# On Fair Allocations and Indivisibilities

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**Abstract**

This paper studies the problem of how to distribute a set of indivisible objects with an amount $M$ of money among a number of agents in a fair way. We allow any number of agents and objects. Objects can be desirable or undesirable and the amount of money can be negative as well. In case $M$ is negative, it can be regarded as costs to be shared by the agents. The objects with the money will be completely distributed among the agents in a way that each agent gets a bundle with at most one object if there are more agents than objects, and gets a bundle with at least one object if objects are no less than agents. We prove via an advanced fixed point argument that under certain conditions the set of envy-free and efficient allocations is nonempty. We show that those conditions are also satisfied by all the existing conditions. Furthermore we demonstrate that if the total amount of money varies in an interval $[X, Y]$, then there exists a connected set of fair allocations whose end points are allocations with sums of money equal to $X$ and $Y$, respectively. Welfare properties are also analyzed when the total amount of money is modeled as a continuous variable.

**Keywords:**

Indivisibility, money, equity, fairness, Pareto optimality, resource allocation.

# 1 Introduction

The objective of this paper is to address the existence problem of fair allocation or fair division. In particular, we are concerned with the problem of how to distribute a number of indivisible objects with an amount of money among a group of people (or agents) in a fair way. As defined by Foley (1967), an allocation is said to be *envy-free* if no agent prefers another agent's consumption bundle to his own, and an allocation is said to be *fair* if it is both envy-free and Pareto optimal. Following Foley (1967) there has been an extensive literature dealing with the problem of fair division but most of this work was directed to the case where all goods are assumed to be perfectly divisible (see Varian (1974)).

The existence problem of fair allocation with indivisible objects was investigated by Svensson (1983), Maskin (1987), and Alkan, Demange and Gale (1991). A common and crucial assumption in these papers is that each agent consumes exactly one indivisible object. More precisely, both Svensson and Maskin required the same number of agents and objects in their models. Svensson showed that if utility functions are continuous and strictly increasing in money, and for every agent and every object, money can be distributed in a way that the agent prefers the object with its corresponding amount of money to any other object with its corresponding amount of money, then fair allocations exist. Maskin obtained his existence result under the condition that utility functions are continuous and strictly increasing in money, all objects are desirable, and every agent prefers any object with an equal share of money to any other object without money. Alkan et al presented a model without restriction on the number of agents and objects and by allowing agents to pay money if necessary. In case there are more objects than agents, Alan et al introduced fictitious agents into their model so that there are the same number of agents as objects. At a fair allocation they required that the total amount of money obtained by real agents be equal to the initial amount of money. Because of this requirement it appeared that the fixed point methods would not work for proving existence. Nevertheless, they showed, by means of an ingenious argument, the existence of fair allocation under certain conditions imposed on their model. The major tool they used is the duality theorem of linear programming. More recently, Aragones (1995) and Klijn (2000) focused their attention to the algorithmic aspect of the above models and presented two polynomial-time algorithms for finding a fair allocation in a setting where agents have quasi-linear utilities in money.

The purpose of this paper is twofold. One the one hand, we will consider a more general model by relaxing some restrictive assumptions in the existing models. In particular, we allow any number of agents and objects. All objects and money will be completely allocated among agents so that no objects will be left unassigned. Recall that in the models mentioned above each agent is assumed to consume exactly one object. So, if there are more objects than agents, some objects will have to be left either unassigned or assigned to fictitious agents. Obviously, this assumption is unrealistic and cannot be satisfied in some real life situations. On the other hand, we will give a unified approach to prove all the existing existence results. This is useful and interesting since the approaches of Svensson (1983), Maskin (1987), and Alkan et al (1991), appeared to be so different from each other. Two mathematical tools we use for the existence are a fundamental fixed point theorem of Browder (1960) and the well-known theorem of Birkhoff and von Neumann from discrete mathematics. Indeed, Alkan et al (1991) state it seems impossible to obtain their existence result by the usual fixed point argument. Nevertheless here we will show that it is still possible to prove their existence result by a more advanced fixed point argument. In fact, we will obtain existence results for their model under even weaker conditions.

The rest of the paper is organized as follows. In Section 2 we formulate the economic model and introduce the main solution concepts: envy-free and fair allocations. The model consists of a vector $W \in \mathbb{Z}^n_{++}$ of indivisible objects, an amount $M$ of money and $m$ agents. Thus, there are $W_j$ units of indivisible object $j = 1, \cdots, n$. We allow any number of agents and objects. Objects could be desirable or undesirable and the amount $M$ of money could be negative as well. In case $M$ is negative, it can be regarded as costs to be shared by agents. The objects with the money will be completely distributed among the agents in a way that each agent gets a bundle with at most one object if there are more agents than objects, and gets a bundle with at least one object if objects are no less than agents. We show by a simple example that if there are more objects than agents, then neither the set of envy-free allocations is a subset of Pareto optimal allocations nor the converse is true. This is in contrast to the case in which there are the same number of agents as objects, where, as shown by Svensson (1983), an envy-free allocation must be Pareto optimal, too. Furthermore, we illustrate via two examples that in general, without proper assumptions, fair allocations may fail to exist in case there are more objects than agents.

In Sections 3 and 4 sufficient conditions are given for the existence of fair allocations. All existence theorems presented here are based upon a mathematical theorem (i.e. Theorem 3.4), which states in a quite general form that a collection of covering sets in $\mathbb{R}^n$ will have a connected set of intersection points under certain mild conditions. We further show that the sufficient conditions introduced by the previous authors aslo satisfy our conditions. In the simplest case, it is shown that if utility functions are quasi-linear in money, then a fair allocation always exists without any additional condition. This result indicates that the conditions for the existence of fair allocations can be much weaker than those for the existence of Walrasian equilibria.

In Section 5 we obtain a strict monotonicity property of fair allocations. This result states that if utility functions are continuous and strictly increasing in money, then for any given fair allocation there exists a connected set of fair allocations in which each allocation makes every agent strictly better (worse) off when the total amount of money strictly increases (decreases). This result strengthens the strict monotonicity result of Alan et al (1991). Finally in Section 6 we give several new sufficient conditions for the existence of fair allocation in the model of Alan et al and then we show that their condition also satisfies the new conditions. Furthermore, a strict monotonicity property of fair allocations is derived. As a result, we provide a new proof for the results of Alan et al.

## 2 The model of fair allocations

We first introduce some notation. Let $I_k$ be the set of first $k$ positive integers, $\mathbb{R}^k$ the $k$-dimensional Euclidean space, and $\mathbb{Z}^k$ the set of integral vectors in $\mathbb{R}^k$. Given a positive integer $k$, $e(i)$ denotes the $i$-th unit vector in $\mathbb{R}^k$ for $i \in I_k$, $0^k$ and $1^k$ the all-zero vector and the all-one vector in $\mathbb{R}^k$, respectively. For any given number $t$, define

$$H(t) = \{x \in \mathbb{R}^m \mid \sum_{i=1}^{m} x_i = t\}.$$

Given two real numbers $X$ and $Y$ with $X \leq Y$, let $H(X,Y) = \{x \in \mathbb{R}^m \mid X \leq \sum_{i=1}^{m} x_i \leq Y\}$. Furthermore, let $S^n = \{x \in \mathbb{R}^n_+ \mid \sum_{j=1}^{n} x_j = 1\}$ be the $(n-1)$-dimensional unit simplex.

Our model consists of a finite number of agents, denoted by $\mathcal{P} = \{1, 2, \cdots, m\}$, a bundle $W \in \mathbb{Z}^n_{++}$ of $n$ types of indivisible objects, and a fixed amount of money, denoted by $M$. The $j$th component $W_j$ of $W$ means that there are $W_j$ units of indivisible object $j$. Here $M$ can be any real number. If $M$ is negative, this will be the case in cost sharing problems. Each agent $i \in \mathcal{P}$ is assumed to have a preference relation $u_i : \mathbb{Z}^n_+ \times \mathbb{R} \mapsto \mathbb{R}$ which is continuous and nondecreasing in money. This assumption is weaker than those made in the literature. Let $\mathcal{E} = (W, M, \mathcal{P}, [u_i])$ represent the economy described here. The goal is to allocate the objects with the money to the agents as fairly as possible. Here money will be treated as a perfectly divisible good. Without loss of generality it is assumed $2 \leq m \leq \sum_{j=1}^{n} W_j$. In case $\sum_{j=1}^{n} W_j < m$, we may add $m - \sum_{j=1}^{n} W_j$ units of dummy object (worthless and harmless objects) to the model. A *feasible allocation or distribution of objects* is an assignment of objects to agents in such a way that all objects are completely distributed among agents and furthermore each agent gets at least one unit of some object. Thus the set of all feasible allocations of objects can be written as

$$\Theta = \{\Pi = (\Pi(1), \Pi(2), \cdots, \Pi(m)) \mid \sum_{i=1}^{m} \Pi(i) = W \text{ and } 0^n \neq \Pi(i) \in \mathbb{Z}^n_+, \ \forall i \in I_m\}.$$

A vector $x \in \mathbb{R}^m$ is called a *distribution of money* among agents. An $m$-vector $x$ will be called $t$-feasible (or feasible) if $\sum_{i \in \mathcal{P}} x_i = t$. Thus the set $H(t)$ is the set of all $t$-feasible distributions of money.

For a pair $(\Pi, x) \in \Theta \times H(t)$, the interpretaion is that agent $i$ receives a bundle of goods $(\Pi(i), x_i)$ consisting of a bundle $\Pi(i)$ of objects and $x_i$ units of money. If $x_i < 0$, then agent $i$ pays others $|x_i|$ units of money. Given such an allocation, some agent may not be pleased with his bundle of goods. We are interested in so called fair allocations in which every agent likes his own bundle at least as well as that of anyone else and no other allocation can make every agent better off. Formally these key solution concepts are defined below.

**Definition 2.1** *An allocation $(\Pi, x) \in \Theta \times H(t)$ is envy-free if for all $i, j \in \mathcal{P}$ it holds*

$$u_i(\Pi(i), x_i) \geq u_i(\Pi(j), x_j).$$

At an envy-free allocation, no agent prefers any other agent's bundle to his own.

**Definition 2.2** *An allocation $(\Pi, x) \in \Theta \times H(t)$ is Pareto optimal if there is no other allocation $(\Xi, y) \in \Theta \times H(t)$ such that it holds*

$$u_i(\Pi(i), x_i) \leq u_i(\Xi(i), y_i).$$

*for every $i \in \mathcal{P}$; and there is some $j \in \mathcal{P}$ satisfying*

$$u_j(\Pi(j), x_j) < u_j(\Xi(j), y_j).$$

Let $PO(\mathcal{E})$ denote the set of all Pareto optimal allocations of the economy $\mathcal{E}$ for $t = M$.

Combining the above concepts leads to the notion of fairness which guarantees both equity of individuls and social optimality.

**Definition 2.3** *An allocation $(\Pi, x) \in \Theta \times H(t)$ is fair or t-fair if it is both envy-free and Pareto optimal.*

With respect to the economy $\mathcal{E}$, we can construct its subeconomies as follows. For any $\Pi \in \Theta$, the economy $E = (\Pi, M, \mathcal{P}, [\bar{u}_i])$ is called a *subeconomy* of the economy $\mathcal{E}$, where $\bar{u}_i : \{\Pi(1), \Pi(2), \cdots, \Pi(m)\} \times \mathbb{R} \mapsto \mathbb{R}$ is defined by $\bar{u}_i(\Pi(j), p) = u_i(\Pi(j), p)$ for each $j \in I_m$. Thus, we regard each set $\Pi(i)$ of objects as a single object for $i \in \mathcal{P}$. In the sequel we will make no difference between $\bar{u}_i$ and $u_i$. Two subeconomies $E^1 = (\Pi^1, M, \mathcal{P}, [u_i])$, $E^2 = (\Pi^2, M, \mathcal{P}, [u_i])$ are said to be *identical* if $\Pi^1$ and $\Pi^2$ are permutation of each other. For example, consider the case in which $W = (1, 1, 1, 1, 1)$ and $\mathcal{P} = \{1, 2, 3\}$. Then the subeconomies constructed from $\Pi^1 = ((1, 1, 0, 0, 0), (0, 0, 1, 1, 0), (0, 0, 0, 0, 1))$, $\Pi^2 = ((0, 0, 0, 0, 1), (1, 1, 0, 0, 0), (0, 0, 1, 1, 0))$ are identical.

It was shown by Svennson (1983) that for the case $m = n$ and $W = 1^n$, an envy-free allocation is also Pareto optimal. However, the following simple example will show that this is not the case if there are more objects than agents.

**Example 1.** Consider the case in which there are two agents 1, 2 and there are three objects $\alpha$, $\beta$, and $\gamma$, and total money (say, dollar) $M$ is equal to zero. Both agents have quasi-linear utilities in money (i.e., $u_i(A, m) = V_i(A) + m$, $i = 1, 2$) and the values of the agents for the different subsets of objects are given in Table 1.

In this example when agent 1 gets object $\beta$ with 2\$ and agent 2 gets objects $\alpha\gamma$ by paying 2\$, this allocation is envy-free but not Pareto optimal, because another allocation in which agent 1 gets objects $\alpha\beta$ by paying 2.5\$ and agent 2 gets object $\gamma$ with 2.5\$ makes both agents strictly better off.

On the other hand, a Pareto optimal allocation is not necessarily envy-free. For example, when agent 1 gets objects $\alpha\beta$ by paying 6\$ and agent 2 gets object $\gamma$ with 6\$, this allocation is clearly Pareto optimal but not envy-free, because agent 1 envies agent 2.

Table 1: The values of objects for both agents

| $V \setminus A$ | $\alpha$ | $\beta$ | $\gamma$ | $\alpha\beta$ | $\alpha\gamma$ | $\beta\gamma$ |
|---|---|---|---|---|---|---|
| $V_1(A)$ | 10 | 8 | 2 | 13 | 11 | 9 |
| $V_2(A)$ | 8 | 5 | 10 | 13 | 14 | 13 |

In general, without proper assumptions on the economy, fair allocations may fail to exist as indicated by the following two examples.

**Example 2.** Consider the case in which there are two agents 1, 2 and there are three objects $\alpha$, $\beta$, and $\gamma$, and total money (say, dollar) $M$ is equal to zero. The values of the agents for the different subsets of objects are given in Table 2, and utility functions are given by $u_1(A, m) = V_1(A) + f(m)$, and $u_2(A, m) = V_2(A) + f(m)$, where $f(m) = 2m$ for $m \geq 0$ and $f(m) = m$ for $m \leq 0$.

In this example the set of all envy-free allocations is equal to $\{(\Pi, 0^2) \mid \Pi \in \Theta\}$. We will show that each envy-free allocation is dominated by another feasible allocation and thus is not Pareto optimal. Consider the case in which agent 1 gets object $\gamma$ (or $\alpha\beta$) with 0\$ and agent 2 gets objects $\alpha\beta$ (or $\gamma$) with 0\$. But this allocation is strictly dominated by the following allocation in which agent 1 gets object $\alpha$ (or $\beta\gamma$) with $m$\$ and agent 2 gets objects $\beta\gamma$ (or $\alpha$) by paying $m$\$ with $m \in (1/2, 1)$. The remaining cases can be verified in a similar way.

**Example 3.** Consider the case in which there are two agents 1, 2 and there are three objects $\alpha$, $\beta$, and $\gamma$, and total money (say, dollar) $M$ is equal to zero. Utility functions are given by $u_i(A, m) = V_i(A) + k_i(A)m$, $i = 1, 2$, which are different from those given in Examples 2, and the values of the agents for the different subsets of objects are given in Table 3.

Table 2: The values of objects for both agents

| $V \setminus A$ | $\alpha$ | $\beta$ | $\gamma$ | $\alpha\beta$ | $\alpha\gamma$ | $\beta\gamma$ |
|---|---|---|---|---|---|---|
| $V_1(A)$ | 0 | $-1$ | 1 | 1 | $-1$ | 0 |
| $V_2(A)$ | 1 | $-1$ | 0 | 0 | $-1$ | 1 |

In this example the set of all envy-free allocations is equal to $\{(\Pi, 0^2) \mid \Pi \in \Theta\}$. We will show that each envy-free allocation is dominated by another feasible allocation and thus is not Pareto optimal. Consider the case in which agent 1 gets object $\gamma$ (or $\alpha\beta$) with 0\$ and agent 2 gets objects $\alpha\beta$ (or $\gamma$) with 0\$. But this allocation is strictly dominated by the following allocation in which agent 1 gets object $\alpha$ (or $\beta\gamma$) with $m$\$ and agent 2 gets objects $\beta\gamma$ (or $\alpha$) by paying $m$\$ with $m \in (1/2, 1)$. The remaining cases can be verified in a similar way.

Table 3: The values of objects for both agents

| $V, k \setminus A$ | $\alpha$ | $\beta$ | $\gamma$ | $\alpha\beta$ | $\alpha\gamma$ | $\beta\gamma$ |
|---|---|---|---|---|---|---|
| $V_1(A)$ | 0 | $-1$ | 1 | 1 | $-1$ | 0 |
| $k_1(A)$ | 2 | 0.5 | 1 | 1 | 0.5 | 2 |
| $V_2(A)$ | 1 | $-1$ | 0 | 0 | $-1$ | 1 |
| $k_2(A)$ | 1 | 0.5 | 2 | 2 | 0.5 | 1 |

# 3 Existence of fair allocations: a basic case

In this section we will establish several existence theorems for fair allocations. We first deal with the case where there are the same number of agents as objects ($m = n$ and $W = 1^n$) and then we move to more general cases in the next section. As mentioned earlier, one of our basic tools used in our analysis is a fixed point theorem due to Browder (1960) and Mas-Colell (1974) to be stated below. We remark that Browder proved the continuous function case and Mas-Colell extended the result to the upper semi-continuous correspondence case.

**Theorem 3.1** *Let $\psi : S^n \times [0,1] \mapsto S^n$ be an upper semi-continuous correspondence with nonempty, convex and compact values. Then the set $T = \{(x,t) \in S^n \times [0,1] \mid x \in \psi(x,t)\}$ contains a connected set $T^c$ such that $T^c \cap (S^n \times \{0\}) \neq \emptyset$ and $T^c \cap (S^n \times \{1\}) \neq \emptyset$.*

In Yang (1999) it is shown that the above theorem can be constructively proved by using the method of Herings, Talman and Yang (1996). By this result we will see that all existence results in this paper can be demonstrated in a constructive manner as well.

Before proceeding to our basic condition for the existence of fair allocations, we introduce one simplified notation. In the case of $m = n$ and $W = 1^n$, we use $j$ to represent $e(j)$ for every $j \in I_n$. Thus $u_i(j,p)$ stands for $u_i(e(j),p)$. Let , denote the family of all permutations $\pi = (\pi(1), \pi(2), \cdots, \pi(n))$ of the elements of $I_n$.

**Assumption 3.2** *For any $(i,j) \in I_n \times I_n$, there exists a real number $B(i,j)$ such that if $x \in H(t)$ and its jth component $x_j \le B(i,j)$, then*

$$u_i(j, x_j) < \max_{k \in I_n} u_i(k, x_k).$$

This assumption says that if $B(i,j)$ is negative (positive), then $|B(i,j)|$ could be loosely seen as the maximum (minimum) amount of money agent $i$ is willing to pay (accept) for getting object $j$. We remark that $B(i,j)$ is in general dependent on $t$. For ease of notation we omit $t$. The following theorem says that if the total amount $t$ of money is a variable in an interval $[X,Y]$ and if Assumption 3.2 is satisfied, then there exists a connected set of fair allocations linking both $H(X)$ and $H(Y)$. In contrast, all previous existence results in the literature are only concerned with a single fair allocation.

**Theorem 3.3** *If Assumption 3.2 holds for each $t \in [X,Y]$ with $X \le Y$, then there exists a connected set $\mathcal{H}$ in $H(X,Y)$ such that $\mathcal{H} \cap H(X) \neq \emptyset$, $\mathcal{H} \cap H(Y) \neq \emptyset$, and for each $x \in \mathcal{H}$, $(\pi, x)$ is a $(\sum_{i=1}^{n} x_i)$-fair allocation for some $\pi \in$ , .*

Proof: For each $t \in [X,Y]$ and each $(i,j) \in I_n \times I_n$, define $M_i^j(t) = \{x \in H(t) \mid u_i(j,x_j) \ge u_i(k,x_k), \forall k \in I_n\}$. Clearly Assumptions (B1), (B2), and (B3) in Theorem 3.4 are satisfied by the assumptions made here. We will prove that Assumption (B4) is also satisfied. For any given sequence $\{t^\nu\} \subset [X,Y]$ with $t^\nu \to t^*$, suppose that $x^\nu \in M_i^j(t^\nu)$ and $x^\nu \to x^*$. Note that $x^\nu \in M_i^j(t^\nu)$ means $u_i(j, x_j^\nu) \ge u_i(k, x_k^\nu)$. So by the continuity of utility functions we have $u_i(j, x_j^*) \ge u_i(k, x_k^*)$. That is $x^* \in M_i^j(t^*)$. Thus we proved that Assumption (B4) holds.

It follows from Theorem 3.4 that there exists a connected set $\mathcal{H}$ in $H(X,Y)$ such that $\mathcal{H} \cap H(X) \neq \emptyset$, $\mathcal{H} \cap H(Y) \neq \emptyset$, and for each $x \in \mathcal{H}$, $(\pi, x)$ is a $(\sum_{i=1}^{n} x_i)$-equitable allocation for some $\pi \in$ , . Let $t = \sum_{j=1}^{n} x_j$. We will show that $(\pi, x)$ is $t$-fair. Since $(\pi, x)$ is $t$-equitable, this implies that for all $i, j \in I_n$ it holds

$$u_i(\pi(i), x_{\pi(i)}) \ge u_i(\pi(j), x_{\pi(j)}). \tag{3.1}$$

Thus $(\pi, x)$ is envy-free. Note that at $(\pi, x)$ agent $i$ gets object $\pi(i)$ with $x_{\pi(i)}$ units of money for each $i \in I_n$. Now suppose to the contrary that $(\pi, x)$ is not Pareto optimal. Then there would exist a feasible allocation $(\xi, y)$ such that it holds

$$u_i(\pi(i), x_{\pi(i)}) \le u_i(\xi(i), y_{\xi(i)}); \tag{3.2}$$

for every $i \in \mathcal{P}$; and there is some $j \in \mathcal{P}$ satisfying

$$u_j(\xi(j), y_{\xi(j)}) > u_j(\pi(j), x_{\pi(j)}). \tag{3.3}$$

Note that $\sum_{j=1}^n y_j = t$. The inequlities (3.1), (3.2) and (3.3) imply that for all $i \in \mathcal{P}$,

$$u_i(\xi(i), y_{\xi(i)}) \geq u_i(\xi(i), x_{\xi(i)})$$

and

$$u_j(\xi(j), y_{\xi(j)}) > u_j(\xi(j), x_{\xi(j)}).$$

Since $u_i(j, \cdot)$, $i \in I_n$, are nondecreasing in money, we have that for all $i \in I_n$, $y_{\xi(i)} \geq x_{\xi(i)}$ and $y_{\xi(j)} > x_{\xi(j)}$. Note that $\sum_{j=1}^n y_{\xi(j)} = \sum_{j=1}^n y_j$ and $\sum_{j=1}^n x_{\xi(j)} = \sum_{j=1}^n x_j$. It follows that $t = \sum_{j=1}^n y_j > \sum_{j=1}^n x_j = t$, yielding a contradiction. Therefore, $(\pi, x)$ must be Pareto optimal as well. □

It is worth mentioning that in the above proof we only need Assumption 3.2 and continuity for the existence of envy-free allocations. The weak monotonicity is for an envy-free allocation to be Pareto-optimal. It will be shown that all previous sufficient conditions in the literature satisfy Assumption 3.2. We note that previous authors all required strong monotonicity.

Now we are going to introduce our main mathematical instrument upon which all existence theorems presented are based. It will be proved by applying both the theorem of Browder and Mas-Colell and the theorem of Birkhoff and von Neumann. Given two real numbers $X$ and $Y$ with $X \leq Y$, recall that $H(X,Y) = \{x \in \mathbb{R}^n \mid X \leq \sum_{i=1}^n x_i \leq Y\}$. For each $i \in I_n$ and $t \in [X, Y]$, there is a class $\{M_i^j(t) \mid j \in I_n\}$ of subsets $M_i^j(t)$ of $H(t)$. The interpretation is as follows. If $x \in M_i^k(t)$, this means that agent $i$ likes the bundle $(k, x_k)$ at least as well as any other bundle $(j, x_j)$. A pair $(\pi, x) \in$ , $\times H(t)$ is called *$t$-equitable* if $x \in \cap_{i \in I_n} M_i^{\pi(i)}(t)$. Thus at a $t$-equitable allocation $(\pi, x)$ the bundle $(\pi(i), x_{\pi(i)})$ is given to agent $i$ for $i \in I_n$. For the existence of equitable allocations, the following assumptions will be made on the sets $M_i^j(t)$:

(B1) For each $i \in I_n$, $\cup_{j \in I_n} M_i^j(t) = H(t)$;

(B2) $M_i^j(t)$ is a closed set for every $i \in I_n$, $j \in I_n$, and $t \in [X, Y]$;

(B3) For any $(i,j) \in I_n \times I_n$, if $x \in H(t)$ and its jth component $x_j \leq B(i,j)$, then $x \notin M_i^j(t)$;

(B4) For each $(i,j) \in I_n \times I_n$, $M_i^j(t)$ is upper semi-continuous in $t \in [X, Y]$.

**Theorem 3.4** *Under Assumptions* (B1), (B2), (B3) and (B4)*, there exists a connected set $\mathcal{H}$ in $H(X,Y)$ such that $\mathcal{H} \cap H(X) \neq \emptyset$, $\mathcal{H} \cap H(Y) \neq \emptyset$, and for each $x \in \mathcal{H}$, $(\pi, x)$ is a $(\sum_{i=1}^n x_i)$-equitable allocation for some $\pi \in$ , .*

Proof: Without loss of generality, we may assume that $B(i,j)=0$ for all $i,j\in I_n$. So from Assumption (B1) we see that $0<X\le Y$. Recall $S^n=\{x\in \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i=1\}$. Define $C^i=\{x\in S^n \mid x_i\ge 1/n\}$ for each $i\in I_n$. Let $V$ denote the set $S^n\times S^n$. For each $(i,j)\in I_n\times I_n$ and $t\in[X,Y]$, define

$$N_i^j(t)=\{y=\frac{x}{t} \mid x\in M_i^j(t)\},$$

and

$$C^{(i,j)}(t)=C^i\times N_i^j(t).$$

Clearly, for each $t\in[X,Y]$, $C^{(i,j)}(t)$ is a closed set, and the collection of sets $\{\,C^{(i,j)}(t) \mid (i,j)\in I_n\times I_n\,\}$ is a covering of $V$. Moreover, it is not difficult to show that $C^{(i,j)}(t)$ is upper semi-continuous in $t$. If $x\in C^{(i,j)}(t)$ then $x_{1,i}>0$ and $x_{2,j}>0$.
For each $i\in I_n$, let $a^i$ denote the vector $\sum_{h=1}^n e(h)/n-e(i)$ in $\mathbb{R}^n$ and let $e^n=\sum_{h=1}^n e(h)/n$. For each $(i,j)\in I_n\times I_n$, define a vector $c^{(i,j)}\in\mathbb{R}^n\times\mathbb{R}^n$ by

$$c^{(i,j)}=(a^i,a^j).$$

Now let the point-to-set mapping $F$ from $[X,Y]\times V$ to the collection of subsets of $\mathbb{R}^n\times\mathbb{R}^n$ be given by

$$F(t,x)=Conv(\{\,c^{(i,j)} \mid x\in C^{(i,j)}(t)\}),$$

where $Conv(D)$ denotes the convex hull of a set $D$. It is easy to see that $F$ is upper semi-continuous. Moreover, $\cup_{t\in[X,Y],x\in V}F(t,x)$ is compact, and for each $t\in[X,Y]$ and $x\in V$ the set $F(t,x)$ is nonempty, convex and compact. Let $W$ be a compact, convex set in $\mathbb{R}^n\times\mathbb{R}^n$ containing $\cup_{t\in[X,Y],x\in V}F(t,x)$. Then we define the point-to-set mapping $G$ from $W$ to the collection of subsets of $V$ by

$$G(y)=\{\,x^*\in V \mid x_1^\top y_1\le (x_1^*)^\top y_1,\ \forall\, x_1\in S^n \quad x_2^\top y_2\le (x_2^*)^\top y_2, \forall\, x_2\in S^n\}.$$

Again, $G$ is upper semi-continuous. Moreover, for any $y\in W$ the set $G(y)$ is nonempty, compact and convex. For $(t,x,y)\in[X,Y]\times V\times W$, let $\phi(t,x,y)$ be defined as

$$\phi(t,x,y)=G(y)\times F(t,x).$$

Then $\phi$ is an upper semi-continuous mapping from the set $[X,Y]\times V\times W$ into the collection of nonempty subsets of $V\times W$ satisfying that for every $(t,x,y)\in[X,Y]\times V\times W$, the set $\phi(t,x,y)$ is nonempty, convex and compact. According to Browder's fixed point theorem, the set

$$T=\{(t,x,y)\in[X,Y]\times V\times W \mid (x,y)\in\phi(t,x,y)\}$$

contains a connected set $T^c$ such that $T^c \cap \{X\} \times V \times W \neq \emptyset$ and $T^c \cap \{Y\} \times V \times W \neq \emptyset$. Let $(t, x(t), y(t))$ be an element in $T$. Then we can write $x(t) = (x_1(t), x_2(t))$ where $x_1(t), x_2(t) \in S^n$. In the following we will show that for each $t \in [X, Y]$, $(\pi, tx_2(t))$ is $t$-equitable for some $\pi \in$ , . Since $t$ is fixed below, we denote $x(t)$, $y(t)$ by $x^*$ and $y^*$. So it holds that

$$x_1^\top y_1^* \leq (x_1^*)^\top y_1^*, \ \forall \ x_1 \in S^n,$$
$$x_2^\top y_2^* \leq (x_2^*)^\top y_2^*, \ \forall \ x_2 \in S^n.$$

Let $\beta_1$ be equal to $(x_1^*)^\top y_1^*$ and $\beta_2$ equal to $(x_2^*)^\top y_2^*$. Then by taking $x_1$ equal to $e^n$, it follows that $\beta_1 \geq 0$, since $\sum_{i=1}^n y_{1,i}^* = 0$. When we take $x_1$ succesively equal to $e(i)$ for every $i \in I_n$, we obtain

$$y_{1,i}^* \leq \beta_1, \forall \ i \in I_n.$$

On the other hand, if for some $i \in I_n$ it holds that $x_{1,i}^* > 0$, by taking $x_1$ equal to $x_1^* + \lambda(x_1^* - e(i))$ for arbitrarily small $\lambda > 0$, we obtain that $y_{1,i}^* \geq \beta_1$. Hence $y_{1,i}^* = \beta_1$ when $x_{1,i}^* > 0$.

Let the collection $\mathcal{L}$ of elements of $I_n \times I_n$ be defined by

$$\mathcal{L} = \{L = (i_1, i_2) \in I_n \times I_n \mid x^* \in C^L(t)\}.$$

Suppose that $\mathcal{L} = \{L^1, \cdots, L^l\}$, where $L^k = (i_1^k, i_2^k)$. Since $y^* \in F(t, x^*)$ there exist some nonnegative numbers $\mu_1$, $\cdots$, $\mu_l$ with sum equal to 1 such that

$$y^* = \sum_{k=1}^{l} \mu_k c^{L^k}.$$

Suppose that $x_{1,h}^* = 0$ for some $h \in I_n$. Then it implies that $h \neq i_1^k$ for every $k = 1, \cdots$, $l$ and hence $y_{1,h}^* \geq 0$. Since $\sum_{i=1}^n y_{1,i}^* = 0$, we have that $y_1^* = \mathbf{0}$. Similarly, we can prove $y_2^* = \mathbf{0}$. So,

$$\sum_{k=1}^{l} \mu_k c^{L^k} = \mathbf{0}. \tag{3.4}$$

It follows from (3.4) that

$$\sum_{(i,j) \in \mathcal{L}} \mu_{(i,j)}(a^i, a^j) = \mathbf{0}$$

and that

$$\sum_{(i,j) \in \mathcal{L}} \mu_{(i,j)} = 1$$

for certain $\mu_{(i,j)} \geq 0$ for $(i,j) \in \mathcal{L}$. Moreover, it holds that for each $i \in I_n$, $\sum_j \mu_{(i,j)} = 1/n$ and that for each $j \in I_n$, $\sum_i \mu_{(i,j)} = 1/n$. From this property it follows that the $n \times n$ matrix $U$ with entries $\nu_{(i,j)}$ defined by $\nu_{(i,j)} = n\mu_{(i,j)}$ if $(i,j) \in \mathcal{L}$ and $\nu_{(i,j)} = 0$ if $(i,j) \notin \mathcal{L}$ is a doubly stochastic matrix and therefore $U$ is a convex combination of permutation matrices according to the theorem of Birkhoff and von Neumann. So, there exists an elment $\pi = (\pi(1), \pi(2), \cdots, \pi(n)) \in$ , such that for every $i \in I_n$, it holds

$$\nu(i, \pi(i)) > 0.$$

Equivalently, for every $i \in I_n$, it holds

$$\mu(i, \pi(i)) > 0.$$

Consequently, for every $i \in I_n$, it holds

$$(i, \pi(i)) \in \mathcal{L}.$$

Since $x(t) = x^* \in \cap_{h=1}^{l} C^{L^h}(t)$, we have

$$tx_2(t) \in \cap_{i=1}^{n} M_i^{\pi(i)}(t).$$

This completes the proof. □

The above theorem can be equivalently stated as follows.

**Theorem 3.5** *Let $X$ and $Y$ be any real numbers with $X \leq Y$ and let $\{M_i^j \mid i, j \in I_n\}$ be a collection of closed subsets of $H(X, Y)$. Suppose that* (i) *it holds $\cup_{j=1}^{n} M_i^j = H(X, Y)$ for every $i \in I_n$;* (ii) *for any $(i, j) \in I_n \times I_n$, $x \in H(X, Y)$ with its jth component $x_j \leq B(i, j)$ implies $x \notin M_i^j$. Then there exists a connected set $\mathcal{H}$ in $H(X, Y)$ such that $\mathcal{H} \cap H(X) \neq \emptyset$, $\mathcal{H} \cap H(Y) \neq \emptyset$, and for each $x \in \mathcal{H}$, $(\pi, x)$ is a $(\sum_{i=1}^{n} x_i)$-equitable allocation for some $\pi \in$ , by defining $M_i^j(t) = M_i^j \cap H(t)$.*

Following immediately from Theorems 3.4 or 3.5, we have the next theorem which can be seen as a generalization of Theorem 5 of Svensson (1983), Lemma of Gale (1984) and the classic lemma of Knaster, Kuratowski and Mazurkewicz (KKM) in combinatorial topology.

**Theorem 3.6** *Let $\{M_i^j(t) \mid i, j \in I_n\}$ be a collection of subsets of $H(t)$ which satisfy Assumptions* (B1), (B2) and (B3). *Then there exists a $t$-equitable allocation $(\pi, x) \in$ , $\times H(t)$.*

# 4 Existence of fair allocations: a general case

In this section we deal with the existence problem of fair allocation in more general cases. Our first theorem states that there exists an fair allocation if for some subeconomy its Pareto optimal allocations are also Pareto optimal with respect to the original economy and if Assumption 3.2 holds for the subeconomy. The analysis seems significantly more difficult without these assumptions.

**Theorem 4.1** *There exists a fair allocation if there is an element $\Pi \in \Theta$ such that for the subeconomy $E = (\Pi, M, \mathcal{P}, [u_i])$ it holds $PO(E) \subseteq PO(\mathcal{E})$ and if the following condition is satisfied: For any $(i,j) \in I_m \times I_m$, there exists a real number $B(i,j)$ such that if $x \in H(M)$ and its jth component $x_j \leq B(i,j)$, then $u_i(\Pi(j), x_j) < \max_{k \in I_m} u_i(\Pi(k), x_k)$.*

Proof: Following the proof of Theorem 3.3 and applying Theorem 3.6 will yield the result. □

Next we will give two different sufficient conditions by requiring utility functions to be continuous and strictly increasing in money. In the first theorem, the second part of the condition is reminiscent of inequality (16) of Alan et al (1991, p.1030), while in the second theorem, the second part of the condition can be seen as a dual counterpart of the condition in the first theorem.

**Theorem 4.2** *There exists a fair allocation if there is an element $\Pi \in \Theta$ such that for the subeconomy $E = (\Pi, M, \mathcal{P}, [u_i])$ it holds $PO(E) \subseteq PO(\mathcal{E})$ and if the following condition is satisfied: The utility functions are continuous and strictly increasing in money and there exists $L > |M|$ such that $u_i(\Pi(j), L) > u_i(\Pi(k), 0)$ for all $i, j, k \in I_m$.*

Proof: Let $Y = mL + |M|$. We will prove that for any $i \in \mathcal{P}$ and for any $x \in H(M)$, if $x_j < -Y$, then

$$u_i(\Pi(j), x_j) < \max_{h \in I_m} u_i(\Pi(h), x_h).$$

Since $u_i$ is strictly increasing in money, we have

$$u_i(\Pi(j), x_j) < u_i(\Pi(j), -Y) < u_i(\Pi(j), 0).$$

Because $u_i(\Pi(l), L) \geq u_i(\Pi(h), 0)$ for all $h, i, l \in I_m$, then we have

$$u_i(\Pi(j), x_j) < u_i(\Pi(l), L)$$

for all $l \in I_m$. Since $\sum_{h=1}^{m} x_h = M$ and $x_j < -Y$, then there exists some $k \in I_m$ such that $x_k \geq L$. Thus,

$$u_i(\Pi(j), x_j) < u_i(\Pi(k), x_k) \leq \max_{h \in I_m} u_i(\Pi(h), x_h).$$

□

**Theorem 4.3** *There exists a fair allocation if there is an element $\Pi \in \Theta$ such that for the subeconomy $E = (\Pi, M, \mathcal{P}, [u_i])$ it holds $PO(E) \subseteq PO(\mathcal{E})$ and if the following condition is satisfied: The utility functions are continuous and strictly increasing in money and there exists $L < -|M|$ such that $u_i(\Pi(j), L) < u_i(\Pi(k), 0)$ for all $i, j, k \in I_m$.*

Proof: Let $Y = -m|M| + L$. Clearly, $Y \leq L < 0$. We will prove that for any $i \in \mathcal{P}$ and for any $x \in H(M)$, if $x_j \leq Y$, then

$$u_i(\Pi(j), x_j) < \max_{h \in I_m} u_i(\Pi(h), x_h).$$

Since $\sum_{h=1}^{m} x_h = M$ and $x_j \leq Y$, then there exists some $k \in I_m$ such that $x_k > 0$. Thus,

$$u_i(\Pi(j), x_j) \leq u_i(\Pi(j), L) < u_i(\Pi(k), 0) < u_i(\Pi(k), x_k) \leq \max_{h \in I_m} u_i(\Pi(h), x_h).$$

□

Now we turn to discuss a simple but interesting case where agents have quasi-linear utility functions. That is, utilities functions are given by $u_i(x, m) = V_i(x) + m$, where $V_i$ are called *reservation value functions*.

**Corollary 4.4** *There always exists a fair allocation in quasi-linear utility cases.*

Proof: Let $\tilde{\Pi}$ be a solution of the following optimization problem

$$\max_{\Pi \in \Theta} \sum_{i=1}^{m} V_i(\Pi(i)).$$

Construct a subeconomy $E = (\tilde{\Pi}, M, \mathcal{P}, [u_i])$. Since $u_i$ is quasi-linear, it is easy to see that every Pareto optimal allocation of $E$ is also an Pareto optimal allocation of $\mathcal{E}$. Furthermore, Assumption 3.2 holds for $E$. Then the result follows immediately from Theorem 4.1. □

Corollary 4.4 shows that in the quasi-linear utility cases fair allocations always exist no matter what reservation value functions may be. It is in striking constrast to the equilibrium results (see e.g. Bevia, Quinzii and Selia (1999) and Gul and Stacchetti (1999)) where strong conditions are required to impose on reservation value functions.

In the above discussions the distribution vector $x$ of money could have positive or negative components. This means that at a fair allocation some agents may have to pay others some amount of money. In other words, agents must initially have a certain amount of money. In some situations one has to require that the distribution vector $x$ of money be nonnegative, for example, when all agents initially have no money at all, or are not willing to pay any money. In this case we have the following theorem, stating that there exists an fair allocation with a nonnegative distribution of money if for some subeconomy its Pareto optimal allocations are also Pareto optimal with respect to the original economy and if any state in the subeconomy without consumption is not preferable. Let $M > 0$. Define

$$\Delta(M) = \{ x \in \mathbb{R}^m_+ \mid \sum_{i=1}^{m} x_i = M \}.$$

**Theorem 4.5** *There exists a fair allocation with a nonnegative distribution of money if there is an element* $\Pi \in \Theta$ *such that for the subeconomy* $E = (\Pi, M, \mathcal{P}, [u_i])$ *it holds* $PO(E) \subseteq PO(\mathcal{E})$ *and the following condition is satisfied: For any* $(i,j) \in I_m \times I_m$ *and any* $x \in \Delta(M)$ *if* $x_j = 0$*, then* $u_i(\Pi(j), x_j) < \max_{k \in I_m} u_i(\Pi(k), x_k)$.

This result generalizes the results of Svensson (1983) and Maskin (1987). One can easily verify that their conditions satisfy the one stated in Theorem 4.5. In their models it is required that there should be the same number of agents as objects and each agent consume exactly one object.

# 5 Strict monotonicity in welfare

In this section we give a comparative statics analysis on the welfare of each agent when the total amount of money is regarded as a variable. The main result says that if utility functions are continuous and strictly increasing in money, then for any given $X$-fair allocation $(\pi^*, x^*)$ there exists a connected set $\mathcal{H}$ in $H(Z, Y)$ with $(Z \le X \le Y)$ such that for each $x \in \mathcal{H}$, $(\pi, x)$ is a $(\sum_{i=1}^n x_i)$-fair allocation for some $\pi \in$ , , and $(\pi, x)$ makes every agent strictly better (worse) off when $\sum_{i=1}^n x_i > (<) \sum_{i=1}^n x_i^*$. In other words, as the total amount $t$ of money increases (decreases), the welfare of each agent will be better (worse) off compared with the initial fair allocation $(\pi^*, x^*)$.

**Theorem 5.1** *Let* $X$, $Y$, $Z$ *be three real numbers with* $Z \le X \le Y$. *Suppose that the utility functions are continuous and strictly increasing in money and that* $(\pi^*, x^*)$ *is an* $X$*-fair allocation. Then there exists a connected set* $\mathcal{H}$ *in* $H(Z, Y)$ *such that* $\mathcal{H} \cap H(Z) \ne \emptyset$, $\mathcal{H} \cap H(X) = \{x^*\}$, $\mathcal{H} \cap H(Y) \ne \emptyset$, *and for each* $x \in \mathcal{H}$, $(\pi, x)$ *is a* $(\sum_{i=1}^n x_i)$*-fair allocation for some* $\pi \in$ , , *and* $(\pi, x)$ *makes every agent strictly better (worse) off when* $\sum_{i=1}^n x_i > (<) \sum_{i=1}^n x_i^*$.

Proof: Without loss of generality, suppose that $\pi^*$ satisfies $\pi^*(i) = i$ for all $i \in I_n$. Let $u_i^* = u_i(i, x_i^*)$. Then we see that $u_i^* \ge u_i(j, x_j^*)$ for all $j \in I_n$. Using the initial fair allocation $x^*$, we define functions $\phi$ and $\psi : \mathcal{R} \to \mathcal{R}$ by

$$\phi(x) = \max\{u_i(j, x_j^* + x) - u_i^* \mid i, j \in I_n\},$$

$$\psi(x) = \min\{u_i(i, x_i^* + x) - u_i^* \mid i \in I_n\}.$$

It is clear that $\phi$ and $\psi$ are strictly increasing, and $\phi(0) = \psi(0) = 0$. Let $\phi^{-1}$ denote the inverse function of $\phi$. Define $\xi(x) = \phi^{-1} \circ \psi(x)$. We see that $\xi$ is strictly increasing and $\xi(0) = 0$ as well. And, since $\phi(\xi(x)) = \phi \circ \xi(x) = \psi(x) \le \phi(x)$, we have that $\xi(x) \le x$. Using the above functions, we define function $\eta : \mathcal{R} \to \mathcal{R}$ by

$$\eta(x) = \psi \circ \overbrace{\xi \circ \cdots \circ \xi}^{n}(x) = \psi \circ \xi^n(x).$$

Again, $\eta$ is continous and strictly increasing with $\eta(0) = 0$ and $\eta(x) \leq \psi(x)$. Now define new utility functions $f_i(j, x_j)$ for all $i, j \in I_n$ by

$$f_i(j, x_j) = \begin{cases} \max\{u_i(j, x_j), u_i^* + \frac{1}{2}\eta(x_j - x_j^*)\} & (\text{for } x_j > x_j^*) \\ u_i^* & (\text{for } x_j = x_j^*) \\ \max\{u_i(j, x_j), u_i^* + 2\eta(x_j - x_j^*)\} & (\text{for } x_j < x_j^*). \end{cases}$$

Clearly, these functions are strictly increasing and continuous. In particular, we have that

$$\begin{aligned} f_i(i, x_i) &= \begin{cases} \max\{u_i(i, x_i), u_i^* + \frac{1}{2}\eta(x_i - x_i^*)\} & (\text{for } x_i > x_i^*) \\ u_i^* & (\text{for } x_i = x_i^*) \\ \max\{u_i(i, x_i), u_i^* + 2\eta(x_i - x_i^*)\} & (\text{for } x_i < x_i^*). \end{cases} \\ &\leq \max\{u_i(i, x_i), u_i^* + \eta(x_i - x_i^*)\} \\ &\leq \max\{u_i(i, x_i), u_i^* + \psi(x_i - x_i^*)\} \\ &\leq \max\{u_i(i, x_i), u_i(i, x_i)\} \\ &= u_i(i, x_i). \end{aligned}$$

This is implies that $f_i(i, x_i) = u_i(i, x_i)$ for all $i \in I_n$ and $x_i \in \mathcal{R}$. With these utility functions, we obtain a new economy. We will show this new economy satisfies Assumption 3.2 for every $t$. In fact, we can choose $B(i, j) = \min\{x_j^*, t - 1 - \sum_{k \neq j} x_k^*\}$ for all $i$ and $j$. If $x \in H(t)$ and $x_j \leq B(i, j)$, then there is some $j' \in I_n$ such that $x_{j'} > x_{j'}^*$. Thus we have

$$f_i(j, x_j) \leq u_i^* < f_i(j', x_{j'}) \leq \max_{k \in I_n} f_i(k, x_k).$$

By Theorem 3.3, we see that there exists a connected set $\mathcal{H}$ in $H(Z, Y)$ such that $\mathcal{H} \cap H(Z) \neq \emptyset$, $\mathcal{H} \cap H(Y) \neq \emptyset$, and for each $x \in \mathcal{H}$, $(\pi, x)$ is a $(\sum_{i=1}^n x_i)$-fair allocation of the new economy for some $\pi \in$ , . Notice that any allocation $x$ of money satisfying $x_j > x_j^*$ and $x_k \leq x_k^*$ (or $x_j \geq x_j^*$ and $x_k < x_k^*$) for some $j$ and $k$ cannot be a fair allocation of money in the new economy. This can be seen as follows. Consider such an allocation $(x, \pi)$ for any $\pi$. Then we have $f_i(k, x_k) \leq f_i(k, x_k^*) = u_i^* = f_i(j, x_j^*) < f_i(j, x_j)$ for all $i$. This means that no agent likes to have the bundle $(x_k, k)$.

It follows from the above discussion that for any $x \in \mathcal{H} \cap H(t)$ and $j \in I_n$ it holds $x_j > x_j^*$ for $t > X$; $x_j = x_j^*$ for $t = X$; and $x_j < x_j^*$ for $t < X$. Thus we have that $\mathcal{H} \cap H(X) = \{x^*\}$. Now it remains to show that a fair allocation of the new economy is also a fair allocation of the original economy.

Let $x$ be an arbitrary element in $\mathcal{H}$. Then $(\pi, x)$ is a $(\sum_{i=1}^n x_i)$-fair allocation of the new economy for some $\pi \in$ , . We first consider the case $\sum_{i=1}^n x_i > \sum_{i=1}^n x_i^*$. We will show that $(\pi, x)$ must be a fair allocation of the original economy. For this purpose, it is enough to show that $f_i(\pi(i), x_{\pi(i)}) = u_i(\pi(i), x_{\pi(i)})$ for all $i \in I_n$. Suppose to the contrary that there is some $i$ such that $f_i(\pi(i), x_{\pi(i)}) \neq u_i(\pi(i), x_{\pi(i)})$. Recall that $f_i(i, x_i) = u_i(i, x_i)$. This means $\pi(i) \neq i$. Then there exists the smallest integer $k$ $(1 \leq k \leq n)$ such that

$$f_{\pi^k(i)}(\pi^{k+1}(i), x_{\pi^{k+1}(i)}) \neq u_{\pi^k(i)}(\pi^{k+1}(i), x_{\pi^{k+1}(i)}),$$

where, $\pi^k(i) = \overbrace{\pi \circ \cdots \circ \pi}^{k}(i)$. Thus we have that

$$u_i^* + \frac{1}{2}\eta(x_{\pi(i)} - x_{\pi(i)}^*) = f_i(\pi(i), x_{\pi(i)})$$

$$\geq f_i(\pi^{k+1}(i), x_{\pi^{k+1}(i)}) \geq u_i^* + \frac{1}{2}\eta(x_{\pi^{k+1}(i)} - x_{\pi^{k+1}(i)}^*).$$

That is, $x_{\pi(i)} - x_{\pi(i)}^* \geq x_{\pi^{k+1}(i)} - x_{\pi^{k+1}(i)}^*$. In the same way we can show that $x_{\pi(i)} - x_{\pi(i)}^* \leq x_{\pi^{k+1}(i)} - x_{\pi^{k+1}(i)}^*$. Consequently, we have that $x_{\pi(i)} - x_{\pi(i)}^* = x_{\pi^{k+1}(i)} - x_{\pi^{k+1}(i)}^*$.

Note that for any integer $l$ with $1 \leq l < k$ we have

$$f_{\pi^l(i)}(\pi^{l+1}(i), x_{\pi^{l+1}(i)}) = u_{\pi^l(i)}(\pi^{l+1}(i), x_{\pi^{l+1}(i)}).$$

Then by the definition of envy free allocations, we obtain that

$$u_{\pi^l(i)}^* + \psi(x_{\pi^l(i)} - x_{\pi^l(i)}^*) \leq u_{\pi^l(i)}(\pi^l(i), x_{\pi^l(i)}) = f_{\pi^l(i)}(\pi^l(i), x_{\pi^l(i)})$$

$$\leq f_{\pi^l(i)}(\pi^{l+1}(i), x_{\pi^{l+1}(i)}) = u_{\pi^l(i)}(\pi^{l+1}(i), x_{\pi^{l+1}(i)}) \leq u_{\pi^l(i)}^* + \phi(x_{\pi^{l+1}(i)} - x_{\pi^{l+1}(i)}^*).$$

The first inequality and the last follow from the definition of $\psi$ and $\phi$, respectively. It follows that

$$\psi(x_{\pi^l(i)} - x_{\pi^l(i)}^*) \leq \phi(x_{\pi^{l+1}(i)} - x_{\pi^{l+1}(i)}^*),$$

and so

$$x_{\pi^{l+1}(i)} - x_{\pi^{l+1}(i)}^* \geq \xi(x_{\pi^l(i)} - x_{\pi^l(i)}^*).$$

Repeating $k-1$ times the last inequality above leads to

$$x_{\pi^k(i)} - x_{\pi^k(i)}^* \geq \xi^{k-1}(x_{\pi(i)} - x_{\pi(i)}^*) = \xi^{k-1}(x_{\pi^{k+1}(i)} - x_{\pi^{k+1}(i)}^*). \qquad (*)$$

On the other hand, we have that

$$u_{\pi^k(i)}^* + \psi(x_{\pi^k(i)} - x_{\pi^k(i)}^*) \leq u_{\pi^k(i)}(\pi^k(i), x_{\pi^k(i)}) = f_{\pi^k(i)}(\pi^k(i), x_{\pi^k(i)})$$

$$\leq f_{\pi^k(i)}(\pi^{k+1}(i), x_{\pi^{k+1}(i)}) = u_{\pi^k(i)}^* + \frac{1}{2}\eta(x_{\pi^{k+1}(i)} - x_{\pi^{k+1}(i)}^*).$$

This implies that

$$0 < \psi(x_{\pi^k(i)} - x_{\pi^k(i)}^*) \leq \frac{1}{2}\eta(x_{\pi^{k+1}(i)} - x_{\pi^{k+1}(i)}^*).$$

Consequently, we obtain that

$$x_{\pi^k(i)} - x_{\pi^k(i)}^* \leq \psi^{-1}(\frac{1}{2}\eta(x_{\pi^{k+1}(i)} - x_{\pi^{k+1}(i)}^*)) < \psi^{-1}(\eta(x_{\pi^{k+1}(i)} - x_{\pi^{k+1}(i)}^*))$$

$$= \xi^n(x_{\pi^{k+1}(i)} - x_{\pi^{k+1}(i)}^*) \leq \xi^{k-1}(x_{\pi^{k+1}(i)} - x_{\pi^{k+1}(i)}^*).$$

This contradicts the fact $(*)$. Therefore we have that for all $i, j \in I_n$

$$u_i(\pi(i), x_{\pi(i)}) = f_i(\pi(i), x_{\pi(i)}) \geq f_i(j, x_j) \geq u_i(j, x_j).$$

That is, $(\pi, x)$ is a fair allocation of the original economy. Note that in this fair allocation, each agent's utility is strictly better off compared with its utility in the initial fair allocation. For the case $\sum_{i=1}^n x_i < \sum_{i=1}^n x_i^*$ we can prove the result in a similar way by noting that $x_j < x_j^*$ for all $j \in I_n$. □

In the above proof we see that each fair allocation in the new economy must be an fair allocation in the original economy. Note that the converse is in general not true. For example, there may be multiple fair allocations at $H(X)$ but the above proof shows that there is only one fair allocation at $H(X)$ in the new economy. In other words, due to the introduction of new utility functions the concept of fair allocation is refined in some way.

# 6 The model of Alan, Demange and Gale

In this section we consider the model of Alan, Demange and Gale (1991). This model consists of $m$ agents and $n$ objects, and money $M$ (called an $m \times n$-economy) and is similar to the models of Svensson (1983) and Maskin (1987) in the case that the number of objects is equal to or less than the number of agents and thus is similar to the model presented in Section 2. When the number of objects is greater than the number of agents (i.e. $m < n$), Alan et al introduce $n - m$ *fictitious agents* in their model who value only mony, so that there will be the same number of agents as objects. If agent $i$ is fictitious, then $u_i(j, x) = x$ for all $j$. The $m \times n$-economy plus $n - m$ fictitious agents will be called an $n \times n$-economy. An allocation of the $m \times n$-economy is strongly fair (strongly envy-free) in the sense of Alan et al if and only if it is fair (envy-free) in the $n \times n$-economy such that the total amount of money obtained by real agents is equal to $M$. Thus, at a strongly fair allocation, each (including fictitious) agent gets exactly one object with some money and the total amount of money obtained by real agents is equal to $M$. Let $\mathcal{P}_R$ and $\mathcal{P}_F$ denote real and fictitious agents, respectively. Hence an allocation $(\pi, x)$ of the $m \times n$-economy is strongly $X$-fair (strongly envy-free) in the sense of Alan et al if and only if it is a $Y$-fair (envy-free) allocation with $\sum_{i \in \mathcal{P}_R} x_{\pi(i)} = X$ for some $Y > X$ in the $n \times n$-economy. In what follows we will give several sufficient conditions for the existence of strongly fair allocations in this model based on Theorems 3.3 and 5.1 and then we show that the condition given by Alan et al also satisfies these conditions. Furthermore, we derive a strict monotonicity property of strongly fair allocation. Consequently we obtain an alternative proof for the results of Alan et al. This proof is interesting and useful in the sense that it appeared impossible to use a fixed point argument. To be precise, Alan et al (1991, p.1024) point out: "The main advantage to our proof, however, is that because of its constructive nature we are able to derive qualitative properties of fair allocations which do not seem to be obtainable by the usual fixed point methods (in fact it seems the fixed point methods will not work even for proving existence for the case when there are more objects than people)".

Now we introduce our first sufficient condition for the existence of a strongly fair allocation in the model of Alan et al.

**Assumption 6.1** *For each $i \in \mathcal{P}_R$ and $p \in \mathbb{R}$, there exists a real number $A(i,p) > p$ such that if $x \geq A(i,p)$ and $y \leq p$, then $u_i(j,x) > u_i(j',y)$ for all $j$ and $j' \in I_n$.*

This assumption says that no object is infinitely desirable compared with money. It is clear that Assumption 6.1 holds when each utility funtion $u_i(j,x)$ satisfies $\lim_{x\to+\infty} u_i(j,x) = +\infty$ and $\lim_{x\to-\infty} u_i(j,x) = -\infty$.

**Theorem 6.2** *If Assumption 6.1 holds for all $i \in \mathcal{P}_R$, then there exists a $Z$-fair allocation $(\rho, z)$ in the $n \times n$-economy with $\sum_{i\in\mathcal{P}_R} z_{\rho(i)} = M$ for some $Z > M$.*

Proof: Choose $Y$ so that $Y > M$ and $\frac{Y-M}{n-m} \geq \max_{i\in\mathcal{P}_R} A(i, \frac{M}{m})$. We first show that Assumption 3.2 is satisfied by Assumption 6.1 for each $t \in [M,Y]$. For each $(i,j) \in I_n \times I_n$, choose $B(i,j) = \min\{0, M-(n-1)A(i,0)\}$. Then for any $t \in [M,Y]$ and $x \in H(t)$ with $x_j \leq B(i,j)$, there exists some $k \in I_n$ such that $x_k \geq \frac{t-x_j}{n-1} \geq \frac{M-B(i,j)}{n-1} \geq A(i,0)$. Hence, by Assumption 6.1 we have that $u_i(j,x_j) < u_i(k,x_k) \leq \max_{h\in I_n} u_i(h,x_h)$. That is, Assumption 3.2 holds for each $t \in [M,Y]$. It follows from Theorem 3.3 that there exists a connected set $\mathcal{H}$ in $H(M,Y)$ such that $\mathcal{H} \cap H(M) \neq \emptyset$, $\mathcal{H} \cap H(Y) \neq \emptyset$, and for each $x \in \mathcal{H}$, $(\pi, x)$ is a $(\sum_{i=1}^n x_i)$-fair allocation for some $\pi \in$ , . Thus, there exists a continuous function $x(\theta)$ for $\theta \in [0,1]$ such that $x(0) \in \mathcal{H} \cap H(M)$, $x(1) \in \mathcal{H} \cap H(Y)$, and $(\pi(\theta), x(\theta))$ is a $(\sum_{i=1}^n x_i(\theta))$-fair allocation for some $\pi(\theta) \in$ , . It remains to to show that there is some $\theta \in [0,1]$ satisfying $\sum_{i\in\mathcal{P}_R} x_{\pi(\theta,i)}(\theta) = M$.

First, by the definition of envy-freeness and the utility functions of fictitious agents, we see that $\sum_{i\in\mathcal{P}_R} x_{\pi(0,i)}(0) \leq M$. Second, we will show that $\sum_{i\in\mathcal{P}_R} x_{\pi(1,i)}(1) \geq M$. Suppose it is not true. Then there will be some $i \in \mathcal{P}_R$ with $x_{\pi(1,i)}(1) \leq \frac{M}{m}$ and some $j \in \mathcal{P}_F$ with $x_{\pi(1,j)}(1) \geq \frac{Y-M}{n-m} \geq \max_{i\in\mathcal{P}_R} A(i, \frac{M}{m})$. Thus, it follows from Assumption 6.1 that $u_i(\pi(1,i), x_{\pi(1,i)}(1)) < u_i(\pi(1,j), x_{\pi(1,j)}(1))$. This contradicts the fact that $(\pi(1), x(1))$ is an envy-free allocation of the $n\times n$-economy. So we have that $\sum_{i\in\mathcal{P}_R} x_{\pi(1,i)} \geq M$. Finally, by the definition of envy-freeness and the utility functions of fictitious agents, we have that for each $i \in \mathcal{P}_F$ and $\theta \in [0,1]$, $x_{\pi(\theta,i)}(\theta) = \max_{j\in I_n} x_j(\theta)$ which is a continuous function of $\theta$. Therefore, the function

$$\sum_{i\in\mathcal{P}_R} x_{\pi(\theta,i)}(\theta) = \sum_{i\in I_n} x_{\pi(\theta,i)}(\theta) - \sum_{i\in\mathcal{P}_F} x_{\pi(\theta,i)}(\theta) = \sum_{i\in I_n} x_{\pi(\theta,i)}(\theta) - (n-m)\max_{j\in I_n} x_j(\theta)$$

is continuous in $\theta \in [0,1]$. Then by the intermediate value theorem, we see that there exists some $\theta \in [0,1]$ satisfying $\sum_{i\in\mathcal{P}_R} x_{\pi(\theta,i)}(\theta) = M$. This completes the proof. $\square$

Note that in the above theorem besides Assumption 6.1 we only require utility functions to be continuous and nondecreasing. For the exsitence of strongly envy free allocation we do not need any monotonicity.

The next theorem says that if the utility functions of real agents are continuous and strictly increasing in money, then for any given $t^*$-fair allocation $(\pi^*, x^*)$, by increasing (or decreasing) the total money to a proper level $Z$ we can obtain a $Z$-fair allocation $(\rho, z)$ with $\sum_{i\in\mathcal{P}_R} z_{\rho(i)} = M$ which makes every agent strictly better (worse) off, when $\sum_{i\in\mathcal{P}_R} x_{\pi^*(i)} < (>) M$.

**Theorem 6.3** *Suppose that $(\pi^*, x^*)$ is a $t^*$-fair allocation in the $n\times n$-economy. Moreover if the utility function $u_i$ is continuous and strictly increasing in money for every $i \in \mathcal{P}_R$, then the economy has a $Z$-fair allocation $(\rho, z)$ with $\sum_{i\in\mathcal{P}_R} z_{\rho(i)} = M$ for some $Z > M$. Furthermore, we have $u_i(\rho(i), z_{\rho(i)}) > (<) u_i(\pi^*(i), x_{\pi^*(i)})$ for all $i \in \mathcal{P}_R$ when $\sum_{i\in\mathcal{P}_R} x_{\pi^*(i)} < (>) M$.*

Proof: Without loss of generality, we may assume $\sum_{i\in\mathcal{P}_R} x_{\pi^*(i)} < M$. Choose an arbitrary $Y$ with

$$Y \geq (n-m+1)M + (n-m)\sum_{j=1}^{n} |x_j^*|.$$

Then, by Theorem 5.1, we have a connected set $\mathcal{H} \subset H(t^*, Y)$ such that $\mathcal{H} \cap H(t^*) = x^*$, $\mathcal{H} \cap H(Y) \neq \emptyset$, and for each $x \in \mathcal{H}$, $(\pi, x)$ is a $(\sum_{i=1}^n x_i)$-fair allocation for some $\pi \in$ , . Moreover, it satisfies that $x_i > x_i^*$, $u_i(\pi(i), x_{\pi(i)}) > u_i(\pi^*(i), x_{\pi^*(i)})$ for all $i \in I_n$, when $\sum_{j=1}^n x_j > t^* = \sum_{j=1}^n x_j^*$. We have to consider the following two cases.

Case 1: $\sum_{i\in\mathcal{P}_R} x_{\pi(i)} \geq M$ for every fair allocation $(\pi, x)$ of the $n\times n$-economy with $x \in \mathcal{H} \cap H(Y)$. In this case, following the proof of Theorem 6.2 we can show the existence of a $Z$-fair allocation $(\rho, z)$ with $\sum_{i\in\mathcal{P}_R} z_{\rho(i)} = M$ in the $n\times n$-economy.

Case 2: There is some $Y$-fair allocation $(\sigma, y)$ of the $n\times n$-economy in $\mathcal{H} \cap H(Y)$ such that $\sum_{i\in\mathcal{P}_R} y_{\sigma(i)} = L < M$. In this case, let us consider the $m\times m$-economy consisting of $m$ real agents and $m$ indivisible objects $J = \{\sigma(i) | i \in \mathcal{P}_R\}$. Let $\bar{y} \in \mathbb{R}^m$ be the vector whose components are $y_{\sigma(i)}$ for all $i \in \mathcal{P}_R$ and $\bar{\sigma}$ whose components are $\sigma(i)$ for all $i \in \mathcal{P}_R$. Let us consider the fair allocation $(\bar{\sigma}, \bar{y})$ in this $m\times m$-economy. Obviously, it is an $L$-fair allocation of the $m\times m$ economy. Then by Theorem 5.1, we can show that there exists an $M$-fair allocation $(\pi, x)$ in the $m\times m$-economy with the total money $M$ satisfying that $x_j > y_j > x_j^*$ and $u_i(\pi(i), x_{\pi(i)}) > u_i(\sigma(i), y_{\sigma(i)}) > u_i(\pi^*(i), x_{\pi^*(i)})$ for all $j \in J$ and $i \in \mathcal{P}_R$. Note that $\pi(i) \in J$ for all $i \in \mathcal{P}_R$ as well. Let $Z = M + (Y - L)$. From $(\pi, x)$ and $(\sigma, y)$ we will construct a $Z$-fair allocation $(\rho, z)$ of the $n\times n$-economy with $\sum_{i\in\mathcal{P}_R} z_{\rho(i)} = M$. In fact, we can take $z_{\rho(i)} = x_{\pi(i)}$ and $\rho(i) = \pi(i)$ for all $i \in \mathcal{P}_R$ and $z_{\rho(j)} = y_{\sigma(j)}$ and $\rho(j) = \sigma(j)$ for all $j \in \mathcal{P}_F$. On the one hand, we have

$$u_i(\rho(i), z_{\rho(i)}) = u_i(\pi(i), x_{\pi(i)}) \geq u_i(j, x_j) = u_i(j, z_j)$$

for all $i \in \mathcal{P}_R$ and $j \in J$. On the other hand, it holds

$$u_i(\rho(i), z_{\rho(i)}) = u_i(\pi(i), x_{\pi(i)}) > u_i(\sigma(i), y_{\sigma(i)}) \geq u_i(j, y_j) = u_i(j, z_j)$$

for all $i \in \mathcal{P}_R$ and $j \in I_n \setminus J$.

Finally let $\bar{x} = \max\{x_j | j \in J\}$. We see that

$$\bar{x} \leq M + \sum_{j \in J} |x_j^*| \leq M + \sum_{j=1}^{n} |x_j^*|.$$

Futhermore, for $j \in I_n \setminus J = \{\sigma(h) | h \in \mathcal{P}_F\}$, we have that

$$y_j \geq \frac{Y - M}{n - m} \geq \frac{(n-m)M + (n-m)\sum_{j=1}^{n} |x_j^*|}{n - m} = M + \sum_{j=1}^{n} |x_j^*| \geq \bar{x}.$$

Thus, $z_{\rho(i)} \geq z_j$ for all $i \in \mathcal{P}_F$ and $j \in I_n$. This implies that the allocation $(\rho, z)$ is a $Z$-fair allocation of the $n \times n$-economy with $\sum_{i \in \mathcal{P}_R} z_{\rho(i)} = M$. □

It follows from Theorem 6.3 that given a strongly $X$-fair allocation of the $m \times n$-economy and $Y > X$, then there exists a strongly $Y$-fair allocation in the economy which makes every agent better off. Thus we also obtain an alternative proof for Theorem 4 of Alan et al (1991, p.1033).

By specifying a sufficient condition for the existence of a $t$-fair allocation we immediately obtain the following existence theorem from Theorem 6.3.

**Theorem 6.4** *If, for every $i \in \mathcal{P}_R$, utility function $u_i$ is continuous and strictly increasing in money and Assumption 3.2 holds for some $t$, then there exists a $Z$-fair allocation $(\rho, z)$ in the $n \times n$-economy such that $\sum_{i \in \mathcal{P}_R} z_{\rho(i)} = M$.*

One can easily verify that Assumption 3.2 holds for the utility functions of fictitious agents. Since the sufficient conditions in Theorem 4.2 also satisfy Assumption 3.2, we are led to the following result.

**Theorem 6.5** *If utility function $u_i$ is continuous and strictly increasing in money and there exists $L > M$ such that $u_i(j, L) > u_i(k, 0)$ for all $i \in \mathcal{P}_R$, $j, k \in I_n$, then there exists a $Z$-fair allocation $(\rho, z)$ in the $n \times n$-economy such that $\sum_{i \in \mathcal{P}_R} z_{\rho(i)} = M$.*

Finally we remark that the main existence theorem (i.e. Theorem 2) of Alan et al (1991) follows from Theorem 6.5, since the above conditions are satisfied by their conditions (1) and (2) on page 1026. This can be seen from their inequality (16) on page 1030. It should be noted that they define fair allocations based on value functions instead of utility functions.

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