between the $O(T^{-1})$ bias terms in $\hat{\rho}$ and $\hat{\rho}_\mu$ will therefore effect a left-shift in the lower tail of both the finite-sample and limiting null distributions of the de-meaned DF statistics $T(\hat{\rho}_\mu - 1)$ and $\tau_\mu \equiv (\hat{\rho}_\mu - 1)/se(\hat{\rho}_\mu)$, vis-à-vis the corresponding statistics $T(\hat{\rho} - 1)$ and $t_\rho \equiv (\hat{\rho} - 1)/se(\hat{\rho})$. This effect will be further amplified in the case of DF tests based on $\hat{\rho}_\tau$; see also Rule 2 of Campbell and Perron (1991, p.150).

Given that, to a large part, this bias problem, together with its implications, stems from the correlation between the de-trended regressor and the disturbance (innovation) term, it seems worthwhile to explore methods of de-trending the series in a way which induces a lesser correlation. A simple example, recently employed by Shin and So (1999) in the context of deriving confidence intervals for the parameter estimates from autoregressive models, is to use the (OLS) recursively de-trended process, $x_{t-1} - \mathsf{D}'_{t-1}\left(\sum_{j=0}^{t-1} \mathsf{D}_j \mathsf{D}'_j\right)^{-1} \sum_{j=0}^{t-1} \mathsf{D}_j x_j$, which is straightforwardly seen to be uncorrelated with $\epsilon_t$. In what follows we will refer to this approach as recursive mean adjustment. Shin and So (1998) also use recursive mean adjustment in developing instrumental variable-based seasonal unit root tests. Other alternatives to OLS de-trending have also been suggested in the literature. A well-known example is the quasi-differencing (QD) de-trending of ERS and Elliott (1999), who demonstrate that, under certain conditions on the initial value of the process, DF tests based on QD de-trending are more powerful under the alternative than those based on OLS de-trending. In the case where $\mathsf{D}_t = 1$, for all $t$, the ERS test, for example, has a standard DF limiting null distribution, while the $\tau_\mu$ test has a standard DF limiting distribution for a regression with intercept. In this case the QD de-trended series is asymptotically uncorrelated with the innovation; see Burridge and Taylor (2000) who provided a detailed assessment of the inter-relationship between the method of de-trending used and its efficiency, the statistical properties of the initial value, and test power.

The plan of this paper is as follows. In Section 2 we outline a general approach for constructing regression-based (seasonal) unit root tests. In Section 3 we discuss the use of recursive mean adjustment, and show that full-sample de-meaning is a special case of the procedure which we outline. We show that this approach maintains similar tests of the (seasonal) unit root hypothesis. In Section 4 we derive representations for the limiting distributions of our proposed test statistics under the (seasonal) unit root null hypothesis and delineate the asymptotic local power functions of both these statistics and those of DF and

HEGY under near (seasonal) integration. For the non-seasonal case, $S = 1$, a comparison of the asymptotic local power function of our proposed statistics with those of ERS and Elliott (1999) is provided. Section 5 contains a small selection of finite-sample critical values for our unit root test statistics, relevant for quarterly data, together with an investigation into the size and power properties of these quarterly tests, relative to conventional quarterly seasonal unit root (HEGY) tests and variants of the HEGY tests constructed using the simple symmetric least squares (SSLS) and weighted symmetric least squares (WSLS) estimators of, *inter alia*, Fuller (1996) and Pantula *et al.* (1994) [PGF]. Section 6 concludes the paper. Proofs are provided in an Appendix.

# 2 Constructing (Seasonal) Unit Root Tests

Denote the periodicity of the data by $S$; for example, if the time-series is quarterly, $S = 4$, while for the non-seasonal case, $S = 1$. Consider the pure AR process $\{x_{St+s}\}$:

$$\alpha(L)\left[x_{St+s} - \mu_{St+s}\right] = u_{St+s}, \quad s = 1 - S, ..., 0, \; t = 1, 2, \ldots, T, \tag{2.1}$$

$$\mu_{St+s} = \gamma_s^* + \delta_s^*(St + s), \tag{2.2}$$

where $\alpha(L) \equiv 1 - \sum_{j=1}^{S} \alpha_j^* L^j$, $L$ the conventional lag operator, and the error process $\{u_{St+s}\}$ is assumed to follow an AR($p$), $0 \leq p < \infty$ process, *viz.*, $\phi(L)u_{St+s} = \epsilon_{St+s}$, the roots of $\phi(z) \equiv 1 - \sum_{j=1}^{p} \phi_j z^j = 0$ all lying outside the unit circle, $|z| = 1$, with $\epsilon_{St+s} \sim iid(0, \sigma_\epsilon^2)$. The (observed) initial values of the process are assumed to satisfy, $x_s = \mu_s + u_s^*$, $s = 1 - S, \ldots, 0$, with exact conditions on the initial innovations $\{u_s^*\}_{s=1-S}^{0}$ delayed until later. The specification (2.1)-(2.2) allows for the presence of deterministic mean effects in $\{x_{St+s}\}$ through $\mu_{St+s}$. For the purposes of this paper, we follow ST2 and consider the following six cases of interest:

Case 1. no intercept, no trend: $\gamma_s^* = 0$, $\delta_s^* = 0$, $s = 1 - S, ..., 0$;

Case 2. constant intercept, no trend: $\gamma_s^* = \gamma$, $\delta_s^* = 0$, $s = 1 - S, ..., 0$;

Case 3. seasonal intercepts, no trend: $\delta_s^* = 0$, $s = 1 - S, ..., 0$;

Case 4. constant intercept, constant trend: $\gamma_s^* = \gamma$, $\delta_s^* = \delta$, $s = 1 - S, ..., 0$;

Case 5. seasonal intercepts, constant trend: $\delta_s^* = \delta$, $s = 1 - S, ..., 0$;

Case 6. seasonal intercepts, seasonal trends: as in (2.2) with $\gamma_s^*$ and $\delta_s^*$ unrestricted.

Notice, for example, that Cases 4,5 and 6 are all identical in the non-seasonal case, $S = 1$.

Our approach to testing for seasonal unit roots in $\alpha(L)$ consists of two stages. First we subtract a linear unbiased estimate of $\mu_{St+s}$ of (2.2) from $x_{St+s}$, viz, $\hat{x}^{\kappa}_{St+s} = x_{St+s} - \hat{\mu}^{\kappa}_{St+s}$, where $\kappa \in \{1, \ldots, 6\}$ is used to indicate which of Cases 1-6 is assumed to hold in estimating the parameters of (2.2). Notice that for Case 1, $\hat{\mu}^{1}_{St+s}$ will be zero, and hence $\hat{x}^{1}_{St+s} = x_{St+s}$, by definition. Provided $\mu_{St+s}$ is not estimated under an overly restrictive case, the resulting unit root tests will provide similar (asymptotically similar if $p > 0$, exact similar otherwise) inference. We will discuss this stage in detail in Section 3.

Following ST2 (Equation (2.11), page 6), we then linearise $\alpha(L)$ in (2.1) around the seasonal unit roots $\exp(\pm i2\pi k/S)$, $k = 0, ..., [S/2]$, to obtain the auxiliary regression equation,

$$\Delta_S \hat{x}^{\kappa}_{St+s} = \sum_{j=0}^{S-1} \pi_j \hat{x}^{\kappa}_{j,St+s-1} + \sum_{j=1}^{p^*} \phi^*_j \Delta_S \hat{x}^{\kappa}_{St+s-j} + \hat{\epsilon}^{\kappa}_{St+s}, \tag{2.3}$$

which may be estimated along $St+s = p^*+1, ..., 4T$, $p^* \geq p$, omitting the term $\pi_{S/2}\hat{x}^{\kappa}_{S/2,St+s-1}$ if $S$ is odd, and where corresponding to the zero and, where $S > 1$, seasonal frequencies $\omega_k = 2\pi k/S$, $k = 0, ..., [S/2]$, $[\cdot]$ denoting the integer part of its argument,

$$\hat{x}^{\kappa}_{0,St+s} \equiv \sum_{j=0}^{S-1} \hat{x}^{\kappa}_{St+s-j}, \quad \hat{x}^{\kappa}_{S/2,St+s} \equiv \sum_{j=0}^{S-1} \cos[(j+1)\pi] \hat{x}^{\kappa}_{St+s-j},$$

$$\hat{x}^{\kappa}_{k,St+s} \equiv \sum_{j=0}^{S-1} \cos[(j+1)\omega_k] \hat{x}^{\kappa}_{St+s-j}, \quad \hat{x}^{\kappa}_{S-k,St+s} \equiv -\sum_{j=0}^{S-1} \sin[(j+1)\omega_k] \hat{x}^{\kappa}_{St+s-j}, \tag{2.4}$$

$k = 1, ..., S^*$, where $S^* = (S/2) - 1$ (if $S$ is even) and $[S/2]$ (if $S$ is odd), together with $\Delta_S \hat{x}^{\kappa}_{St+s} \equiv \hat{x}^{\kappa}_{St+s} - \hat{x}^{\kappa}_{S(t-1)+s}$. For the case of quarterly, $S = 4$, data the relevant transformations are

$$\hat{x}^{\kappa}_{0,St+s} \equiv (1 + L + L^2 + L^3)\hat{x}^{\kappa}_{St+s}, \quad \hat{x}^{\kappa}_{2,St+s} \equiv -\left(1 - L + L^2 - L^3\right)\hat{x}^{\kappa}_{St+s},$$

$$\hat{x}^{\kappa}_{1,St+s} \equiv -L(1 - L^2)\hat{x}^{\kappa}_{St+s}, \quad \hat{x}^{\kappa}_{3,St+s} \equiv -(1 - L^2)\hat{x}^{\kappa}_{St+s},$$

while for the non-seasonal case, $S = 1$, $\hat{x}^{\kappa}_{0,t} \equiv \hat{x}^{\kappa}_t$.

The parameters $\{\pi_j\}_{j=0}^{S-1}$ of (2.3) are of focal interest. As demonstrated in HEGY and ST2, the (seasonal) unit root null hypothesis,

$$\mathsf{H}_0 : \alpha(L) = \Delta_S = 1 - L^S, \tag{2.5}$$

is the intersection of the subsidiary null hypotheses: $\mathsf{H}_{0,0} : \pi_0 = 0$, a unit root at $\omega_0 \equiv 0$; $\mathsf{H}_{0,S/2} : \pi_{S/2} = 0$, a unit root at $\omega_{S/2} \equiv \pi$ ($S$ even), and $(\mathsf{H}_{0,k} : \pi_k = 0) \cap (\mathsf{H}_{0,S-k} : \pi_{S-k} = 0)$, a pair of complex conjugate roots at the $k$th seasonal harmonic frequency $\omega_k \equiv (2\pi k/S)$, $k = 1, \ldots, S^*$.

Consequently, and in order to test $\mathsf{H}_0$ of (2.5) against the alternative of stationarity at at least one of the zero and seasonal frequencies, we follow HEGY and ST1, inter alia, and propose the following regression statistics in (2.3): $t_0$ (left-sided) for the exclusion of $\hat{x}^{\kappa}_{0,St+s-1}$; $t_{S/2}$ (left-sided) for the exclusion of $\hat{x}^{\kappa}_{S/2,St+s-1}$ ($S$ even); $t^{\alpha}_k$ (left-sided) and $t^{\beta}_k$ (two-sided) for the exclusion of $\hat{x}^{\kappa}_{k,St+s-1}$ and $\hat{x}^{\kappa}_{S-k,St+s-1}$ respectively, and $F_k$ for the exclusion of both $\hat{x}^{\kappa}_{k,St+s-1}$ and $\hat{x}^{\kappa}_{S-k,St+s-1}$, $k = 1, ..., S^*$. Following GLN, Taylor (1998), ST1 and ST2 we also consider the joint frequency $F$-statistics, $F_{1\ldots[S/2]}$, for the exclusion of $\{\hat{x}^{\kappa}_{j,St+s-1}\}^{S-1}_{j=1}$ and $F_{0\ldots[S/2]}$, for $\mathsf{H}_0$, the exclusion $\{\hat{x}^{\kappa}_{j,St+s-1}\}^{S-1}_{j=0}$. In the non-seasonal case, $S = 1$, we need only consider $t_0$, the Dickey-Fuller-type statistic.

# 3 Recursive Mean Adjustment

In order to remove the dependency of the data $\{x_{St+s}\}$ under $\mathsf{H}_0$ of (2.5) on the nuisance parameters $\{\gamma^*_s, \delta^*_s, x_s\}^0_{s=1-S}$ existing approaches, adopted by, inter alia, HEGY, BM, ST1 and ST2, use full-sample mean adjustment of $\{x_{St+s}\}$. The resulting mean-adjusted regressors in (2.3) are then correlated with the error term $u_{St+s}$; see Taylor (1999) for detailed discussion on this point.

Following the discussion in Section 1, we now explore recursive mean adjustment, whereby $x_{St+s}$ is (seasonally) de-trended and/or (seasonally) de-meaned using only the observations $\{x_{1-S}, \ldots, x_{St+s}\}$, in order to reduce the inherent biases present in the least squares estimators of the $\pi_j$, $j = 0, \ldots, S-1$, regression parameters from (2.3), whilst maintaining similar tests. Taking Case 6 of (2.2), first, we consider the recursive seasonally de-meaned and seasonally de-trended process,

$$\hat{x}^6_{St+s} = \hat{x}^3_{St+s} - (t - \bar{t}) \left( \sum_{k=0}^{t} (k - \bar{t})^2 \right)^{-1} \sum_{k=0}^{t} (k - \bar{t})[x_{Sk+s} - \bar{x}_{t,s}], \tag{3.1}$$

$s = 1 - S, \ldots, 0$, $t = 0, \ldots, T$, where $\bar{x}_{t,s} = (t+1)^{-1} \sum_{k=0}^{t} x_{Sk+s}$, $s = 1 - S, \ldots, 0$, and $\bar{t} \equiv (t+1)^{-1} \sum_{k=0}^{t} k = t/2$. In defining (3.1) we have introduced the recursive seasonally de-meaned process $\{\hat{x}^3_{St+s}\}$, appropriate to Case 3 of (2.2),

$$\hat{x}^3_{St+s} = x_{St+s} - \bar{x}_{t,s}, \tag{3.2}$$

$s = 1-S, \ldots, 0$, $t = 0, \ldots, T$. Notice that, for $t > 0$, $\hat{x}^6_{St+s} \equiv x_{St+s} - \hat{\gamma}_{s,t} - \hat{\delta}_{s,t} t$, where $\hat{\gamma}_{s,t}$ and $\hat{\delta}_{s,t}$ are respectively the estimated coefficients from OLS regression of $x_{Sv+s}$ on an intercept and $v$, for $v = 0, \ldots, t$. Similarly, the recursive seasonally de-meaned series $\hat{x}^3_{St+s} \equiv x_{St+s} - \tilde{\gamma}_{s,t}$, where $\tilde{\gamma}_{s,t}$ is the estimated coefficient from OLS regression of $x_{Sv+s}$ on an intercept, for $v = 0, \ldots, t$, $t \geq 0$. For the remaining cases of (2.2) we consider: Case 1: no de-meaning, $\hat{x}^1_{St+s} \equiv x_{St+s}$, $s = 1-S, \ldots, 0$, $t = 1, \ldots, T$; Case 2: recursive (non-seasonal) de-meaning, $\hat{x}^2_{St+s} \equiv x_{St+s} - \tilde{\gamma}_{St+s}$, where $\tilde{\gamma}_{St+s}$ is the estimated coefficient from an OLS regression of $x_{Sv+j}$ on an intercept, $Sv+j = 1-S, \ldots, St+s$, $St+s \geq 1-S$; Case 4: recursive (non-seasonal) de-meaning and de-trending, $\hat{x}^4_{St+s} \equiv x_{St+s} - \hat{\gamma}_{St+s} - \hat{\delta}_{St+s}(St+s)$, where $\hat{\gamma}_{St+s}$ and $\hat{\delta}_{St+s}$ are the estimated coefficients from an OLS regression of $x_{Sv+j}$ on an intercept and the linear time-trend $(Sv+j)$, $Sv+j = 1-S, \ldots, St+s$, $St+s \geq 2-S$; Case 5: recursive seasonal de-meaning and (non-seasonal) de-trending, $\hat{x}^5_{St+s} \equiv x_{St+s} - \hat{\gamma}^s_{St+s} - \hat{\delta}_{St+s}(St+s)$, where $\{\hat{\gamma}^j_{St+s}\}^0_{j=1-S}$, and $\hat{\delta}_{St+s}$ are the estimated coefficients from an OLS regression of $x_{Sv+j}$ on $S$ conventional seasonal indicator variables and the linear time-trend $(Sv+j)$, $Sv+j = 1-S, \ldots, St+s$, $St+s \geq 1$. In all of the above cases it is trivially seen that, in contrast to the full-sample de-meaning of, inter alia, HEGY and ST1 and ST2, the first lag of the transformations of (2.4) are uncorrelated with the innovation $u_{St+s}$.

Under $\mathsf{H}_0$ of (2.5), the recursive seasonally de-meaned process $\{\hat{x}^3_{St+s}\}$ may be written as

$$\hat{x}^3_{St+s} = \gamma_s(t/2) + \sum_{v=1}^{t} u_{Sv+s} - (t+1)^{-1} \sum_{k=1}^{t} \sum_{v=1}^{k} u_{Sv+s}, \tag{3.3}$$

which is invariant to the initial conditions $\{x_s\}^0_{s=1-S}$, but not to the seasonal drifts $\{\gamma_s\}^0_{s=1-S}$, where $\gamma_s \equiv S(\delta^*_s - \sum_{j=1}^{p} \phi_j \delta^*_{S-j})$, unless $\gamma_s = 0$, $s = 1-S, \ldots, 0$, as occurs under each of Cases 1-3 of (2.2). From (3.3), we may correspondingly express the recursive seasonally de-meaned and seasonally de-trended process $\{\hat{x}^6_{St+s}\}$ under $\mathsf{H}_0$ as

$$\hat{x}^6_{St+s} = \sum_{v=1}^{t} u_{Sv+s} - (t+1)^{-1} \sum_{k=1}^{t} \sum_{v=1}^{k} u_{Sv+s} - (t/2) \left( \sum_{k=0}^{t} (k - \bar{t})^2 \right)^{-1} \tag{3.4}$$
$$\times \sum_{k=0}^{t} (k - \bar{t}) \left( \sum_{v=1}^{k} u_{Sv+s} - (t+1)^{-1} \sum_{j=1}^{t} \sum_{v=1}^{j} u_{Sv+s} \right),$$

which is invariant to the initial conditions $\{x_s\}^0_{s=1-S}$ and the seasonal drifts $\{\gamma_s\}^0_{s=1-S}$. In a similar manner, it is straightforward to show that the recursive seasonally de-meaned and (non-seasonally) de-trended process $\{\hat{x}^5_{St+s}\}$ is invariant under $\mathsf{H}_0$ to the initial conditions

$\{x_s\}_{s=1-S}^{0}$, but depends on the quantity $(\gamma_s - \bar{\gamma})$, where $\bar{\gamma} \equiv S^{-1} \sum_{s=1-S}^{0} \gamma_s$. Similarly, the recursive de-meaned and de-trended process $\{\hat{x}_{St+s}^{4}\}$ depends on the quantities $(\gamma_s - \bar{\gamma})$, and $(x_s - \bar{x}_0)$, where $\bar{x}_0 \equiv S^{-1} \sum_{s=1-S}^{0} x_s$, and hence on the seasonal intercepts $\{\gamma_s^*\}_{s=1-S}^{0}$, while the recursive de-meaned process $\{\hat{x}_{St+s}^{2}\}$ depends on the seasonal drifts $\{\gamma_s\}_{s=1-S}^{0}$ and the quantity $(x_s - \bar{x}_0)$. Corresponding invariance properties to those detailed above can be shown to hold under the alternative except that in all cases the mean-adjusted data now depend, in general, on the initial innovations $\{u_s^*\}_{s=1-S}^{0}$; cf. ERS.

In the above analysis we have suggested using recursive mean adjustment methods whereby $x_{St+s}$ is mean adjusted using only the sample observations $\{x_{1-S}, \ldots, x_{St+s}\}$, for $St + s = 1 - S, \ldots, ST$. However, one might also consider placing some minimum bound on the fraction of sample data used to estimate the required mean function. Specifically, under Case 6, we might therefore also consider using the auxiliary regression (2.3), appropriately re-defining the terms in (2.4) as the corresponding transformations of

$$\hat{x}_{St+s}^{6} = \hat{x}_{St+s}^{3} - \frac{(t-\bar{\lambda}) \sum_{k=0}^{\lambda^*} (k - \bar{\lambda}) \left( x_{Sk+s} - (\lambda^* + 1)^{-1} \sum_{j=0}^{\lambda^*} x_{Sj+s} \right)}{\sum_{k=0}^{\lambda^*} (k - \bar{\lambda})^2} \tag{3.5}$$

$$\hat{x}_{St+s}^{3} = x_{St+s} - (\lambda^* + 1)^{-1} \sum_{k=0}^{\lambda^*} x_{Sk+s}, \tag{3.6}$$

$s = 1 - S, \ldots, 0$, $t = 0, \ldots, T$, where $\lambda^* = \max(t, [T\lambda])$, $\lambda \in [0, 1]$ and $\bar{\lambda} = (\lambda^* + 1)^{-1} \sum_{k=0}^{\lambda^*} k$. Notice that the end case, $\lambda = 0$, coincides with (3.1), while $\lambda = 1$ gives the full-sample seasonal de-meaning and seasonal de-trending approach of HEGY, ST1 and ST2, discussed at the start of this Section. Where $t \geq [T\lambda]$, $\hat{x}_{St+s}^{6}$ of (3.5) clearly coincides with $\hat{x}_{St+s}^{6}$ of (3.1). However, where $t < [T\lambda]$, $\hat{x}_{St+s}^{6}$ of (3.5) is $x_{St+s}$ de-meaned and de-trended using the sample data $\{x_{1-S}, \ldots x_{S[T\lambda]+s}\}$. Cases 2-5 of (2.3) follow directly, *mutatis mutandis*. Case 1 is unaffected. In all cases it is easily seen that tests based on this approach share the same invariance properties (except with regard to the initial innovations) as the corresponding tests formed from $\lambda = 0$, discussed above. However, notice that it is only where $\lambda = 0$ that the first lag of the transformations of (2.4) in (2.3) are uncorrelated with $u_{St+s}$, $St + s = 1, \ldots, ST$.

# 4 Asymptotic Representations

We now derive representations for the limiting distributions of the (seasonal) unit root statistics of Section 2 computed from auxiliary regression (2.3) under each of Cases 1-6 of (2.2).

We derive these representations for the statistics where a minimum (fixed) fraction of the data, $\lambda \in [0,1]$, is used in the recursive mean adjustment. To distinguish between the statistics computed for different values of $\lambda$, we use the notation $t_j(\lambda)$, $j = 0, S/2$, $t_k^{\alpha}(\lambda)$, $t_k^{\beta}(\lambda)$, $F_k(\lambda)$, $k = 1, ..., S^*$, $F_{1...[S/2]}(\lambda)$ and $F_{0...[S/2]}(\lambda)$, in what follows. The end cases $\lambda = 0$ and $\lambda = 1$ then simply obtain as special cases of the stated representations. These representations are derived under $\mathsf{H}_0$ of (2.5) in Theorem 4.1 and also under near (seasonal) unit root alternatives in Theorem 4.2.

In the results given below, the superscript $\xi$ relates to Cases 1-6 of $\mu_{St+s}$ of (2.2), and hence (2.3). For the zero frequency $\omega_0$ tests: Case 1: $\xi = 0$; Cases 2 and 3: $\xi = 1$; Cases 4, 5 and 6: $\xi = 2$. For the seasonal frequency $\omega_k$ tests, $k = 1, ..., [S/2]$: Cases 1, 2 and 4: $\xi = 0$; Cases 3 and 5: $\xi = 1$; Case 6: $\xi = 2$.

Theorem 4.1 is derived under the following assumptions:

Assumption 4.1: The error process $\{u_{St+s}\} \sim iid(0, \sigma_u^2)$.

Assumption 4.2: The initial innovations satisfy $T^{-1/2}u_s^* \to^p 0$, $s = 1 - S, \ldots, 0$.

Theorem 4.1 Under $\mathsf{H}_0$ of (2.5) and Assumptions 4.1-4.2, denoting weak convergence by "$\Rightarrow$",

$$t_j(\lambda) \Rightarrow \frac{\int_0^1 \mathsf{W}_j^{\xi}(r,\lambda) d\mathsf{W}_j^{\xi}(r,\lambda)}{\left[\int_0^1 \mathsf{W}_j^{\xi}(r,\lambda)^2 dr\right]^{1/2}} \equiv \eta_{j,\lambda}, \quad j = 0, S/2, \tag{4.1}$$

$$t_k^{\alpha}(\lambda) \Rightarrow \frac{\int_0^1 \left[\mathsf{W}_{\alpha,k}^{\xi}(r,\lambda) d\mathsf{W}_{\alpha,k}^{\xi}(r,\lambda) + \mathsf{W}_{\beta,k}^{\xi}(r,\lambda) d\mathsf{W}_{\beta,k}^{\xi}(r,\lambda)\right]}{\left[\int_0^1 \left[\mathsf{W}_{\alpha,k}^{\xi}(r,\lambda)^2 + \mathsf{W}_{\beta,k}^{\xi}(r,\lambda)^2\right] dr\right]^{1/2}} \equiv \eta_{k,\alpha,\lambda}, \tag{4.2}$$

$$t_k^{\beta}(\lambda) \Rightarrow -\frac{\int_0^1 \left[\mathsf{W}_{\alpha,k}^{\xi}(r,\lambda) d\mathsf{W}_{\beta,k}^{\xi}(r,\lambda) - \mathsf{W}_{\beta,k}^{\xi}(r,\lambda) d\mathsf{W}_{\alpha,k}^{\xi}(r,\lambda)\right]}{\left[\int_0^1 \left[\mathsf{W}_{\alpha,k}^{\xi}(r,\lambda)^2 + \mathsf{W}_{\beta,k}^{\xi}(r,\lambda)^2\right] dr\right]^{1/2}} \equiv \eta_{k,\beta,\lambda}, \tag{4.3}$$

$$F_k(\lambda) \Rightarrow \frac{1}{2}\left[\eta_{k,\alpha,\lambda}^2 + \eta_{k,\beta,\lambda}^2\right], \; k = 1, \ldots, S^* \tag{4.4}$$

$$F_{1...[S/2]}(\lambda) \Rightarrow \frac{1}{S-1}\left[\eta_{S/2,\lambda}^2 + \sum_{k=1}^{S^*}\left(\eta_{k,\alpha,\lambda}^2 + \eta_{k,\beta,\lambda}^2\right)\right], \tag{4.5}$$

$$F_{0...[S/2]}(\lambda) \Rightarrow \frac{1}{S}\left[\eta_{0,\lambda}^2 + \eta_{S/2,\lambda}^2 + \sum_{k=1}^{S^*}\left(\eta_{k,\alpha,\lambda}^2 + \eta_{k,\beta,\lambda}^2\right)\right], \tag{4.6}$$

omitting terms involving $\eta_{S/2,\lambda}$ if $S$ is odd. The limiting processes $\mathsf{W}_j^{\xi}(r,\lambda)$, $j = 0, S/2$ and $\mathsf{W}_{\alpha,k}^{\xi}(r,\lambda)$ and $\mathsf{W}_{\beta,k}^{\xi}(r,\lambda)$, $k = 1, ..., S^*$, are defined in the Appendix, noting that for $\xi = 0$,

$W_j^0(r,\lambda) \equiv W_j^0(r)$, $j = 0, S/2$, $W_{\alpha,k}^0(r,\lambda) \equiv W_{\alpha,k}^0(r)$ and $W_{\beta,k}^0(r,\lambda) \equiv W_{\beta,k}^0(r)$, $k = 1, ..., S^*$, are independent standard Brownian motions which do not depend on $\lambda$.

Remark 4.1: Although Theorem 4.1 was derived under Assumption 4.1, (4.1) and (4.4)-(4.6) can also be shown to hold under the more general condition that $\{u_{St+s}\}$ follows a stationary $AR(p)$ process with martingale difference innovations satisfying the assumptions given in Chan and Wei (1988), provided $p^* \geq p$ in (2.3); cf. Burridge and Taylor (2001).

Remark 4.2: In Theorem 4.1 we have assumed, via Assumption 4.2, that the initial conditions $\{x_s\}_{s=1-S}^0$ are all random variables with finite (possibly zero) variances. Under Cases 3,5 and 6, this assumption may be dropped due to the similarity properties of the statistics, as detailed in Section 3. Moreover, under Cases 2 and 4 this assumption is also not needed for (4.1) to hold for $j = 0$ because $\sum_{s=1-S}^0 (x_s - \bar{x}_0) \equiv 0$.

Remark 4.3: The limiting representations (4.1)-(4.6) of Theorem 4.1 generalise those given in ST2, reducing to the latter where $\lambda = 1$.

Remark 4.4: It can be seen from (4.1) of Theorem 4.1 that for $\xi = 0$, $t_0(\lambda)$ and $t_{S/2}(\lambda)$ have identical standard Dickey-Fuller (1979) limiting null distributions, regardless of the value of $\lambda$. Moreover, for $\xi \in \{1, 2\}$ the limiting representations of $t_0(\lambda)$ and $t_{S/2}(\lambda)$ are identical in distribution. In each of the above cases the $t_0(\lambda)$ and $t_{S/2}(\lambda)$ statistics are also asymptotically independent, by virtue of the independence of $\mathsf{W}_0^\xi(r,\lambda)$ and $\mathsf{W}_{S/2}^\xi(r,\lambda)$, $\xi \in \{0, 1, 2\}$.

Remark 4.5: Representations (4.2)-(4.3) of Theorem 4.1 demonstrate that, under $\mathsf{H}_0$ of (2.5), $t_k^\alpha(\lambda)$ and $t_l^\alpha(\lambda)$, $k \neq l$, and $t_k^\beta(\lambda)$ and $t_l^\beta(\lambda)$, $k \neq l$, $k, l = 1, \ldots, S^*$, possess identical limiting distributions, are mutually asymptotically independent and are also asymptotically independent of $t_0(\lambda)$ and $t_{S/2}(\lambda)$.

Remark 4.6: The representations given in Theorem 4.1 allow an explanation for the similarity between critical values which occurs in different scenarios and between different statistics; cf. Section 5.1. E.g., from (4.1) it is seen that $t_{S/2}(\lambda)$ has an identical limiting distribution under Cases 3 and 4, which coincides with that of $t_0(\lambda)$ under both Cases 2 and 3.

Remark 4.7: The $F$-statistics, $F_k(\lambda)$, $k = 1, ..., S^*$, are asymptotically mutually independent and are asymptotically independent of $t_0(\lambda)$ and $t_{S/2}(\lambda)$ under $\mathsf{H}_0$ of (2.5). Moreover, the $F$-statistic $F_{1\ldots[S/2]}(\lambda)$ is asymptotically independent of $t_0(\lambda)$ under $\mathsf{H}_0$ of (2.5).

Having derived the limiting null distributions of the (seasonal) unit root statistics from Cases 1 through 6 of the auxiliary regression (2.3), we now turn in Theorem 4.2 to derive the corresponding limiting distributions under local alternatives to the (seasonal) unit root hypothesis. Specifically, we will consider the local alternative:

$$\mathsf{H}_c : \alpha(L) = 1 - (1 + c/T)L^S, \ c < 0. \tag{4.7}$$

**Theorem 4.2** Under $\mathsf{H}_c$ of (4.7), and Assumptions 4.1-4.2, with $\xi$ defined as for Theorem 4.1,

$$
\begin{aligned}
t_j(\lambda) &\Rightarrow c\left[\int_0^1 \mathsf{W}^{\xi}_{j,c}(r,\lambda)^2 dr\right]^{1/2} + \frac{\int_0^1 \mathsf{W}^{\xi}_{j,c}(r,\lambda) d\mathsf{W}^{\xi}_{j,c}(r,\lambda)}{\left[\int_0^1 \mathsf{W}^{\xi}_{j,c}(r,\lambda)^2 dr\right]^{1/2}} \equiv \tau_{j,\lambda}(c), \ j = 0, S/2, & (4.8)\\
t^{\alpha}_k(\lambda) &\Rightarrow c\left[\int_0^1 \left[\mathsf{W}^{\xi}_{\alpha,k,c}(r,\lambda)^2 + \mathsf{W}^{\xi}_{\beta,k,c}(r,\lambda)^2\right] dr\right]^{1/2} \\
&+ \frac{\int_0^1 \left[\mathsf{W}^{\xi}_{\alpha,k,c}(r,\lambda) d\mathsf{W}^{\xi}_{\alpha,k,c}(r,\lambda) + \mathsf{W}^{\xi}_{\beta,k,c}(r,\lambda) d\mathsf{W}^{\xi}_{\beta,k,c}(r,\lambda)\right]}{\left[\int_0^1 \left[\mathsf{W}^{\xi}_{\alpha,k,c}(r,\lambda)^2 + \mathsf{W}^{\xi}_{\beta,k,c}(r,\lambda)^2\right] dr\right]^{1/2}} \equiv \tau_{k,\alpha,\lambda}(c), & (4.9)\\
t^{\beta}_k(\lambda) &\Rightarrow -\frac{\int_0^1 \left[\mathsf{W}^{\xi}_{\alpha,k,c}(r,\lambda) d\mathsf{W}^{\xi}_{\beta,k,c}(r,\lambda) - \mathsf{W}^{\xi}_{\beta,k,c}(r,\lambda) d\mathsf{W}^{\xi}_{\alpha,k,c}(r,\lambda)\right]}{\left[\int_0^1 \left[\mathsf{W}^{\xi}_{\alpha,k,c}(r,\lambda)^2 + \mathsf{W}^{\xi}_{\beta,k,c}(r,\lambda)^2\right] dr\right]^{1/2}} \equiv \tau_{k,\beta,\lambda}(c), & (4.10)
\end{aligned}
$$

$$
\begin{aligned}
F_k(\lambda) &\Rightarrow \frac{1}{2}\left\{[\tau_{k,\alpha,\lambda}(c)]^2 + [\tau_{k,\beta,\lambda}(c)]^2\right\}, \ k = 1, \ldots, S^*, & (4.11)\\
F_{1\ldots[S/2]}(\lambda) &\Rightarrow \frac{1}{S-1}\left\{[\tau_{S/2,\lambda}(c)]^2 + \sum_{k=1}^{S^*}\left([\tau_{k,\alpha,\lambda}(c)]^2 + [\tau_{k,\beta,\lambda}(c)]^2\right)\right\}, & (4.12)\\
F_{0\ldots[S/2]}(\lambda) &\Rightarrow \frac{1}{S}\left\{[\tau_{0,\lambda}(c)]^2 + [\tau_{S/2,\lambda}(c)]^2 + \sum_{k=1}^{S^*}\left([\tau_{k,\alpha,\lambda}(c)]^2 + [\tau_{k,\beta,\lambda}(c)]^2\right)\right\}, & (4.13)
\end{aligned}
$$

omitting terms involving $\tau_{S/2,\lambda}(c)$ if $S$ is odd. The limiting processes $\mathsf{W}^{\xi}_{j,c}(r,\lambda)$, $j = 0, S/2$ and $\mathsf{W}^{\xi}_{\alpha,k,c}(r,\lambda)$ and $\mathsf{W}^{\xi}_{\beta,k,c}(r,\lambda)$, $k = 1, ..., S^*$, are defined in the Appendix.

**Remark 4.8**: Under Case 1 of (2.3) and $S = 4$, the limiting representations (4.8)-(4.13) specialise to those given for the quarterly case in Rodrigues (2001), our results therefore generalise his to allow for an arbitrary seasonal aspect, $S$, and for deterministic components, via Cases 2 through 6 of (2.2).

**Remark 4.9**: The representations given in Theorems 4.1 and 4.2 delineate the asymptotic local power functions of both the recursive mean adjusted ($\lambda = 0$) and full-sample mean adjusted ($\lambda = 1$) seasonal unit root statistics from (2.3), and indeed for all $\lambda \in [0, 1]$, in all cases indexed by a common non-centrality parameter $c$, $c < 0$. In practice, we would probably want to permit the non-centrality parameter to vary across the seasonal frequencies $\omega_k \equiv 2\pi k/S$, $k = 0, ..., [S/2]$. Following the same development as in Rodrigues (2001) one obtains the representations given in Theorem 4.2, but with $c$ now indexed by the frequency under test; i.e. we have a set of non-centrality parameters $c_k < 0$, $k = 0, \ldots, [S/2]$, attached to the representations (4.8)-(4.11), rather than the common value $c$.

Remark 4.10: Assumption 4.2 ensures that the limiting representations in (4.8)-(4.13) are invariant to the initial innovations $\{u_s^*\}_{s=1-S}^{0}$. If, however, $u_s^* = \sum_{k=0}^{[T\kappa]}(1+c/T)^k u_{-Sk}$, $\kappa > 0$, $s = 1-S, \ldots, 0$, which violates Assumption 4.2, then $(\sigma_u^2 T)^{-1/2} u_s^* \Rightarrow \eta_s \sim N(0, (-2c)^{-1}(1-\exp\{2c\kappa\}))$, $s = 1-S, ..., 0$, which are mutually independent. In this case the left member of (A.6) weakly converges to $\mathsf{J}_c^{s*}(r) \equiv \mathsf{J}_c^{s}(r) + \exp\{rc\}\eta_s$, $r \in [0,1]$, $s = 1-S, \ldots, 0$. Consequently, the representations in (4.8)-(4.13) hold as the corresponding functionals of the $\{\mathsf{J}_c^{s*}(r)\}_{s=1-S}^{0}$ processes. Where $\kappa \to \infty$, so that $(\sigma_u^2 T)^{-1/2} u_s^* \Rightarrow N(0, (-2c)^{-1})$, $s = 1-S, \ldots, 0$, we have what is referred to as the unconditional case, since the starting condition is drawn from the stationary distribution of the process.

It would be interesting to derive the limiting power envelope of tests for unit roots at each of the seasonal frequencies and for each of $\xi \in \{0,1,2\}$, along the lines of ERS for the non-seasonal case. However, when $S = 1$ we are concerned only with a simple power curve. In the seasonal case, where we are faced with an $S$-dimensional power hypersurface, the problem is clearly much more involved. For the same reasons, generalisations of the ERS and Elliott (1999) QD de-trended DF statistics to the seasonal unit root tests of Section 2 are problematic and certainly beyond the scope of the present paper. This is clearly an avenue which merits further research and it would be instructive to compare the asymptotic local power functions of the recursively de-meaned seasonal unit root statistics of this paper with such statistics. Seasonal first-difference de-trending, along the lines developed for $S = 1$ by Bhargava (1986) and Schmidt and Phillips (1992), has been developed in Rodrigues (2000) and is not considered further here. However, as demonstrated in Elliott (1994,p.44) and Stock (1994,p.2763), for $S = 1$ these statistics and DF tests based on simple symmetric estimation (see PGF) have equivalent asymptotic power properties. Moreover, Elliott (1994) finds no differences between their finite-sample properties. We conjecture that this property will extend to the seasonal case. The finite sample simulations reported in Rodrigues (2000) appear supportive of this view.

However, for $S = 1$ we may compare the asymptotic local power function of our $t_0(0)$

statistic with the QD-de-trended DF tests of ERS and Elliott (1999). The role of the initial innovation $u_0^*$ forms the distinction between these tests. ERS assume that $u_0^*$ satisfies Assumption 4.2, in particular they assume that $u_0^* \sim (0, \sigma_u^2)$, what is referred to as the **conditional** case in the literature, while Elliott (1999) develops QD-de-trended DF statistics for the **unconditional** case where $T^{-1/2} u_0^* \Rightarrow N(0, (-2c)^{-1} \sigma_u^2)$; cf. Remark 4.10. The limiting power envelope and the local limiting power of the ERS test are altered in the unconditional case **vis-à-vis** the conditional case; cf. ERS (pp.819-820) and Elliott (1999). Precise details on the form of these statistics are given in ERS and Elliott (1999). We will denote the statistics of ERS and Elliott (1999) as $t_{GLS}$ and $t_{GLSU}$, respectively. In Table 4.1 we tabulate the limiting powers of these statistics, together with those of the $t_0(0)$ and $t_0(1)$ (DF) statistics for $\xi = 1, 2$, and the non-centrality parameter $c \in \{-1, -5, -7, -10, -13, -20\}$, in each case for tests conducted at each of the 0.01, 0.05 and 0.10 levels. We tabulate the limiting powers of these statistics under both the conditional and unconditional environments. The results were obtained by direct simulation of the limiting functionals, as detailed in ERS (p.821), but using $100,000$ replications and a sample size of $1,000$. The limiting power envelopes for $\xi = 1, 2$ on a nominal 0.05 significance level, are tabulated in Table 9.3 of Tanaka (1996, p.348) for the 'conditional' case and Elliott (1999) for the 'unconditional' case.

Table 4.1 about here

We see from the results in Table 4.1 that in both the conditional and unconditional cases the standard DF test, $t_0(1)$, displays limiting power well below the other statistics for both $\xi = 1$ and $\xi = 2$. Echoing findings in Elliott (1999), the $t_{GLS}$ statistic has higher local power than $t_{GLSU}$ in the conditional case, as expected, but still generally outperforms the $t_{GLSU}$ statistic in the unconditional case, except for larger negative values of $c$. In the conditional case the $t_{GLS}$ statistic is also more powerful than the recursive de-meaned DF statistic $t_0(0)$, although $t_0(0)$ is always superior to the $t_{GLSU}$ statistic. However, in the **unconditional** case, where the QD de-trending of ERS is **not** efficient, $t_0(0)$ outperforms **both** $t_{GLS}$ and $t_{GLSU}$ for all values of $c$ and for both $\xi = 1$ and $\xi = 2$. Where there are differences between the asymptotic local power of the statistics in Table 4.1, these are larger for $\xi = 1$ **vis-à-vis** $\xi = 2$; ERS and Elliott (1999) also observe this phenomenon.

If, as argued by PGF "... there are a modest number of situations in which one is willing to assume that the first observation has variance equal to the residual variance ..." (p.459), then the results in Table 4.1 suggest that the $t_0(0)$ statistic should be preferred to the QD de-trended DF tests of ERS and Elliott (1999).

# 5 Finite-Sample Results

In this Section we use Monte Carlo methods to produce finite-sample critical values for the $t_0(0)$, $t_2(0)$, $F_1(0)$, $F_{[1...2]}(0)$ and $F_{[0...2]}(0)$ test-statistics of Section 3 for quarterly data, $S = 4$. We also investigate the small sample properties (size under autocorrelated errors and power under stationary alternatives) of these statistics and compare these with the conventional quarterly HEGY tests $t_0(1)$, $t_2(1)$, $F_1(1)$, $F_{[1...2]}(1)$ and $F_{[0...2]}(1)$, and also such (full-sample de-meaned) HEGY tests where the auxiliary regression (2.3) is estimated not by OLS but using the SSLS and WSLS estimators detailed in Fuller (1996,pp.414-416 and 568-573). The SSLS and WSLS estimators have been applied in the case of $S = 1$ by PGF, who demonstrate that DF tests based on SSLS and, in particular WSLS, estimation of the test regression display considerably better finite-sample power properties under the alternative than DF tests based on OLS estimation. Elliott (1999) also finds that, for $S = 1$, the WSLS DF test performs very well in the unconditional case. It therefore seems a useful exercise to compare seasonal extensions of the PGF procedures with the standard OLS-HEGY tests and also the recursive de-meaned HEGY tests proposed in this paper. In assessing the finite-sample size and power properties of these statistics we have focused on the sample sizes $4T = 100, 200$, and on Cases 3 and 6 of (2.3), where $\xi = 1$ and $\xi = 2$ respectively for all reported statistics. All tests were run at the nominal 0.05 level. Cases 2, 4 and 5 of (2.3) and other nominal levels were also considered, as were the corresponding statistics for other values of $S$, including $S = 1$, but in each case yielded qualitatively similar results to those reported. In all experiments, the lag truncation order $p^*$ in (2.3) was chosen via a data-dependent rule. As is commonly done in practice, we followed the general-to-specific approach outlined in BM (pp.318-19), starting with an initial four lags of $\Delta_4 \hat{x}^{\kappa}_{4t+s}$, ($p_{\max} = 4$) and progressively deleting lags found to be insignificant at the 0.10 level.

All reported simulations were programmed using the RNDN function of GAUSS 3.1 on a Pentium 233Mhz micro-computer using 40,000 replications for each simulation. These programs, available from the author on request, can replicate all experiments, including the calculation of critical values, reported in this paper for arbitrary $S$ (including $S = 1$), $T$

and $\lambda$ and also allow for other methods of selecting $p^*$, the lag truncation order, (including deterministic) in (2.3), and also allow one to compute critical values and size/power results for the $t_k^{\alpha}(\lambda)$ and $t_k^{\beta}(\lambda)$, $k = 1, \ldots, S^*$, statistics which are not reported here.

## 5.1 Finite Sample Critical Values

The simulations computed in this Section were based on the DGP $\Delta_4 x_{4t+s} = u_{4t+s} \sim IN(0,1)$, $s = -3, \ldots, 0$, $t = 1, \ldots, T$, with $u_{4j+s} = x_{4j+s} = 0$, $j \leq 0$. We report results for the sample sizes $4T = 48, 100, 136, 200, 400$. The test-statistics $t_k(0)$, $k = 0, 2$, $F_1(0)$, $F_{1\ldots2}(0)$ and $F_{0\ldots2}(0)$ were computed from Cases 2 to 6 of (2.3) in Tables 5.1a-5.1e, respectively, with $p^* = 0$ in (2.3). Critical values for Case 1 of (2.3) may be found in HEGY (Tables 1a and 1b, pp.226-7) and GLN.

Tables 5.1a – 5.1e about here

The finite sample critical values reported in Tables 5.1a-5.1e reflect the asymptotic predictions of Remark 4.6, even at relatively small sample sizes. It is also interesting to compare the critical values in Tables 5.1a-5.1e with the corresponding conventional HEGY, ($\lambda = 1$), tests. For example, for $4T = 100$, from ST1 (Tables 1a-1b, p.276) the 0.05 critical values for the conventional HEGY tests, $t_0(1)$ and $t_2(1)$ statistics are $-3.39$ and $-3.38$ respectively, all of which are considerably left-shift, vis-à-vis the corresponding critical value for the recursive mean adjusted statistic in Table 5.1e. Corresponding right-shifts are seen in the upper tails of each of the $F_1(1)$, $F_{[1\ldots2]}(1)$ and $F_{[0\ldots2]}(1)$ statistics, relative to the $F_1(0)$, $F_{[1\ldots2]}(0)$ and $F_{[0\ldots2]}(0)$ statistics.

## 5.2 Size Properties

Table 5.2 compares the finite-sample properties of the $t_0(\lambda)$, $t_{S/2}(\lambda)$, $F_1(\lambda)$, $F_{[1\ldots2]}(\lambda)$ and $F_{[0\ldots2]}(\lambda)$, $\lambda \in \{0, 1\}$, statistics when the error process displays weak parametric autocorrelation. In the case of $\lambda = 1$, (2.3) was estimated in three ways: OLS, SSLS and WSLS.

Table 5.2 about here

Table 5.2 reports the actual size of the above statistics (nominal 0.05 level) when the true DGP for $\{x_{4t+s}\}$ is:

$$(1 - \phi L)\Delta_4 x_{4t+s} = (1 + \theta L^4)u_{4t+s}, \quad s = -3, \ldots, 0, t = 1, \ldots, T, \tag{5.1}$$

with the initial conditions $x_s$ set to zero, and $u_{4t+s} \sim IN(0, 1)$, $s = -3, \ldots, 0$, $t = -100, \ldots, T$. We consider the effects of $(\phi, \theta) = (0.6, 0)$, and $(\phi, \theta) = (0, -0.4)$. Other parameter values were considered but qualitatively do not add to or contradict what is reported here.

It is evident from Table 5.2 that for $(\phi, \theta) = (0.6, 0)$ all of the reported statistics are approximately on their nominal levels, as might be hoped given the choice of $p_{\text{max}} = 4$. In the case of $(\phi, \theta) = (0, -0.4)$, where we might expect to see some size distortions in the statistics, we see from Table 5.2 that for $\xi = 1$ the patterns of distortions from the nominal level seen in the conventional OLS-HEGY tests and their recursive mean adjusted counterparts are virtually identical, but that the HEGY tests based on SSLS and WSLS estimation of (2.3) display rather larger size distortions. In the case of $\xi = 2$ the recursively de-meaned HEGY statistics outperform all other variants, including the OLS-HEGY statistics. All of the statistics display size distortions that increase between $\xi = 1$ and $\xi = 2$ but decrease between $4T = 100$ and $4T = 200$.

## 5.3 Empirical Power

This sub-section compares the empirical finite sample power properties of the statistics considered in Section 5.2 against seasonal (trend) stationary autoregressive processes.

Tables 5.3 about here

Table 5.3 reports the power (nominal 0.05 level) of the above statistics when the true DGP for $\{x_{4t+s}\}$ is the stationary seasonal AR model:

$$\alpha(L)x_{4t+s} = u_{4t+s} \sim IN(0, 1), s = -3, \ldots, 0, t = 1, \ldots, T, \tag{5.2}$$

with $\alpha(L) = (1 - \alpha L^4)$, $u_{4j+s} = 0$, $j \leq 0$, and the initial conditions drawn from the stationary distribution of the $\{x_{4t+s}\}$ process; that is, $x_s \sim IN(0, (1 - \alpha^2)^{-1})$, $s = -3, \ldots, 0$. We investigate the effects of $\alpha \in \{0.95, 0.90, 0.80\}$ in our experiments.

The results in Table 5.3 highlight a number of points. Firstly, for all of the reported statistics, power at a specific seasonal frequency, or set of frequencies, is the greater, *ceteris* *paribus*: (i) the greater is the sample size, $4T$; (ii) the smaller is the seasonal AR parameter, $\alpha$, and (iii) for $\xi = 1$, relative to $\xi = 2$. These results are fully consistent with what we might expect and with earlier simulation studies of conventional HEGY tests; *e.g.*, GLN.

It is quite clear from the results in Table 5.3 that for both $\xi = 1$ and $\xi = 2$, the recursive mean adjusted seasonal unit root statistics proposed in this paper enjoy considerable finite-sample power advantages over the corresponding conventional OLS-HEGY tests and, to a lesser extent, the SSLS-HEGY tests. In fact, there is not one single example in Table 5.3 where either an OLS-HEGY or SSLS-HEGY test displays superior power than the corresponding recursive mean-adjusted seasonal unit root statistic. The same can be said of the SSLS-HEGY statistics *vis-à-vis* the OLS-HEGY statistics. The WSLS-HEGY statistic are, however, much closer competitors on power to the recursively de-meaned HEGY statistics, although in most cases the recursive de-meaned statistics outperform their WSLS-HEGY counterparts. Both clearly dominate the OLS-HEGY and SSLS-HEGY statistics.

Although not reported here, we also considered the conditional case where the initial values were generated according to $x_s \sim IN(0, 1)$, $s = -3, ..., 0$. In this case the SSLS-HEGY, WSLS-HEGY and recursively de-meaned HEGY statistics all showed a small increase in power over the unconditional case, but their relative rankings on power were not altered. The results for the OLS-HEGY statistics were virtually identical to those in Table 5.3. We also experimented with values of the *warm up* parameter $\lambda$ other than zero; see the discussion at the end of Section 3. In general, for the DGPs considered in this Section, varying $\lambda$ from zero reduced the power of the recursive mean adjusted tests, *vis-à-vis* the values reported in Table 5.3 for $\lambda = 0$.

# 6 Conclusions

This paper has been concerned with providing regression-based test statistics for (seasonal) unit roots which are similar both exactly and asymptotically with respect to initial value(s) of the process and (seasonal) drift parameter(s). Existing approaches (seasonally) de-mean and (seasonally) de-trend the series over all available sample data. In contrast, we obtain similar tests *via* recursive (seasonal) de-meaning and (seasonal) de-trending of the data.

Representations for the limiting distributions of these statistics under the (seasonal) unit root null hypothesis and a near (seasonal) integrated alternative are provided, together with finite-sample critical values appropriate to a Gaussian quarterly random walk DGP. By deriving representations for the limiting distributions of statistics associated with other cases of interest, we are also able to provide a rationale for the similarity between the statistics' critical values in different scenarios. For the non-seasonal case we tabulate the asymptotic local power function of the proposed statistics, showing that this compares favourably with the QD de-trended DF tests of ERS and Elliott (1999).

We have also investigated the size and power properties of the recursively de-meaned HEGY statistics, and compared these with conventional full-sample de-meaned HEGY statistics obtained from test regressions estimated by each of OLS, SSLS and WSLS. The results suggest that the recursively de-meaned HEGY statistics simultaneously display both the best size and the best power properties of all of the statistics considered. Given all of the above positive evidence, we therefore strongly recommend their use in practice for both seasonal and non-seasonal unit root testing.

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# Appendix - Proof of Theorems

**Proof of Theorem 4.1**: Define the partial sum processes (PSP) $\mathcal{S}^s_{[Tr]} = \sum_{k=1}^{[Tr]} u_{Sk+s}$, $r \in [0,1)$, where $[Tr]$ denotes the integer part of $Tr$, with the convention $\mathcal{S}^s_T = \sum_{k=1}^{T} u_{Sk+s}$, $s = 1-S, ..., 0$. Under Assumption 4.1, cf. Chan and Wei (1988, Theorem 2.2, p.372), the PSP's $\mathcal{S}^s_{[Tr]}$, $s = 1-S, ..., 0$, satisfy a multivariate invariance principle such that

$$(\sigma_u^2 T)^{-1/2} \mathcal{S}^s_{[Tr]} \Rightarrow \mathsf{W}^s(r), r \in [0,1], \tag{A.1}$$

jointly, where the $\mathsf{W}^s(r)$ are independent standard Brownian motions, $s = 1-S, ..., 0$. Consequently, from (3.6), (A.1) and an application of the continuous mapping theorem (CMT), under $\mathsf{H}_0$ of (2.5),

$$\begin{aligned}(\sigma_u^2 T)^{-1/2} \hat{x}^3_{S[Tr]+s} &= (\sigma_u^2 T)^{-1/2} \left( \mathcal{S}^s_{[Tr]} - \frac{1}{\bar{r}} \int_0^{\bar{r}} \mathcal{S}^s_{[T\tau]} d\tau \right) \\ &\Rightarrow \mathsf{W}^s(r) - \bar{r}^{-1} \int_0^{\bar{r}} \mathsf{W}^s(\tau) d\tau \equiv \mathsf{B}^1_s(r, \lambda), \end{aligned} \tag{A.2}$$

$s = 1-S, ..., 0$, $\bar{r} = \max(r, \lambda)$, $r, \lambda \in [0,1]$. Similarly, from (3.5), (A.1) and the CMT,

$$\begin{aligned}(\sigma_u^2 T)^{-1/2} \hat{x}^6_{S[Tr]+s} &\Rightarrow \mathsf{B}^1_s(r, \lambda) - 12\bar{r}^{-3}(r - \frac{\bar{r}}{2}) \int_0^{\bar{r}} \left[ (u - \frac{\bar{r}}{2}) \left\{ \mathsf{W}^s(u) - \frac{1}{\bar{r}} \int_0^{\bar{r}} \mathsf{W}^s(\tau) d\tau \right\} \right] du \\ &\equiv \mathsf{B}^2_s(r, \lambda), \quad s = 1-S, ..., 0, \quad r, \lambda \in [0,1], \quad u \in [0, \bar{r}]. \end{aligned} \tag{A.3}$$

That is, $\mathsf{B}^1_s(r, \lambda)$ and $\mathsf{B}^2_s(r, \lambda)$ are, respectively, independent recursive de-meaned and recursive de-meaned and de-trended standard Brownian motions, $s = 1-S, ..., 0$. Notice that for $\lambda = 0$, we obtain $\mathsf{B}^2_s(r, 0) = \mathsf{B}^1_s(r, 0) - 6r^{-2} \int_0^r \left[ (u - \frac{r}{2}) \left\{ \mathsf{W}^s(u) - \frac{1}{r} \int_0^r \mathsf{W}^s(\tau) d\tau \right\} \right] du$, $s = 1-S, ..., 0$, where $\mathsf{B}^1_s(r, 0) \equiv \mathsf{W}^s(r) - r^{-1} \int_0^r \mathsf{W}^s(\tau) d\tau$, $s = 1-S, ..., 0$, while for $\lambda = 1$, $\mathsf{B}^2_s(r, 1)$ and $\mathsf{B}^1_s(r, 1)$ are respectively independent de-meaned and de-meaned and de-trended standard Brownian motions, $s = 1-S, ..., 0$; cf. ST2.

In the analysis which follows we will require certain definitions of standard Brownian motions:

$$\begin{aligned}\sqrt{S} \mathsf{W}^0_0(r) \equiv \sum_{s=1-S}^{0} \mathsf{W}^s(r), \sqrt{S} \mathsf{W}^0_{S/2}(r) \equiv \sum_{j=1-S}^{0} (-1)^{-j} \mathsf{W}^j(r), \\ \sqrt{\frac{S}{2}} \mathsf{W}^0_{\alpha,k}(r) \equiv \sum_{j=1-S}^{0} \cos j\omega_k \mathsf{W}^j(r), \sqrt{\frac{S}{2}} \mathsf{W}^0_{\beta,k}(r) \equiv \sum_{j=1-S}^{0} \sin j\omega_k \mathsf{W}^j(r), \end{aligned} \tag{A.4}$$

and of recursive de-meaned ($\xi = 1$) and recursive de-meaned and de-trended ($\xi = 2$) standard Brownian motions:

$$\sqrt{S}\mathsf{W}_0^{\xi}(r,\lambda) \equiv \sum_{j=1-S}^{0} \mathsf{B}_j^{\xi}(r,\lambda), \sqrt{S}\mathsf{W}_{S/2}^{\xi}(r,\lambda) \equiv \sum_{j=1-S}^{0} (-1)^{-j}\mathsf{B}_j^{\xi}(r,\lambda),$$

$$\sqrt{\frac{S}{2}}\mathsf{W}_{\alpha,k}^{\xi}(r,\lambda) \equiv \sum_{j=1-S}^{0} \cos j\omega_k \mathsf{B}_j^{\xi}(r,\lambda), \sqrt{\frac{S}{2}}\mathsf{W}_{\beta,k}^{\xi}(r,\lambda) \equiv \sum_{j=1-S}^{0} \sin j\omega_k \mathsf{B}_j^{\xi}(r,\lambda), \qquad \text{(A.5)}$$

$k = 1, ..., S^*$. Consequently, $\mathsf{W}_0^0(r)$, $\mathsf{W}_{S/2}^0(r)$, $\mathsf{W}_{\alpha,k}^0(r)$, $\mathsf{W}_{\beta,k}^0(r)$, $k = 1, ..., S^*$, $\mathsf{W}_0^1(r,\lambda)$, $\mathsf{W}_{S/2}^1(r,\lambda)$, $\mathsf{W}_{\alpha,k}^1(r,\lambda)$, $\mathsf{W}_{\beta,k}^1(r,\lambda)$, $k = 1, ..., S^*$, and $\mathsf{W}_0^2(r,\lambda)$, $\mathsf{W}_{S/2}^2(r,\lambda)$, $\mathsf{W}_{\alpha,k}^2(r,\lambda)$, $\mathsf{W}_{\beta,k}^2(r,\lambda)$, $k = 1, ..., S^*$, are mutually independent as they are respectively orthogonal transformations of the standard Brownian motions, $\mathsf{W}^s(r)$, $s = 1-S, ..., 0$, the recursive de-meaned standard Brownian motions, $\mathsf{B}_s^1(r,\lambda)$, $s = 1 - S, ..., 0$, and the recursive de-meaned and de-trended standard Brownian motions $\mathsf{B}_s^2(r,\lambda)$, $s = 1 - S, ..., 0$, defined in (A.1)-(A.3).

Consider first the statistics from Case 6 of (2.3). It follows from (3.5), (A.3) and applications of the CMT that:

$$(\sigma_u^2 T)^{-1/2}\hat{x}_{0,S[Tr]+s}^6 \Rightarrow \sqrt{S}\mathsf{W}_0^2(r,\lambda), s = 1 - S, ..., 0,$$

$$(\sigma_u^2 T)^{-1/2}\hat{x}_{S/2,S[Tr]+s}^6 \Rightarrow \cos[(s+1)\pi]\sqrt{S}\mathsf{W}_{S/2}^2(r,\lambda),$$

$$(\sigma_u^2 T)^{-1/2}\hat{x}_{k,S[Tr]+s}^6 \Rightarrow \sqrt{\frac{S}{2}}\left[\cos(s+1)\omega_k \mathsf{W}_{\alpha,k}^2(r,\lambda) + \sin(s+1)\omega_k \mathsf{W}_{\beta,k}^2(r,\lambda)\right],$$

$$(\sigma_u^2 T)^{-1/2}\hat{x}_{S-k,S[Tr]+s}^6 \Rightarrow \sqrt{\frac{S}{2}}\left[\sin(s+1)\omega_k \mathsf{W}_{\alpha,k}^2(r,\lambda) - \cos(s+1)\omega_k \mathsf{W}_{\beta,k}^2(r,\lambda)\right],$$

$r, \lambda \in [0,1]$, $s = 1 - S, ..., 0$. Noting that under $\mathsf{H}_0$ of (2.5), the OLS estimator for $\sigma_u^2$ from the regression (2.3) is consistent for $\sigma_u^2$, and using results directly analogous to (B.11) and (B.19) and the asymptotic orthogonality results, (B.10) and (B.12)-(B.16) of ST2, Appendix B, and by the CMT, we obtain the representation in (4.1) for the limiting distribution of $t_0(\lambda)$. The representation in (4.1) for $t_{S/2}(\lambda)$ follows using analogous expressions to (B.11) and (B.20) of ST2, Appendix B, similarly. Finally, using analogous results to (B.8), (B.17) and (B.9), (B.18) and the asymptotic orthogonality results of ST2, Appendix B, we similarly obtain the asymptotic representations (4.2) and (4.3) respectively for $t_k^{\alpha}(\lambda)$ and $t_k^{\beta}(\lambda)$. The representations (4.4)-(4.6) then follow immediately from the asymptotic orthogonality results, (B.10) of ST2, Appendix B.

We now adapt the above results to derive the limiting null distributions of the corresponding tests statistics derived from Cases 1-5 of (2.2), and hence (2.3).

Case I: In this case it follows from (A.1) that under $\mathsf{H}_0$ of (2.5), $(\sigma_u^2 T)^{-1/2} x_{S[Tr]+s}^1 \Rightarrow \mathsf{W}^s(r)$, $s = 1 - S, \ldots, 0$. Hence replace $\mathsf{B}_s^2(r, \lambda)$ by $\mathsf{W}^s(r)$. It is then easily seen, using the same development as above, that (4.1)-(4.6) hold for the standard Brownian motions $\mathsf{W}_0^0(r)$, $\mathsf{W}_{S/2}^0(r)$ and $\mathsf{W}_{\alpha,k}^0(r)$, $\mathsf{W}_{\beta,k}^0(r)$, $k = 1, ..., S^*$, which do not depend on $\lambda$.

Case II: Under $\mathsf{H}_0$ of (2.5),

$$(\sigma_u^2 T)^{-1/2} \hat{x}_{S[Tr]+s}^2 \Rightarrow \mathsf{W}^s(r) - \bar{r}^{-1} \int_0^{\bar{r}} \left[ \frac{1}{S} \sum_{j=1-S}^{0} \mathsf{W}^j(r) \right] dr \equiv \mathsf{B}_s^{*1}(r, \lambda), s = 1 - S, ..., 0.$$

Hence, replace $\mathsf{B}_s^2(r, \lambda)$ by $\mathsf{B}_s^{*1}(r, \lambda)$ in (4.1)-(4.6). Consequently, the representation corresponding to $t_0(\lambda)$ of (4.1) involves

$$\frac{1}{\sqrt{S}} \sum_{j=1-S}^{0} \mathsf{B}_j^{*1}(r, \lambda) = \frac{1}{\sqrt{S}} \sum_{j=1-S}^{0} \mathsf{B}_j^1(r, \lambda) \equiv \mathsf{W}_0^1(r, \lambda).$$

Turning to the seasonal frequency tests $t_k^\alpha(\lambda)$, $t_k^\beta(\lambda)$ and $F_k(\lambda)$, $k = 1, \ldots, S^*$ and $t_{S/2}(\lambda)$, the corresponding representations (4.2)-(4.4) involve

$$\sqrt{\frac{2}{S}} \sum_{j=1-S}^{0} \cos j\omega_k \mathsf{B}_j^{*1}(r, \lambda) = \sqrt{\frac{2}{S}} \sum_{j=1-S}^{0} \cos j\omega_k \mathsf{W}^j(r), k = 1, \ldots S^*,$$

$$\sqrt{\frac{2}{S}} \sum_{j=1-S}^{0} \sin j\omega_k \mathsf{B}_j^{*1}(r, \lambda) = \sqrt{\frac{2}{S}} \sum_{j=1-S}^{0} \sin j\omega_k \mathsf{W}^j(r), k = 1, \ldots S^*,$$

and

$$\frac{1}{\sqrt{S}} \sum_{j=1-S}^{0} (-1)^{-j} \mathsf{B}_j^{*1}(r, \lambda) = \frac{1}{\sqrt{S}} \sum_{j=1-S}^{0} (-1)^{-j} \mathsf{W}^j(r),$$

each of which are standard Brownian motions; cf. (A.4).

Case III: In this case from (A.2), replace $\mathsf{B}_s^2(r, \lambda)$ by $\mathsf{B}_s^1(r, \lambda)$, $s = 1 - S, ..., 0$. Hence, the representations in (4.1)-(4.6) hold for the recursive de-meaned Brownian motions $\mathsf{W}_0^1(r, \lambda)$, $\mathsf{W}_{S/2}^1(r, \lambda)$ and $\mathsf{W}_{\alpha,k}^1(r, \lambda)$, $\mathsf{W}_{\beta,k}^1(r, \lambda)$, $k = 1, ..., S^*$.

Case IV: Here, under $\mathsf{H}_0$ of (2.5),

$$(\sigma_u^2 T)^{-1/2} \hat{x}_{S[Tr]+s}^4 \Rightarrow \mathsf{B}_s^{*1}(r, \lambda) - 12\bar{r}^{-3}(r - \frac{\bar{r}}{2}) \int_0^{\bar{r}} \left[ S^{-1}(u - \frac{\bar{r}}{2}) \sum_{j=1-S}^{0} \mathsf{B}_j^{***1}(u, \lambda) \right] du \equiv \mathsf{B}_s^{*2}(r, \lambda),$$

$s=1-S,...,0$, where $\mathsf{B}_s^{***1}(u,\lambda)=\mathsf{W}^s(u)-\bar{r}^{-1}\int_0^{\bar{r}}\left[\frac{1}{S}\sum_{j=1-S}^{0}\mathsf{W}^j(\tau)\right]d\tau$, $u\in[0,\bar{r}]$, $s=1-S,...,0$. Hence, replace $\mathsf{B}_s^2(r,\lambda)$ by $\mathsf{B}_s^{*2}(r,\lambda)$ in (4.1)-(4.6). Similarly to Case II, we see that

$$\frac{1}{\sqrt{S}}\sum_{j=1-S}^{0}(-1)^{-j}\mathsf{B}_j^{*2}(r,\lambda)=\frac{1}{\sqrt{S}}\sum_{j=1-S}^{0}(-1)^{-j}\mathsf{W}^j(r),$$

$$\sqrt{\frac{2}{S}}\sum_{j=1-S}^{0}\cos j\omega_k\mathsf{B}_j^{*2}(r,\lambda)=\sqrt{\frac{2}{S}}\sum_{j=1-S}^{0}\cos j\omega_k\mathsf{W}^j(r),k=1,\ldots,S^*,$$

and

$$\sqrt{\frac{2}{S}}\sum_{j=1-S}^{0}\sin j\omega_k\mathsf{B}_j^{*2}(r,\lambda)=\sqrt{\frac{2}{S}}\sum_{j=1-S}^{0}\sin j\omega_k\mathsf{W}^j(r),k=1,\ldots,S^*,$$

each of which is a standard Brownian motion; cf. (A.4). The representation corresponding to $t_0(\lambda)$ of (4.1) involves

$$\frac{1}{\sqrt{S}}\sum_{j=1-S}^{0}\mathsf{B}_j^{*2}(r)=\frac{1}{\sqrt{S}}\sum_{j=1-S}^{0}\mathsf{B}_j^{2}(r,\lambda)\equiv\mathsf{W}_0^2(r,\lambda).$$

Case V: In this case, under $\mathsf{H}_0$ of (2.5),

$$(\sigma_u^2T)^{-1/2}\hat{x}^5_{S[Tr]+s}\Rightarrow\mathsf{B}_s^1(r,\lambda)-12\bar{r}^{-3}(r-\frac{\bar{r}}{2})\int_0^{\bar{r}}\left[S^{-1}(u-\frac{\bar{r}}{2})\sum_{j=1-S}^{0}\mathsf{B}_j^{***2}(u,\lambda)\right]du\equiv\mathsf{B}_s^{**2}(r,\lambda),$$

$s=1-S,...,0$, where $\mathsf{B}_s^{**2}(u,\lambda)=\mathsf{W}^s(u)-\bar{r}^{-1}\int_0^{\bar{r}}\mathsf{W}^s(\tau)d\tau$, $s=1-S,...,0$, $u\in[0,\bar{r}]$. Hence, replace $\mathsf{B}_s^2(r,\lambda)$ by $\mathsf{B}_s^{**2}(r,\lambda)$. Consequently, the limiting distribution of $t_0(\lambda)$ of (4.1) involves

$$\frac{1}{\sqrt{S}}\sum_{j=1-S}^{0}\mathsf{B}_j^{**2}(r,\lambda)=\frac{1}{\sqrt{S}}\sum_{j=1-S}^{0}\mathsf{B}_j^{*2}(r,\lambda)\equiv\mathsf{W}_0^2(r,\lambda).$$

Representations (4.2)-(4.4) involve

$$\frac{1}{\sqrt{S}}\sum_{j=1-S}^{0}(-1)^{-j}\mathsf{B}_j^{**2}(r,\lambda)=\frac{1}{\sqrt{S}}\sum_{j=1-S}^{0}(-1)^{-j}\mathsf{B}_j^{1}(r,\lambda),$$

$$\sqrt{\frac{2}{S}}\sum_{j=1-S}^{0}\cos j\omega_k\mathsf{B}_j^{**2}(r,\lambda)=\sqrt{\frac{2}{S}}\sum_{j=1-S}^{0}\cos j\omega_k\mathsf{B}_j^{1}(r,\lambda),k=1,...,S^*,$$

and

$$\sqrt{\frac{2}{S}}\sum_{j=1-S}^{0}\sin j\omega_k\mathsf{B}_j^{**2}(r,\lambda)=\sqrt{\frac{2}{S}}\sum_{j=1-S}^{0}\sin j\omega_k\mathsf{B}_j^{1}(r,\lambda),k=1,...,S^*,$$

each of which is a recursive de-meaned standard Brownian motion; cf. (A.5).

$\mathsf{Proof\ of\ Theorem\ 4.2}$: Under $\mathsf{H}_c$ of (4.7) and the conditions of Theorem 4.2,

$$(\sigma_u^2 T)^{-1/2} x_{[Tr]+s} \Rightarrow \mathsf{J}_c^s(r), r \in [0,1], \tag{A.6}$$

jointly, where the $\mathsf{J}_c^s(r)$ are independent standard Ornstein-Ullenbeck (OU) processes, $\mathsf{J}_c^s(r) = \mathsf{W}^s(r) + c\int_0^r e^{(r-s)c}\mathsf{W}^s(r)dr$, $s = 1-S, ..., 0$. Cf. Chan and Wei (1987) and Rodrigues (2001). Similarly to the proof of Theorem 4.1, it then follows that under the above conditions,

$$(\sigma_u^2 T)^{-1/2} \hat{x}^3_{S[Tr]+s} \Rightarrow \mathsf{J}_c^s(r) - \bar{r}^{-1}\int_0^{\bar{r}} \mathsf{J}_c^s(\tau)d\tau \equiv \tilde{\mathsf{J}}_c^s(r,\lambda), r, \lambda \in [0,1], s = 1-S, ..., 0, \tag{A.7}$$

$S$ independent recursive de-meaned OU processes, and

$$(\sigma_u^2 T)^{-1/2} \hat{x}^6_{S[Tr]+s} \Rightarrow \tilde{\mathsf{J}}_c^s(r,\lambda) - 12\bar{r}^{-3}(r - \frac{\bar{r}}{2})\int_0^{\bar{r}}\left[(u - \frac{\bar{r}}{2})\left\{\mathsf{J}_c^s(u) - \frac{1}{\bar{r}}\int_0^{\bar{r}}\mathsf{J}_c^s(\tau)d\tau\right\}\right]du \equiv \hat{\mathsf{J}}_c^s(r,\lambda), \tag{A.8}$$

$r, \lambda \in [0,1]$, $u \in [0, \bar{r}]$, $s = 1-S, ..., 0$, $S$ independent recursive de-meaned and de-trended OU processes. The main results then follow along very similar lines to the proof for Theorem 4.1 (see also Rodrigues, 2001), where $\mathsf{W}^{\xi}_{j,c}(r,\lambda)$, $j = 0, S/2$, and $\mathsf{W}^{\xi}_{\alpha,k,c}(r,\lambda)$ and $\mathsf{W}^{\xi}_{\beta,k,c}(r,\lambda)$, $k = 1, \ldots, S^*$, are straightforwardly seen, with an obvious notation, to be independent standard, recursive de-meaned, and recursive de-meaned and de-trended OU processes for $\xi = 0$, $\xi = 1$ and $\xi = 2$, respectively.

**Table 4.1: Asymptotic Local Power of the $t_{GLS}$, $t_{GLSU}$, $t_0(1)$ and $t_0(0)$ Statistics at the 1 %, 5 % anc**

| | | | $c=-1$ | | | $c=-5$ | | | $c=-7$ | | | $c=-10$ | | | $c=$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Test | $\xi$ | C/U | 1 % | 5 % | 10 % | 1 % | 5 % | 10 % | 1 % | 5 % | 10 % | 1 % | 5 % | 10 % | 1 % | 5 |
| $t_{GLS}$ | 1 | C | .017 | .078 | .156 | .081 | .319 | .532 | .150 | .495 | .733 | .315 | .758 | .921 | .529 | .91 |
| | | U | .013 | .064 | .129 | .044 | .186 | .331 | .076 | .278 | .452 | .151 | .435 | .617 | .256 | .57 |
| | 2 | C | .010 | .052 | .105 | .026 | .113 | .215 | .042 | .174 | .312 | .086 | .309 | .499 | .165 | .48 |
| | | U | .010 | .052 | .103 | .021 | .099 | .188 | .033 | .145 | .259 | .063 | .244 | .404 | .117 | .37 |
| $t_{GLSU}$ | 1 | C | .014 | .066 | .126 | .041 | .154 | .271 | .067 | .232 | .382 | .132 | .392 | .580 | .240 | .58 |
| | | U | .013 | .060 | .118 | .038 | .150 | .268 | .063 | .231 | .386 | .129 | .396 | .589 | .239 | .59 |
| | 2 | C | .010 | .051 | .104 | .023 | .103 | .189 | .036 | .148 | .263 | .067 | .244 | .398 | .122 | .37 |
| | | U | .010 | .052 | .105 | .021 | .097 | .183 | .032 | .138 | .248 | .060 | .230 | .384 | .110 | .36 |
| $t_0(1)$ | 1 | C | .012 | .059 | .116 | .027 | .117 | .219 | .042 | .177 | .313 | .086 | .307 | .493 | .164 | .47 |
| | | U | .011 | .057 | .114 | .026 | .124 | .231 | .043 | .190 | .334 | .088 | .325 | .512 | .167 | .49 |
| | 2 | C | .010 | .049 | .101 | .018 | .081 | .157 | .026 | .110 | .206 | .046 | .178 | .314 | .083 | .28 |
| | | U | .010 | .051 | .103 | .018 | .084 | .162 | .025 | .113 | .213 | .044 | .186 | .326 | .082 | .29 |
| $t_0(0)$ | 1 | C | .017 | .076 | .152 | .063 | .235 | .395 | .107 | .347 | .541 | .208 | .552 | .752 | .350 | .74 |
| | | U | .014 | .064 | .128 | .046 | .187 | .333 | .079 | .291 | .482 | .167 | .493 | .706 | .302 | .70 |
| | 2 | C | .011 | .053 | .107 | .023 | .107 | .201 | .036 | .156 | .279 | .071 | .267 | .432 | .128 | .40 |
| | | U | .010 | .053 | .107 | .022 | .099 | .190 | .034 | .146 | .265 | .067 | .248 | .414 | .119 | .38 |

**Notes**: (1) The column headed 'U/C' refers to whether the data was generated according to the unconditional cas
(2) In the column headed 'Test', $t_{GLS}$ refers to the QD de-trended DF test of ERS, $t_{GLSU}$ refers to the QD de-tre
$t_0(1)$ refers to the OLS de-trended DF test, and $t_0(0)$ refers to the recursive de-meaned DF test; (3) $\xi$ is as defined

**Table 5.1: Finite Sample Critical Values, Recursively De-Meaned HEGY Statistics**
**DGP:** $\Delta_4 x_{4t+s} = u_{4t+s} \sim IN(0,1)$
**Auxiliary Regression (2.3),** $p^* = 0$

**Table 5.1a: Case 2**

| | $t_1(0)$ | | | | $t_2(0)$ | | | | $F_1(0)$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $4T$ | 0.010 | 0.025 | 0.050 | 0.100 | 0.010 | 0.025 | 0.050 | 0.100 | 0.900 | 0.950 | 0.975 | 0.990 |
| 48 | −2.95 | −2.61 | −2.32 | −2.01 | −2.54 | −2.19 | −1.90 | −1.56 | 2.38 | 3.11 | 3.83 | 4.80 |
| 100 | −3.00 | −2.67 | −2.40 | −2.09 | −2.56 | −2.21 | −1.90 | −1.58 | 2.44 | 3.16 | 3.89 | 4.88 |
| 136 | −3.02 | −2.71 | −2.42 | −2.11 | −2.53 | −2.19 | −1.91 | −1.59 | 2.38 | 3.12 | 3.87 | 4.74 |
| 200 | −3.02 | −2.70 | −2.43 | −2.13 | −2.53 | −2.20 | −1.91 | −1.59 | 2.41 | 3.11 | 3.80 | 4.79 |
| 400 | −3.02 | −2.70 | −2.44 | −2.15 | −2.54 | −2.23 | −1.93 | −1.62 | 2.42 | 3.13 | 3.87 | 4.81 |

| | $F_{[1...2]}(0)$ | | | | $F_{[0...2]}(0)$ | | | |
|---|---|---|---|---|---|---|---|---|
| $4T$ | 0.900 | 0.950 | 0.975 | 0.990 | 0.900 | 0.950 | 0.975 | 0.990 |
| 48 | 2.24 | 2.81 | 3.39 | 4.15 | 2.38 | 2.89 | 3.41 | 4.06 |
| 100 | 2.22 | 2.79 | 3.37 | 4.13 | 2.40 | 2.92 | 3.38 | 4.00 |
| 136 | 2.22 | 2.77 | 3.29 | 3.96 | 2.40 | 2.86 | 3.35 | 3.97 |
| 200 | 2.22 | 2.73 | 3.27 | 3.95 | 2.41 | 2.89 | 3.34 | 3.88 |
| 400 | 2.22 | 2.76 | 3.29 | 3.97 | 2.41 | 2.89 | 3.33 | 3.86 |

**Table 5.1b: Case 3**

| | $t_1(0)$ | | | | $t_2(0)$ | | | | $F_1(0)$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $4T$ | 0.010 | 0.025 | 0.050 | 0.100 | 0.010 | 0.025 | 0.050 | 0.100 | 0.900 | 0.950 | 0.975 | 0.990 |
| 48 | −2.91 | −2.56 | −2.30 | −1.99 | −2.92 | −2.56 | −2.29 | −1.98 | 3.43 | 4.33 | 5.27 | 6.36 |
| 100 | −2.95 | −2.61 | −2.35 | −2.04 | −2.91 | −2.63 | −2.36 | −2.05 | 3.56 | 4.44 | 5.29 | 6.43 |
| 136 | −2.97 | −2.64 | −2.36 | −2.06 | −2.95 | −2.63 | −2.37 | −2.07 | 3.61 | 4.49 | 5.32 | 6.44 |
| 200 | −2.97 | −2.65 | −2.38 | −2.09 | −2.97 | −2.64 | −2.38 | −2.09 | 3.68 | 4.54 | 5.38 | 6.46 |
| 400 | −3.04 | −2.71 | −2.43 | −2.14 | −2.99 | −2.69 | −2.44 | −2.13 | 3.69 | 4.56 | 5.41 | 6.47 |

| | $F_{[1...2]}(0)$ | | | | $F_{[0...2]}(0)$ | | | |
|---|---|---|---|---|---|---|---|---|
| $4T$ | 0.900 | 0.950 | 0.975 | 0.990 | 0.900 | 0.950 | 0.975 | 0.990 |
| 48 | 3.14 | 3.86 | 4.55 | 5.52 | 2.99 | 3.59 | 4.20 | 5.01 |
| 100 | 3.25 | 3.93 | 4.59 | 5.49 | 3.06 | 3.63 | 4.16 | 4.89 |
| 136 | 3.30 | 3.96 | 4.63 | 5.46 | 3.13 | 3.68 | 4.25 | 4.90 |
| 200 | 3.35 | 4.00 | 4.67 | 5.45 | 3.14 | 3.70 | 4.20 | 4.90 |
| 400 | 3.40 | 4.07 | 4.69 | 5.47 | 3.21 | 3.76 | 4.26 | 4.94 |

**Table 5.1c: Case 4**

| | $t_1(0)$ | | | | $t_2(0)$ | | | | $F_1(0)$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $4T$ | 0.010 | 0.025 | 0.050 | 0.100 | 0.010 | 0.025 | 0.050 | 0.100 | 0.900 | 0.950 | 0.975 | 0.990 |
| 48 | −3.59 | −3.25 | −2.96 | −2.64 | −2.53 | −2.17 | −1.89 | −1.56 | 2.50 | 3.26 | 4.06 | 5.13 |
| 100 | −3.63 | −3.30 | −3.04 | −2.74 | −2.53 | −2.19 | −1.90 | −1.56 | 2.48 | 3.22 | 3.91 | 4.88 |
| 136 | −3.63 | −3.33 | −3.06 | −2.77 | −2.53 | −2.20 | −1.91 | −1.58 | 2.45 | 3.23 | 3.98 | 4.88 |
| 200 | −3.63 | −3.32 | −3.06 | −2.78 | −2.52 | −2.19 | −1.91 | −1.57 | 2.43 | 3.17 | 3.89 | 4.94 |
| 400 | −3.65 | −3.35 | −3.09 | −2.82 | −2.53 | −2.20 | −1.91 | −1.60 | 2.44 | 3.14 | 3.83 | 4.67 |

| | $F_{[1...2]}(0)$ | | | | $F_{[0...2]}(0)$ | | | |
|---|---|---|---|---|---|---|---|---|
| $4T$ | 0.900 | 0.950 | 0.975 | 0.990 | 0.900 | 0.950 | 0.975 | 0.990 |
| 48 | 2.34 | 2.95 | 3.56 | 4.39 | 3.18 | 3.80 | 4.42 | 5.25 |
| 100 | 2.28 | 2.84 | 3.37 | 4.10 | 3.15 | 3.72 | 4.28 | 4.95 |
| 136 | 2.28 | 2.81 | 3.36 | 4.08 | 3.15 | 3.71 | 4.22 | 4.91 |
| 200 | 2.24 | 2.78 | 3.30 | 4.02 | 3.12 | 3.67 | 4.19 | 4.83 |
| 400 | 2.23 | 2.77 | 3.28 | 3.96 | 3.14 | 3.66 | 4.17 | 4.77 |

**Table 5.1d: Case 5**

| | $t_1(0)$ | | | | $t_2(0)$ | | | | $F_1(0)$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $4T$ | 0.010 | 0.025 | 0.050 | 0.100 | 0.010 | 0.025 | 0.050 | 0.100 | 0.900 | 0.950 | 0.975 | 0.990 |
| 48 | −3.46 | −3.13 | −2.83 | −2.53 | −2.95 | −2.58 | −2.30 | −1.98 | 3.44 | 4.36 | 5.37 | 6.66 |
| 100 | −3.53 | −3.21 | −2.94 | −2.66 | −2.94 | −2.62 | −2.36 | −2.05 | 3.51 | 4.35 | 5.19 | 6.28 |
| 136 | −3.54 | −3.25 | −2.98 | −2.69 | −2.96 | −2.65 | −2.37 | −2.05 | 3.58 | 4.46 | 5.31 | 6.37 |
| 200 | −3.58 | −3.29 | −3.03 | −2.74 | −3.00 | −2.67 | −2.40 | −2.08 | 3.64 | 4.51 | 5.37 | 6.54 |
| 400 | −3.62 | −3.31 | −3.07 | −2.78 | −3.03 | −2.69 | −2.41 | −2.11 | 3.70 | 4.57 | 5.41 | 6.45 |

| | $F_{[1...2]}(0)$ | | | | $F_{[0...2]}(0)$ | | | |
|---|---|---|---|---|---|---|---|---|
| $4T$ | 0.900 | 0.950 | 0.975 | 0.990 | 0.900 | 0.950 | 0.975 | 0.990 |
| 48 | 3.17 | 3.90 | 4.68 | 5.68 | 3.56 | 4.23 | 4.91 | 5.82 |
| 100 | 3.22 | 3.89 | 4.54 | 5.44 | 3.66 | 4.27 | 4.91 | 5.67 |
| 136 | 3.26 | 3.93 | 4.59 | 5.44 | 3.71 | 4.32 | 4.90 | 5.64 |
| 200 | 3.32 | 3.99 | 4.68 | 5.51 | 3.81 | 4.41 | 5.00 | 5.71 |
| 400 | 3.40 | 4.03 | 4.67 | 5.56 | 3.87 | 4.45 | 4.99 | 5.71 |

**Table 5.1e: Case 6**

| | $t_1(0)$ | | | | $t_2(0)$ | | | | $F_1(0)$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $4T$ | 0.010 | 0.025 | 0.050 | 0.100 | 0.010 | 0.025 | 0.050 | 0.100 | 0.900 | 0.950 | 0.975 | 0.990 |
| 48 | −3.60 | −3.21 | −2.92 | −2.57 | −3.55 | −3.21 | −2.90 | −2.57 | 6.03 | 7.39 | 8.75 | 10.57 |
| 100 | −3.57 | −3.22 | −2.96 | −2.64 | −3.56 | −3.22 | −2.96 | −2.64 | 6.13 | 7.29 | 8.44 | 9.88 |
| 136 | −3.58 | −3.24 | −2.97 | −2.67 | −3.53 | −3.23 | −2.96 | −2.67 | 6.18 | 7.34 | 8.39 | 9.87 |
| 200 | −3.56 | −3.26 | −3.00 | −2.71 | −3.58 | −3.25 | −2.98 | −2.70 | 6.28 | 7.40 | 8.49 | 9.83 |
| 400 | −3.60 | −3.31 | −3.05 | −2.76 | −3.57 | −3.26 | −3.02 | −2.75 | 6.46 | 7.52 | 8.54 | 9.85 |

| | $F_{[1...2]}(0)$ | | | | $F_{[0...2]}(0)$ | | | |
|---|---|---|---|---|---|---|---|---|
| $4T$ | 0.900 | 0.950 | 0.975 | 0.990 | 0.900 | 0.950 | 0.975 | 0.990 |
| 48 | 5.60 | 6.70 | 7.78 | 9.19 | 5.34 | 6.30 | 7.23 | 8.40 |
| 100 | 5.67 | 6.58 | 7.46 | 8.60 | 5.40 | 6.19 | 6.96 | 7.91 |
| 136 | 5.75 | 6.67 | 7.54 | 8.57 | 5.48 | 6.28 | 6.95 | 7.84 |
| 200 | 5.84 | 6.72 | 7.54 | 8.53 | 5.58 | 6.33 | 7.01 | 7.81 |
| 400 | 6.00 | 6.78 | 7.62 | 8.57 | 5.71 | 6.43 | 7.10 | 7.92 |

**Table 5.2: Empirical Size of Seasonal Unit Root Tests (Nominal $0.05$ level)**
**DGP:** $(1-\phi L)\Delta_4 x_{4t+s} = \epsilon_{4t+s} - \theta\epsilon_{4(t-1)+s}$
**Auxiliary Regression (2.3) with** $p_{\max} = 4$
$\xi = 1$ **indicates Case 3 of (2.3),** $\xi = 2$ **indicates Case 6 of (2.3)**

| $4T$ | $\phi$ | $\theta$ | Variant | $t_0$ | | $t_2$ | | $F_1$ | | $F_{[1...2]}$ | | $F_{[0...2]}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | $\xi=1$ | $\xi=2$ | $\xi=1$ | $\xi=2$ | $\xi=1$ | $\xi=2$ | $\xi=1$ | $\xi=2$ | $\xi=1$ | $\xi=2$ |
| 100 | 0.6 | 0.0 | SSLS | .05 | .04 | .04 | .04 | .05 | .05 | .05 | .05 | .05 | .07 |
| | | | WSLS | .05 | .04 | .04 | .05 | .05 | .05 | .04 | .06 | .05 | .07 |
| | | | OLS | .06 | .05 | .04 | .05 | .04 | .06 | .04 | .07 | .05 | .09 |
| | | | RDM | .05 | .04 | .05 | .05 | .04 | .05 | .05 | .05 | .05 | .05 |
| | 0.0 | $-0.4$ | SSLS | .22 | .29 | .22 | .32 | .23 | .44 | .29 | .57 | .34 | .66 |
| | | | WSLS | .20 | .29 | .20 | .30 | .23 | .42 | .27 | .56 | .32 | .64 |
| | | | OLS | .17 | .28 | .15 | .27 | .18 | .40 | .22 | .50 | .26 | .59 |
| | | | RDM | .17 | .23 | .15 | .23 | .18 | .34 | .24 | .42 | .26 | .46 |
| 200 | 0.6 | 0.0 | SSLS | .05 | .04 | .05 | .04 | .04 | .04 | .04 | .04 | .05 | .04 |
| | | | WSLS | .05 | .04 | .04 | .04 | .05 | .04 | .04 | .04 | .04 | .05 |
| | | | OLS | .05 | .04 | .04 | .03 | .05 | .04 | .05 | .04 | .04 | .04 |
| | | | RDM | .05 | .04 | .04 | .04 | .05 | .05 | .04 | .04 | .05 | .05 |
| | 0.0 | $-0.4$ | SSLS | .15 | .26 | .16 | .23 | .15 | .29 | .18 | .31 | .20 | .32 |
| | | | WSLS | .15 | .23 | .14 | .23 | .15 | .27 | .16 | .30 | .18 | .31 |
| | | | OLS | .12 | .22 | .12 | .21 | .13 | .24 | .15 | .27 | .15 | .27 |
| | | | RDM | .13 | .19 | .13 | .20 | .14 | .20 | .15 | .24 | .16 | .23 |

**Note**: In the column headed 'Variant', SSLS, WSLS and OLS refer to the $t_0(1)$, $t_2(1)$, $F_1(1)$, $F_{[1...2]}(1)$ and $F_{[0...2]}(1)$ statistics from (2.3) estimated by SSLS, WSLS and OLS respectively, while RDM refers to the $t_0(0)$, $t_2(0)$, $F_1(0)$, $F_{[1...2]}(0)$ and $F_{[0...2]}(0)$ statistics from (2.3).

**Table 5.3: Empirical Power of Seasonal Unit Root Tests (Nominal $0.05$ level)**
**DGP: $(1-\alpha L^4)x_{4t+s} = \epsilon_{4t+s}$**
**Auxiliary Regression (2.3) with $p_{\max} = 4$**
**$\xi = 1$ indicates Case 3 of (2.3), $\xi = 2$ indicates Case 6 of (2.3)**

| $4T$ | $\alpha$ | Variant | $t_0$ | | $t_2$ | | $F_1$ | | $F_{[1\ldots2]}$ | | $F_{[0\ldots2]}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\xi=1$ | $\xi=2$ | $\xi=1$ | $\xi=2$ | $\xi=1$ | $\xi=2$ | $\xi=1$ | $\xi=2$ | $\xi=1$ | $\xi=2$ |
| 100 | 0.95 | SSLS | .09 | .05 | .09 | .06 | .10 | .06 | .12 | .06 | .14 | .06 |
| | | WSLS | .09 | .05 | .07 | .06 | .12 | .06 | .12 | .07 | .15 | .05 |
| | | OLS | .07 | .06 | .07 | .05 | .07 | .06 | .08 | .06 | .09 | .06 |
| | | RDM | .09 | .06 | .10 | .06 | .13 | .06 | .15 | .07 | .16 | .07 |
| | 0.90 | SSLS | .13 | .07 | .14 | .07 | .16 | .08 | .22 | .09 | .26 | .09 |
| | | WSLS | .14 | .06 | .13 | .07 | .20 | .07 | .26 | .09 | .33 | .09 |
| | | OLS | .08 | .07 | .07 | .06 | .10 | .07 | .11 | .07 | .12 | .08 |
| | | RDM | .14 | .07 | .14 | .07 | .21 | .08 | .27 | .09 | .32 | .09 |
| | 0.80 | SSLS | .21 | .09 | .20 | .11 | .29 | .13 | .38 | .15 | .46 | .17 |
| | | WSLS | .21 | .09 | .20 | .10 | .31 | .13 | .40 | .16 | .51 | .17 |
| | | OLS | .14 | .09 | .15 | .10 | .19 | .11 | .25 | .13 | .31 | .15 |
| | | RDM | .22 | .10 | .22 | .12 | .35 | .14 | .47 | .18 | .54 | .20 |
| 200 | 0.95 | SSLS | .12 | .07 | .12 | .06 | .16 | .07 | .21 | .08 | .26 | .09 |
| | | WSLS | .14 | .06 | .12 | .07 | .20 | .07 | .27 | .08 | .34 | .09 |
| | | OLS | .07 | .06 | .07 | .05 | .11 | .07 | .12 | .07 | .12 | .07 |
| | | RDM | .14 | .07 | .13 | .08 | .22 | .08 | .28 | .09 | .35 | .10 |
| | 0.90 | SSLS | .21 | .11 | .25 | .09 | .34 | .14 | .47 | .16 | .59 | .20 |
| | | WSLS | .29 | .10 | .27 | .12 | .46 | .15 | .61 | .20 | .76 | .25 |
| | | OLS | .12 | .09 | .13 | .08 | .21 | .11 | .26 | .12 | .29 | .13 |
| | | RDM | .29 | .12 | .28 | .12 | .47 | .18 | .60 | .22 | .73 | .26 |
| | 0.80 | SSLS | .50 | .26 | .51 | .22 | .74 | .39 | .89 | .50 | .96 | .63 |
| | | WSLS | .55 | .25 | .52 | .25 | .80 | .42 | .92 | .58 | .98 | .69 |
| | | OLS | .37 | .22 | .36 | .20 | .64 | .32 | .78 | .43 | .86 | .51 |
| | | RDM | .55 | .28 | .55 | .28 | .82 | .46 | .93 | .60 | .98 | .71 |

**Note**: In the column headed 'Variant', SSLS, WSLS and OLS refer to the $t_0(1)$, $t_2(1)$, $F_1(1)$, $F_{[1\ldots2]}(1)$ and $F_{[0\ldots2]}(1)$ statistics from (2.3) estimated by SSLS, WSLS and OLS respectively, while RDM refers to the $t_0(0)$, $t_2(0)$, $F_1(0)$, $F_{[1\ldots2]}(0)$ and $F_{[0\ldots2]}(0)$ statistics from (2.3).