A Dynamic Model of R and D in Oligopoly with Spillovers

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1 Introduction

We develop a dynamic model of R and D in an oligopolistic industry where firms' R and D investments have spillover effects. d'Aspremont and Jacquemin (1988) examined in a two-stage game the effects of R and D spillovers on subgame perfect equilibria with cooperative and noncooperative R and D scenarios.

Henriques (1990) showed that in order to ensure that the result regarding comparison between cooperative and noncooperative R and D is valid, we need to take account of an appropriate stability condition. These papers, however, are based upon static two-stage game models.

In this paper, we consider a differential game to analyze oligopolistic firms' R and D investment behavior when R and D investments have spillover effects. We will show that the stability condition obtained in Henriques (1990) corresponds to a limit case of the stability condition obtained in our dynamic R and D model when the discount rate tends to zero. Thus our model may be thought of as a natural dynamic extension of two stage-games analyzed by d'Aspremont and Jacquemin to a dynamic game.

The paper is organized as follows. In Section 2, we describe the model. In Section 3, we derive open-loop Nash equilibrium. Stability conditions are also examined. In Section 4, we present two extensions of the basic model analyzed in the previous sections. Section 5 concludes.

2 The Model

In this section we describe the model. There are two firms in an industry where they produce a homogeneous product. Let q_i be firm i's output, $i = 1, 2$. The constant unit cost of production of firm i is denoted c_i.

Let the inverse demand function be given by

$$p = f(\sum q_i) = a - b \sum q_i \tag{1}$$

where p is the price of the product, and $a > c_i \geq 0$ and $b > 0$.

Each firm can undertake R and D investment to reduce its production cost. Let x_i denote the amount of R and D investments undertaken by firm i. The cost of R and D investments is assumed to be $\frac{\gamma}{2}x_i^2$. For simplicity, we assume $\gamma = 1$. Each firm's R and D investment indirectly affects other firm's cost through R and D spillovers. Let β denote an R and D spillover parameter. Let $y(t)$ denote cumulative R and D investment at time t and δ a depreciation parameter.

Then the effects of R and D evolve according to the following differential equation.

$$\dot{y}_i = \frac{dy_i}{dt} = x_i + \beta x_j - \delta y_i \tag{2}$$

$$y_i(0) = y^0 \tag{3}$$

and

$$c_i = \bar{c}_i - y_i \tag{4}$$

for $i = 1, 2$, where $\bar{c}_i$ is an initial constant unit production cost of firm i.

The two firms first undertake R and D investment and then given each firm's R and D, the firms engage in Cournot competition in the product market. We assume, for simplicity, that $b = 1$ in (1). Thus, given c_i and c_j, by finding Cournot equilibrium in the product market, firm i's instantaneous profit is given by

$$\pi_i(c_i, c_j) = \frac{(a - 2c_i + c_j)^2}{9}, i, j = 1, 2, i \neq j. \tag{5}$$

Each firm's objective is to maximize the discounted sum of its instantaneous profit over an infinite time horizon.

Thus the problem of firm i may be stated as follows.

$$Z^i = \max_{x_i} \int_0^\infty \left[\pi_i(c_i, c_j) - \frac{1}{2} x_i^2 \right] e^{-rt} dt \tag{6}$$

subject to (2), (3), and (4), where r is a common discount rate.

To analyze this problem, we employ differential game framework. We consider open-loop strategies. Open-loop strategies and open-loop Nash equilibrium are defined as follows.

Definition 1 *The open-loop strategy space for firm is the set*

$$X_i = \{x_i(t) : x_i(t) \text{ is piecewise continuous and } x_i(t) \geq 0 \text{ for every } t\}.$$

Definition 2 *An open-loop Nash equilibrium is an open-loop strategy selection* $x^* = (x_i^*, x_j^*)$ *such that for* $i, j = 1, 2, i \neq j$,

$$Z^i(x_i^*, x_j^*) \geq Z^i(x_i, x_j^*), \text{ for } \forall x_i \in X_i.$$

3 Equilibrium

In this section, we consider two scenarios regarding firm's R and D investment decisions. One is that the two firms choose R and D investment noncooperatively. The other is that the firms cooperate when they choose R and D investment while they compete in the product market. For the both cases, we derive open-loop Nash equilibrium.

Theorem 3 *There exists a unique stationary open-loop Nash equilibrium under R and D competition. Each firm's equilibrium strategy is given by*

$$x^N = \frac{2(2-\beta)\delta(a-\bar{c})}{9\delta(r+\delta) - 2(2-\beta)(1+\beta)}$$

and the equilibrium value of cumulative R and D investment by

$$y^N = \frac{2(1+\beta)(2-\beta)\delta(a-\bar{c})}{9\delta(r+\delta) - 2(2-\beta)(1+\beta)}.$$

Proof. For $i = 1, 2$, firm i's problem is maximizing (6) subject to (2), (3), and (4), and given the other firm's strategy x_j. Let the current-value Hamiltonian for firm i be

$$H_i = \pi_i(c_i, c_j) - \frac{1}{2}x_i^2 + \lambda_i(x_i + \beta x_j - \delta y_i) + \mu_i(x_j + \beta x_i - \delta y_j) \qquad (7)$$

where λ_i and μ_i are costate variables, $i, j = 1, 2, i \neq j$. The necessary conditions for an open-loop Nash equilibrium are

$$\begin{aligned}
\frac{\partial H_i}{\partial x_i} &= -x_i + \lambda_i + \beta\mu_i = 0 && (8) \\
\dot{\lambda}_i &= r\lambda_i - \frac{\partial H_i}{\partial y_i} \\
&= (r+\delta)\lambda_i - \frac{4\left(a - 2(\bar{c}_i - y_i) + (\bar{c}_j - y_j)\right)}{9} \\
\dot{\mu}_i &= r\mu_i - \frac{\partial H_i}{\partial y_j} \\
&= (r+\delta)\mu_i + \frac{2\left(a - 2(\bar{c}_i - y_i) + (\bar{c}_j - y_j)\right)}{9} \\
\lim_{t\to\infty} e^{-rt}\lambda_i &= 0 \\
&\quad \textit{and} && (9) \\
\lim_{t\to\infty} e^{-rt}\mu_i &= 0.
\end{aligned}$$

At the steady state, we have $\dot{\lambda}_i = 0$, $\dot{\mu}_i = 0$, and $\dot{y}_i = 0$.

Then we have

$$(r+\delta)\lambda_i = \frac{4\left(a - 2(\bar{c}_i - y_i) + (\bar{c}_j - y_j)\right)}{9},$$

$$(r+\delta)\mu_i = -\frac{2\left(a - 2(\bar{c}_i - y_i) + (\bar{c}_j - y_j)\right)}{9},$$

and

$$x_i + \beta x_j - \delta y_i = 0.$$

We consider a symmetric equilibrium. Then we have

$$y = \frac{1+\beta}{\delta}x$$

and

$$x = \frac{2(2-\beta)(a - (\bar{c} - y))}{9(r+\delta)}.$$

Thus we obtain

$$\begin{aligned} x &= \frac{\delta}{1+\beta}y \\ &= \frac{2(2-\beta)\delta(a-\bar{c})}{9\delta(r+\delta) - 2(2-\beta)(1+\beta)} \equiv x^N \end{aligned} \tag{10}$$

and

$$y = \frac{2(1+\beta)(2-\beta)\delta(a-\bar{c})}{9\delta(r+\delta) - 2(2-\beta)(1+\beta)} \equiv y^N. \tag{11}$$

Recall that we have

$$\dot{y}_i = x_i + \beta x_j - \delta y_i.$$

Differentiate this with respect to time and we get

$$\ddot{y}_i = \dot{x}_i + \beta \dot{x}_j - \delta \dot{y}_i.$$

From (8), we obtain

$$x_i = \lambda_i + \beta\mu_i. \tag{12}$$

Differentiate equation (12) with respect to time and we have

$$\begin{aligned} \dot{x}_i &= \dot{\lambda}_i + \beta\dot{\mu}_i \\ &= (r+\delta)x_i - \frac{2(2-\beta)\left(a - 2(\bar{c}_i - y_i) + (\bar{c}_j - y_j)\right)}{9}. \end{aligned}$$

Therefore we have

$$\ddot{y}_i = (r+\delta)(\dot{y}_i+\delta y_i) - \frac{2(2-\beta)\left(a(1+\beta)-(\beta-2)(\bar{c}_i-y_i)+(1-2\beta)(\bar{c}_j-y_j)\right)}{9}.$$

That is,

$$\begin{aligned}\ddot{y}_i &= (r+\delta)\dot{y}_i + \left\{\delta(r+\delta) - \frac{2(2-\beta)^2}{9}\right\} y_i \\ &\quad + \frac{2(2-\beta)(1-2\beta)}{9} y_j + \frac{2(2-\beta)(1+\beta)(a-\bar{c}_i)}{9}.\end{aligned}$$

At a stationary point, we have $\ddot{y}_i = 0$ and $\dot{y}_i = 0$ for $i = 1, 2$. Using $y_i = y_j = y$, we get

$$y = \frac{2(1+\beta)(2-\beta)\delta(a-\bar{c})}{9\delta(r+\delta) - 2(2-\beta)(1+\beta)}.$$

Now let us define

$$A = \delta(r+\delta) - \frac{2(2-\beta)^2}{9} \tag{13}$$

and

$$B = \frac{2(2-\beta)(1-2\beta)}{9}. \tag{14}$$

Then we have the following system of differential equations.

$$\begin{bmatrix} \frac{dy_1}{dt} \\ \frac{d\dot{y}_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{d\dot{y}_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ A & r+\delta & B & 0 \\ 0 & 0 & 0 & 1 \\ B & 0 & A & r+\delta \end{bmatrix} \begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{bmatrix}. \tag{15}$$

The characteristic equation of this system is

$$\begin{vmatrix} -\xi & 1 & 0 & 0 \\ A & r+\delta-\xi & B & 0 \\ 0 & 0 & -\xi & 1 \\ B & 0 & A & r+\delta-\xi \end{vmatrix} = 0.$$

Then we have

$$\left[\xi^2 - (r+\delta)\xi - (A+B)\right]\left[\xi^2 - (r+\delta)\xi - (A-B)\right] = 0.$$

Therefore the characteristic roots are

$$\xi = \frac{(r+\delta) \pm \sqrt{(r+\delta)^2 + 4(A+B)}}{2}$$

and

$$\xi = \frac{(r+\delta) \pm \sqrt{(r+\delta)^2 + 4(A-B)}}{2}.$$

Note that we have

$$A + B = \delta(r+\delta) - \frac{2(2-\beta)(1+\beta)}{9} > 0$$

and

$$A - B = \delta(r+\delta) - \frac{2(2-\beta)(1-\beta)}{3} > 0.$$

Thus there are two positive roots and two negative roots. Therefore the equilibrium is a saddle point. ■

Next we consider the situation where the firms can cooperate when they choose their R and D investment levels. Then the problem becomes, for $\forall i$, to choose x_i to maximize

$$\sum_{i=1}^{2} \int_0^\infty \left[\pi_i(c,c) - \frac{1}{2}x_i^2\right] e^{-rt} dt.$$

The current-value Hamiltonian Φ in this case is

$$\Phi = \sum_{i=1}^{2} \left[\pi_i(c,c) - \frac{1}{2}x_i^2\right] + \lambda(x_i + \beta x_j - \delta y_i) + \mu(x_j + \beta x_i - \delta y_j).$$

The necessary conditions for an open-loop Nash equilibrium are

$$\frac{\partial \Phi_i}{\partial x_i} = -x_i + \lambda + \beta\mu = 0,$$

$$\begin{aligned}
\dot{\lambda} &= r\lambda - \frac{\partial \Phi_i}{\partial y_i} \\
&= (r+\delta)\lambda - \frac{4\left(a - 2(\bar{c}_i - y_i) + (\bar{c}_j - y_j)\right)}{9} + \frac{2\left(a - 2(\bar{c}_i - y_i) + (\bar{c}_j - y_j)\right)}{9},
\end{aligned}$$

$$\begin{aligned}
\dot{\mu} &= r\mu - \frac{\partial \Phi_i}{\partial y_j} \\
&= (r+\delta)\mu + \frac{2\left(a - 2(\bar{c}_i - y_i) + (\bar{c}_j - y_j)\right)}{9} - \frac{4\left(a - 2(\bar{c}_i - y_i) + (\bar{c}_j - y_j)\right)}{9},
\end{aligned}$$

$$\lim_{t\to\infty} e^{-rt}\lambda = 0,$$

and

$$\lim_{t\to\infty} e^{-rt}\mu = 0.$$

At the steady state, we get

$$x^C = \frac{2(1+\beta)\delta(a-\bar{c})}{9\delta(r+\delta) - 2(1+\beta)^2} \tag{16}$$

and

$$\begin{aligned}
y^C &= \frac{9(r+\delta)}{2(1+\beta)}x - (a-\bar{c}) \qquad (17) \\
&= \frac{9\delta(r+\delta)(a-\bar{c})}{9\delta(r+\delta) - 2(1+\beta)^2} - (a-\bar{c}) \\
&= \frac{2(1+\beta)^2(a-\bar{c})}{9\delta(r+\delta) - 2(1+\beta)^2}.
\end{aligned}$$

Now we can make comparison between noncooperative R and D and cooperative R and D. Then we have the following result.

Proposition 4 *At the open-loop Nash equilibrium, we have*

$$x^N \gtreqqless x^C \Longleftrightarrow \beta \lesseqqgtr \frac{1}{2}. \tag{18}$$

Proof. Recall that we have, for the noncooperative R and D case,

$$x^N = \frac{2(2-\beta)\delta(a-\bar{c})}{9\delta(r+\delta) - 2(2-\beta)(1+\beta)}$$

and for the cooperative R and D case,

$$x^C = \frac{2(1+\beta)\delta(a-\bar{c})}{9\delta(r+\delta) - 2(1+\beta)^2}.$$

Then it follows that

$$x^N \gtreqless x^C \Longleftrightarrow \beta \lesseqgtr \frac{1}{2}.$$

■

When the spillover parameter is relatively small, noncooperative R and D investment level is larger than cooperative one. This result is in accord with the one comparing noncooperative R and D with cooperative R and D in d'Aspremont and Jacquemin (1988).

Next we consider a limit case when the discount rate goes to zero. We suppose that depreciation parameter δ equals one, that is,

$$\delta = 1.$$

When the discount rate goes to zero, that is, when $r \longrightarrow 0$, we have, for the noncooperative R and D case,

$$x^N = \frac{2(2-\beta)\delta(a-\bar{c})}{9\delta(r+\delta) - 2(2-\beta)(1+\beta)} \longrightarrow \frac{2(2-\beta)(a-\bar{c})}{9 - 2(2-\beta)(1+\beta)},$$

and for the cooperative R and D case,

$$x^C = \frac{2(1+\beta)\delta(a-\bar{c})}{9\delta(r+\delta) - 2(1+\beta)^2} \longrightarrow \frac{2(1+\beta)(a-\bar{c})}{9 - 2(1+\beta)^2}.$$

Note that these equilibrium values correspond to those for the two-stage game in d'Aspremont and Jacquemin(1988).

Next we examine stability condition. When the discount rate goes to zero, we obtain

$$\begin{aligned} A - B &= \delta(r+\delta) - \frac{2(2-\beta)(1-\beta)}{3} \\ &\rightarrow 1 - \frac{2(2-\beta)(1-\beta)}{3}. \end{aligned}$$

Then the stability for an open-loop Nash equilibrium requires that we have

$$1 - \frac{2(2-\beta)(1-\beta)}{3} > 0.$$

Equivalently, we have

$$2\beta^2 - 6\beta + 1 < 0.$$

It then follows that we have

$$\beta > \frac{3-\sqrt{7}}{2}. \tag{19}$$

Observe that the condition (19) is the one obtained for the two-stage game in Henriques (1990).[1] That is, when the discount rate approaches zero, our dynamic stability condition will be corresponding to the stability condition in a static two-stage game analyzed in d'Aspremont and Jacquemin(1988). We summarize these results in the following proposition.

Proposition 5 *When the discount rate goes to zero, open-loop Nash equilibrium values and the dynamic stability condition for our dynamic game, in the limit, approach Nash equilibrium values and the stability condition in a static two-stage game.*

4 Extensions

In this section, two extensions of our basic model are considered. First we consider the case where the R and D spillover parameters are not necessarily symmetric. Then we have the following differential equations regarding R and D investments.

$$\dot{y}_i = \frac{dy_i}{dt} = x_i + \beta_j x_j - \delta y_i$$

and

$$\dot{y}_j = \frac{dy_j}{dt} = x_j + \beta_i x_i - \delta y_j.$$

Then the Hamiltonian in this case is given by

$$H_i = \pi_i(c_i, c_j) - \frac{1}{2}x_i^2 + \lambda_i(x_i + \beta_j x_j - \delta y_i) + \mu_i(x_j + \beta_i x_i - \delta y_j), i, j = 1, 2, i \neq j. \tag{20}$$

[1] In Henriques(1990), the stablity condition is given as $\beta > \frac{3}{2} - \sqrt{\frac{7}{2}}$. The correct stability condition, however, is $\beta > \frac{3-\sqrt{7}}{2}$.

The necessary conditions for an open-loop Nash equilibrium are

$$\begin{aligned}
\frac{\partial H_i}{\partial x_i} &= -x_i + \lambda_i + \beta_i \mu_i = 0, \qquad (21)\\
\dot{\lambda}_i &= r\lambda_i - \frac{\partial H_i}{\partial y_i}\\
&= (r+\delta)\lambda_i - \frac{4\left(a - 2(\bar{c}_i - y_i) + (\bar{c}_j - y_j)\right)}{9},\\
\dot{\mu}_i &= r\mu_i - \frac{\partial H_i}{\partial y_j}\\
&= (r+\delta)\mu_i + \frac{2\left(a - 2(\bar{c}_i - y_i) + (\bar{c}_j - y_j)\right)}{9},\\
\lim_{t\to\infty} e^{-rt}\lambda_i &= 0,\\
&\quad \textit{and} \qquad (22)\\
\lim_{t\to\infty} e^{-rt}\mu_i &= 0.
\end{aligned}$$

At the steady state, we have $\dot{\lambda}_i = 0$, $\dot{\mu}_i = 0$, $\dot{y}_i = 0$, and $\dot{y}_j = 0$. Thus we have

$$(r+\delta)\lambda_i = \frac{4\left(a - 2(\bar{c}_i - y_i) + (\bar{c}_j - y_j)\right)}{9},$$

$$(r+\delta)\mu_i = -\frac{2\left(a - 2(\bar{c}_i - y_i) + (\bar{c}_j - y_j)\right)}{9},$$

$$x_i = \frac{2(2-\beta_i)}{9(r+\delta)}\left(a - 2(\bar{c}_i - y_i) + (\bar{c}_j - y_j)\right),$$

$$y_i = \frac{x_i + \beta_j x_j}{\delta},$$

and

$$y_j = \frac{x_j + \beta_i x_i}{\delta}.$$

It follows that we get

$$x_j^* = \frac{2F\delta(a-\bar{c})\left[9\delta(r+\delta) - 2E\left(E - (2\beta_i - 1)\right)\right]}{\left(9\delta(r+\delta) - 2F^2\right)\left(9\delta(r+\delta) - 2E^2\right) - 4EF(2\beta_i - 1)(2\beta_j - 1)} \qquad (23)$$

where

$$E = 2 - \beta_i \text{ and } F = 2 - \beta_j, i, j = 1, 2, i \neq j.$$

Next we proceed to analyze the case where there are n firms in the industry. For simplicity, we assume that spillover parameters are symmetric. Then the differential equation regarding R and D investments becomes

$$\dot{y}_i = \frac{dy_i}{dt} = x_i + \beta \sum_j x_j - \delta y_i \tag{24}$$

where $i, j \in \{1, 2, \cdots n\}, i \neq j$.

Each firm's instantaneous profit under Cournot competition in the product market is

$$\pi_i = \frac{\left(a - (n+1)c_i + \sum_{j=1}^{n} c_j\right)^2}{(n+1)^2}. \tag{25}$$

Then the Hamiltonian in this case is given by

$$H_i = \pi_i - \frac{1}{2}x_i^2 + \sum_{k=1}^{n} \lambda_i^k (x_k + \beta \sum_{j \neq k} x_j - \delta y_k). \tag{26}$$

The necessary conditions for an open-loop Nash equilibrium are

$$\begin{aligned}
\frac{\partial H_i}{\partial x_i} &= -x_i + \lambda_i^i + \beta \sum_{j \neq i}^{n} \lambda_i^j = 0, \qquad (27)\\
\dot{\lambda}_i^i &= r\lambda_i^i - \frac{\partial H_i}{\partial y_i}\\
&= (r+\delta)\lambda_i^i - \frac{\partial \pi_i}{\partial y_i},\\
\dot{\lambda}_i^j &= r\lambda_i^j - \frac{\partial H_i}{\partial y_j}\\
&= (r+\delta)\lambda_i^j - \frac{\partial \pi_i}{\partial y_j},\\
\lim_{t\to\infty} e^{-rt}\lambda_i &= 0,\\
& \quad \textit{and} \qquad (28)\\
\lim_{t\to\infty} e^{-rt}\mu_i &= 0,
\end{aligned}$$

where

$$\frac{\partial \pi_i}{\partial y_i} = \frac{2n\left(a-(n+1)c_i+\sum_{j=1}^n c_j\right)}{(n+1)^2}$$

and

$$\frac{\partial \pi_i}{\partial y_j} = -\frac{2n\left(a-(n+1)c_i+\sum_{j=1}^n c_j\right)}{(n+1)^2}.$$

At the steady state, we have $\dot{\lambda}_i^i = 0$, $\dot{\lambda}_i^j = 0$, and $\dot{y}_i = 0$.
We consider a symmetric equilibrium. Then we get

$$y = \frac{(1+(n-1)\beta)\,x}{\delta},$$

$$(r+\delta)\lambda = \frac{2n\left(a-(n+1)c_i+\sum_{j=1}^n c_j\right)}{(n+1)^2},$$

and

$$(r+\delta)\mu = -\frac{2n\left(a-(n+1)c_i+\sum_{j=1}^n c_j\right)}{(n+1)^2}.$$

It follows that we have the following equilibrium values.

$$x^{**} = \frac{2\delta\,(n-(n-1)\beta)\,(a-\bar{c})}{(n+1)^2\delta(r+\delta)-2\,(n-(n-1)\beta)\,(1+(n-1)\beta)} \tag{29}$$

and

$$y^{**} = \frac{2\,(n-(n-1)\beta)\,(1+(n-1)\beta)\,(a-\bar{c})}{(n+1)^2\delta(r+\delta)-2\,(n-(n-1)\beta)\,(1+(n-1)\beta)}. \tag{30}$$

Note that we have

$$\lim_{n\to\infty} x^{**} = 0$$

and

$$\lim_{n\to\infty} n x^{**} = \frac{2\delta(1-\beta)(a-\bar{c})}{\delta(r+\delta)-(1-\beta)\beta} \neq 0.$$

Thus we obtain the following result.

Proposition 6 *When the number of firms becomes infinite, the investment level of each firm at the open-loop Nash equilibrium converges to zero. The total investment level of the firms, however, does not vanish.*

This proposition says that when the number of firms becomes arbitrarily large, the investment level of each firm will be negligible. The investment level as a whole, however, does not vanish as the number of firms becomes infinite. Therefore we may conclude that for arbitrary number of firms, there exist effective R and D investments for the industry at the open-loop Nash equilibrium.

5 Conclusion

In this paper, we have developed a dynamic model of an oligopoly where firms' R and D investments have spillover effects. We have shown that there exists a unique stable open-loop Nash equilibrium. We have compared equilibrium under R and D competition with equilibrium under R and D cooperation and shown that for small spillovers, each firm's R and D investments are larger under R and D competition than under R and D cooperation. The result concerning equilibrium investment levels is in accord with the one comparing noncooperative R and D with cooperative R and D in d'Aspremont and Jacquemin (1988).We have also demonstrated that in the limit when the discount rate approaches zero, the dynamic stability condition obtained in the paper corresponds to the one in standard two-stage games. These results lead to the conclusion that our differential game model of R and D with spillovers may be thought of as a dynamic extension of d'Aspremont and Jacquemin (1988).

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