Application of Shapley Value on Network Industries with Essential Facilities

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1. Model

A network industry is composed of K+1 operators: operator 0 provides the essential facility or input for downstream operators i's (i = 1, 2, ... , K), which with access to the essential factor, supply final services for end-users. Operator i covers a network with size n_i, and obtains gross profits before the payment of access charges, $n_i u(N)$, where $N \equiv \sum_{i=1}^{K} n_i$. The size of network may represent the number of end-users (user groups or service areas). If $u(N)$ represents net benefit per end-user, we expect $u'(N) > 0$ due to network externalities. The upstream operator may be vertically integrated, and serves its own network with size n_0. But, our results do not depend on whether the essential facility provider is vertically integrated or not. To save notation, we will assume the initial state is vertical separation, $n_0 = 0$. We do not consider the costs of facilities investment; they are sunk. Then, operator 0's gross profit is zero, which does not take access charges into account. Now, suppose that each downstream operator i pays access charge a_i to operator 0. Then, operators' net profits are: $\sum_{i=1}^{K} a_i$ for 0, and $n_i u(N) - a_i$ for i = 1, 2, ... , K. We do not model explicitly the process of bargaining that determines access charges. Instead we assume that each obtains its share according to the Shapley value. Given Shapley values Sh_i (i = 0, 1, ..., K), we can determine access charges by:

$$Sh_0 = \sum_{i=1}^{K} a_i \text{ and } Sh_i = n_i u(N) - a_i \ (i = 1, 2, \dots , K). \qquad \text{--- (1)}$$

Subsequently, we will focus on determining each operator's Shapley value.

Denote the grand coalition as $I \equiv \{0,1,2,....,K\}$ and the value function as $v(S)$ for any partial coalition such as $S \subseteq I$. Any coalition's value is the aggregate of its members' profits. The network structure, specified as above, simplifies it to a great extent.

Especially, we can reduce the value function as follows. First of all, due to the nature of essentiality of operator 0's facilities, we have:

$$v(S) = 0 \qquad \forall S \subseteq I \text{ such as } 0 \notin S. \qquad \text{--- (2)}$$

The value for a partial coalition such as $S = \{0, i_1, ..., i_k\}$ will be the aggregate of $n_{i_1}\ u(n_{i_1} + ... + n_{i_k})\ + ... + n_{i_k}\ u(n_{i_1} + ... + n_{i_k})$. That is,

$$v(S)\ =\ (n_{i_1} + ... + n_{i_k})\ u(n_{i_1} + ... + n_{i_k}) \quad \forall S \subseteq I \ \text{ such as } \ S = \{0, i_1, ..., i_k\} \qquad \text{--- (3)}$$

For example, the value of the grand coalition is $v(I) = Nu(N)$. To simplify notations, we denote:

$$\pi(n) \equiv n u(n); \qquad \text{--- (4)}$$
$$v(S)\ =\ \pi(n_{i_1} + ... + n_{i_k}) \quad \forall S \subseteq I \ \text{ such as } \ S = \{0, i_1, ..., i_k\}.$$

Obviously, $\pi(0) = 0$. The assumption of network externalities, $u'(n) > 0$, implies $\pi'(n) > 0$.

Given the characteristic value function $v(S)$, the Shapley values are defined as follows:

$$Sh_i\ =\ \sum_{\substack{i \in S \\ S \subseteq I}} \frac{(s-1)!(K+1-s)!}{(K+1)!} [\, v(S) - v(S/\{i\})\,] \quad (i = 0,\ 1,\ ....,\ K), \qquad \text{--- (5)}$$

where *s* and *K*+1 are the numbers of members in coalition *S* and *I*, respectively. As is well known, the Shapley values, as defined by (5), can be given the following heuristic interpretation. Suppose K+1 players line up in a random order. It is assumed that all orders of lining up have the same probability: viz., 1/(*K*+1)!. Suppose that if a player, *i*, finds the members of coalition *S*/{*i*} (and no others) in front of him, he receives the amount $v(S) - v(S/\{i\})$, i.e., the marginal amount which he contributes to the coalition, as payoff. Then, Shapley value Sh_i is the expected payoff to player *i* under this randomization scheme.[1]

In general, the Shapley values are too complicated to work with when *K* is large. However, in our set-up of network industries, we can exploit the properties of value function $v(S)$ in (2) and (3) to characterize them in a manageable way as follows.

$$Sh_0\ =\ \sum_{i \in I/\{0\}} \frac{(K-1)!}{(K+1)!} \pi(n_i)$$

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[1] This explanation is adapted from Owen (1982, p. 197).

$$+ \;.......$$

$$+ \sum_{\substack{i_1<...<i_k \\ \{i_1,...,i_k\}\subseteq I/\{0\}}} \frac{k!(K-k)!}{(K+1)!}\pi(n_{i_1}+....+n_{i_k})^{2} \qquad \text{--- (6)}$$

$$+ \;.........$$

$$+ \sum_{\substack{i_1<...<i_k \\ \{i_1,...,i_k\}\subseteq I/\{0\}}} \frac{(K-k)!k!}{(K+1)!}\pi(N-n_{i_1}-....-n_{i_k})^{3}$$

$$+ \;..........$$

$$+ \sum_{i\in I/\{0\}} \frac{(K-1)!}{(K+1)!}\pi(N-n_i)$$

$$+ \frac{K!}{(K+1)!}\pi(N)$$

$$Sh_1 = \frac{(K-1)!}{(K+1)!}\pi(n_1)$$

$$+ \sum_{i\in I/\{0,1\}} \frac{2!(K-2)!}{(K+1)!}\{\pi(n_1+n_i)-\pi(n_i)\}$$

$$+ \;...........$$

$$+ \sum_{\substack{i_1<...<i_k \\ \{i_1,...,i_k\}\subseteq I/\{0,1\}}} \frac{(k+1)!(K-k-1)!}{(K+1)!}\{\pi(n_1+n_{i_1}+....+n_{i_k})-\pi(n_{i_1}+....+n_{i_k})\}^{4}$$

$$+ \;............ \qquad \text{--- (7)}$$

$$+ \sum_{\substack{i_1...<i_k \\ \{i_1,...,i_k\}\subseteq I/\{0,1\}}} \frac{(K-k)!k!}{(K+1)!}\{\pi(N-n_{i_1}-....-n_{i_k})-\pi(N-n_1-n_{i_1}-....-n_{i_k})\}^{5}$$

$$+ \;.............$$

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[2] To interpret this in terms of (5), consider i = 0 and $S = \{0, i_1, ..., i_k\}$. The summation is over all $(i_1, ..., i_k)$ such that $i_1 < .... < i_k$ and $\{i_1, ..., i_k\} \subseteq I/\{0\}$. All other summations in the paper should be read similarly.

[3] To interpret this in terms of (5), consider i = 0 and $S = I/\{i_1, ..., i_k\}$.

[4] To interpret this in terms of (5), consider i = 1 and $S = \{0, 1, i_1, ..., i_k\}$.

[5] To interpret this in terms of (5), consider i = 1 and $S = I/\{i_1, ..., i_k\}$.

$$+ \sum_{i \in I/\{0,1\}} \frac{(K-1)!}{(K+1)!}\{\pi(N-n_i)-\pi(N-n_1-n_i)\}$$
$$+ \frac{K!}{(K+1)!}\{\pi(N)-\pi(N-n_1)\}$$

We can determine Sh_i (i = 2, ..., K) in general by adapting (7) slightly.

2. Incentives of Vertical Integration

Now we consider the incentives of vertical integration, i.e., whether operator 0 and 1 can gain by merging. Denote the merged operator and its share with 0+1 and Sh_{0+1}, respectively. In this case, the grand coalition is $\{0+1,2,....,K\}$. We can derive Sh_{0+1} by modifying (6) slightly:

$$\begin{aligned} Sh_{0+1} = & \frac{(K-1)!}{K!}\pi(n_1) \\ & + \sum_{i \in I/\{0,1\}} \frac{(K-2)!}{K!}\pi(n_1+n_i) \\ & + \ .... \\ & + \sum_{\substack{i_1<...<i_k \\ \{i_1,...,i_k\}\subseteq I/\{0,1\}}} \frac{k!(K-k-1)!}{K!}\pi(n_1+n_{i_1}+....+n_{i_k})^{6} \\ & + \ .... \\ & + \sum_{\substack{i_1<...<i_k \\ \{i_1,...,i_k\}\subseteq I/\{0,1\}}} \frac{(K-k-1)!k!}{K!}\pi(N-n_{i_1}-....-n_{i_k})^{7} \\ & + \ .... \\ & + \sum_{i \in I/\{0,1\}} \frac{(K-2)!}{K!}\pi(N-n_i) \\ & + \frac{(K-1)!}{K!}\pi(N) \end{aligned} \qquad \text{--- (8)}$$

Then, we can say that vertical integration is beneficial for the integrated parties if and only if $Sh_0 + Sh_1 < Sh_{0+1}$.

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[6] To interpret this in terms of (5), consider i = 0+1 and $S = \{0+1, i_1, ..., i_k\}$.

[7] To interpret this in terms of (5), consider i = 0+1 and $S = I/\{i_1, ..., i_k\}$.

Suppose that there do not exist dominant downstream operators whose network sizes are considerably larger than others. More specifically, we assume:

Assumption 1. If $s > s'$, then $\sum_{i\in S} n_i \geq \sum_{i'\in S'} n_{i'} \quad \forall S, S' \subseteq \{1,2,.....K\}$.

In the above, s and s' denote the number of members in coalitions S and S', respectively. Hence, Assumption 2 means that any coalition with s members has a network size no less than does any other coalition with $s-1$ members. An extreme version of this assumption is that all operators have the same size of networks. With Assumption 1, we can establish the following result on the necessary and sufficient condition for beneficial vertical integration.

Theorem 1. Suppose Assumption 1. Then, we have:

$$Sh_0 + Sh_1 > Sh_{0+1} \quad \Leftrightarrow \quad \pi''(n) > 0.$$

Proof. Decompose (6) into:

$$\begin{aligned}
Sh_0 = \; & \frac{(K-1)!}{(K+1)!}\pi(n_1) + \sum_{i\in I/\{0,1\}} \frac{(K-1)!}{(K+1)!}\pi(n_i) \\
& + ... \\
& + \sum_{\substack{i_1<...<i_{k-1} \\ \{i_1,..,i_{k-1}\}\subseteq I/\{0,1\}}} \frac{k!(K-k)!}{(K+1)!}\pi(n_1 + n_{i_1} + .. + n_{i_{k-1}}) \\
& \qquad + \sum_{\substack{i_1<...<i_k \\ \{i_1,...,i_k\}\subseteq I/\{0,1\}}} \frac{k!(K-k)!}{(K+1)!}\pi(n_{i_1} + ... + n_{i_k}) \\
& + ... \qquad\qquad \text{--- (9)} \\
& + \sum_{\substack{i_1<...<i_k \\ \{i_1,...,i_k\}\subseteq I/\{0,1\}}} \frac{(K-k)!k!}{(K+1)!}\pi(N - n_{i_1} - .... - n_{i_k}) \\
& \qquad + \sum_{\substack{i_1<...<i_{k-1} \\ \{i_1,...,i_{k-1}\}\subseteq I/\{0,1\}}} \frac{(K-k)!k!}{(K+1)!}\pi(N - n_1 - n_{i_1} - .... - n_{i_{k-1}}) \\
& + ... \\
& + \sum_{i\in I/\{0,1\}} \frac{(K-1)!}{(K+1)!}\pi(N - n_i) + \frac{(K-1)!}{(K+1)!}\pi(N - n_1) \\
& + \frac{K!}{(K+1)!}\pi(N)
\end{aligned}$$

Denote $I/\{0,1\}$ (={2,3,...,K}) $\equiv I'$. From (7), (8) and (9), we have:

$$(Sh_0 + Sh_1) - Sh_{0+1} = \left\{2\frac{(K-1)!}{(K+1)!} - \frac{(K-1)!}{K!}\right\}\pi(n_1) \quad \text{-- (10.1)}$$

$$+ \left\{2\frac{2!(K-2)!}{(K+1)!} - \frac{(K-2)!}{K!}\right\}\sum_{i\in I'}\pi(n_1+n_i) \quad \text{-- (10.2)}$$

+ ....

$$+ \left\{2\frac{k!(K-k)!}{(K+1)!} - \frac{(k-1)!(K-k)!}{K!}\right\}\sum_{\substack{i_1<\ldots<i_{k-1} \\ \{i_1,\ldots,i_{k-1}\}\subseteq I'}}\pi(n_1+n_{i_1}+\ldots+n_{i_{k-1}})$$

+ .... -- (10.3)

$$+ \left\{2\frac{(K-k)!k!}{(K+1)!} - \frac{(K-k-1)!k!}{K!}\right\}\sum_{\substack{i_1<\ldots<i_k \\ \{i_1,\ldots,i_k\}\subseteq I'}}\pi(N-n_{i_1}-\ldots-n_{i_k}) \quad \text{-- (10.4)}$$

+ ....

$$+ \left\{2\frac{(K-1)!}{(K+1)!} - \frac{(K-2)!}{K!}\right\}\sum_{i\in I'}\pi(N-n_i) \quad \text{-- (10.5)}$$

$$+ \left\{\frac{(K-1)!}{(K+1)!} - \frac{2!(K-2)!}{(K+1)!}\right\}\sum_{i\in I'}\pi(n_i) \quad \text{-- (10.6)}$$

+ ....

$$+ \left\{\frac{k!(K-k)!}{(K+1)!} - \frac{(k+1)!(K-k-1)!}{(K+1)!}\right\}\sum_{\substack{i_1<\ldots<i_k \\ \{i_1,\ldots,i_k\}\subseteq I'}}\pi(n_{i_1}+\ldots+n_{i_k}) \quad \text{-- (10.7)}$$

+ ....

$$+ \left\{\frac{(K-k)!k!}{(K+1)!} - \frac{(K-k+1)!(k-1)!}{(K+1)!}\right\}\sum_{\substack{i_1<\ldots<i_{k-1} \\ \{i_1,\ldots,i_{k-1}\}\subseteq I'}}\pi(N-n_1-n_{i_1}-\ldots-n_{i_{k-1}}) \quad \text{-- (10.8)}$$

+ ....

$$+ \left\{\frac{(K-2)!2!}{(K+1)!} - \frac{(K-1)!}{(K+1)!}\right\}\sum_{i\in I'}\pi(N-n_1-n_i) \quad \text{-- (10.9)}$$

$$+ \left\{\frac{(K-1)!}{(K+1)!} - \frac{K!}{(K+1)!}\right\}\pi(N-n_1) \quad \text{-- (10.10)}$$

$$+ \left\{2\frac{K!}{(K+1)!} - \frac{(K-1)!}{K!}\right\}\pi(N) \quad \text{-- (10.11)}$$

We can check that the coefficients of (10.1) and (10.10) are equivalent. Similarly, notice that the coefficients of (10.2) and (10.9), those of (10.3) and (10.8), those of (10.4) and (10.7), and those of (10.5) and (10.6) are equivalent pair-wise. To confirm the equivalence in general terms, compare two sequences that are composed of terms in equation (10) – one with (10.1) through (10.5) and the other with (10.6) through (10.10); each sequence is composed of (K-1) elements. The *k*-th term of the former sequence from the top is (10.3), while the *k*-th term of the latter from the bottom is (10.8). Their coefficients of (10.3) and (10.8) are the same as:

$$-(K+1-2k)\frac{(K-k)!(k-1)!}{(K+1)!}.$$

Hence, equation (10) can be rearranged into:

$$(Sh_0 + Sh_1) - Sh_{0+1} = -(K-1)\frac{(K-1)!}{(K+1)!}\{\pi(n_1)+\pi(N-n_1)\} \quad \text{-- (11.1)}$$

$$-(K-3)\frac{(K-2)!}{(K+1)!}\sum_{i\in I'}\{\pi(n_1+n_i)+\pi(N-n_1-n_i)\} \quad \text{-- (11.2)}$$

$$+ \ldots\ldots\ldots\ldots$$

$$-(K+1-2k)\frac{(K-k)!(k-1)!}{(K+1)!}\sum_{\substack{i_1<\ldots<i_{k-1}\\ \{i_1,\ldots,i_{k-1}\}\subseteq I'}}\{\pi(n_1+n_{i_1}+..+n_{i_{k-1}})+\pi(N-n_1-n_{i_1}-..-n_{i_{k-1}})\}$$

$$+ \ldots\ldots\ldots\ldots \quad \text{-- (11.3)}$$

$$+(K+1-2k)\frac{(k-1)!(K-k)!}{(K+1)!}\sum_{\substack{i_1<\ldots<i_{k-1}\\ \{i_1,\ldots,i_{k-1}\}\subseteq I'}}\{\pi(n_{i_1}+..+n_{i_{k-1}})+\pi(N-n_{i_1}-..-n_{i_{k-1}})\}$$

$$+ \ldots\ldots\ldots\ldots \quad \text{-- (11.4)}$$[8]

$$+ \quad (K-3)\frac{(K-2)!}{(K+1)!}\sum_{i\in I'}\{\pi(n_i)+\pi(N-n_i)\} \quad \text{-- (11.5)}$$

$$+ \quad (K-1)\frac{(K-1)!}{(K+1)!}\pi(N). \quad \text{-- (11.6)}$$[9]

Finally, we can reduce (11) into:

$$(Sh_0 + Sh_1) - Sh_{0+1} =$$

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[8] This term is the sum of the terms that result from plugging *k*-1 in (10.4) and (10.7) in place of *k*.

[9] This term can be expressed as $(K-1)\frac{(K-1)!}{(K+1)!}\{\pi(0)+\pi(N)\}$ since $\pi(0)=0$.

$$\sum_{k<\frac{K+1}{2}} \sum_{\substack{i_1<\ldots<i_{k-1} \\ \{i_1,..,i_{k-1}\}\subseteq I'}} (K+1-2k)\frac{(k-1)!(K-k)!}{(K+1)!}[\{\pi(n_{i_1}+..+n_{i_{k-1}})+\pi(N-n_{i_1}-..-n_{i_{k-1}})\}$$
$$-\{\pi(n_1+n_{i_1}+..+n_{i_{k-1}})+\pi(N-n_1-n_{i_1}-..-n_{k-1})\}] \qquad \text{-- (12)}$$

Now we can show that if Assumption 1 holds true,

$$\{\pi(n_{i_1}+..+n_{i_{k-1}})+\pi(N-n_{i_1}-..-n_{i_{k-1}})\}$$
$$-\{\pi(n_1+n_{i_1}+..+n_{i_{k-1}})+\pi(N-n_1-n_{i_1}-..-n_{k-1})\} \geq 0 \Leftrightarrow \pi''(n)>0 \qquad \text{-- (13)}$$

for any $k<\frac{K+1}{2}$ and for any $(i_1,.....,i_{k-1})$ such that $i_1<.....<i_{k-1}$ and $\{i_1,.....,i_{k-1}\}\subseteq I'$, and with strict inequalities for at least one k's. Denote $n_{i_1}+..+n_{i_{k-1}}\equiv M$. Then, we have:

$$\int_0^{N-2M-n_1}\int_0^{n_1}\pi''(M+x+y)dxdy = \{\pi(N-M)-\pi(n_1+M)\}-\{\pi(N-n_1-M)-\pi(M)\}$$
$$= \{\pi(M)+\pi(N-M)\}-\{\pi(n_1+M)+\pi(N-n_1-M)\}. \qquad \text{-- (14)}$$

Since the number of members in S = $\{1,2,...,K\}/\{i_1,.....,i_{k-1}\}$ is $s=K-k+1$ and that in $S'=\{1,i_1,.....,i_{k-1}\}$ is s'=k, we know $s>s'$ for any $k<\frac{K+1}{2}$. Assumption 1 implies $\sum_{i\in S} n_i = N-M \geq \sum_{i'\in S'} n_{i'} = n_1+M$: i.e., $N-2M-n_1\geq 0$.[10] Therefore, from expression (14), we know that (13) holds true for $k<\frac{K+1}{2}$. Moreover, it holds true with strict inequality for k = 1. It is because M = 0 and $N-2M-n_1$ > 0 when k = 1. Therefore, (Sh_0 + Sh_1) − Sh_{0+1} in (12) is greater (less) than 0 when $\pi(n)$ is convex (concave). This proves Theorem 1. Q.E.D.

Assumption 1 is not necessary for Theorem 1; it is just a sufficient condition. In fact, it is not necessary at all when K = 2 and 3.

Theorem 2. For K = 2 and 3, even without Assumption 1, we have:

$$Sh_0 + Sh_1 > Sh_{0+1} \quad \Leftrightarrow \quad \pi''(n)>0.$$

Proof. When K = 2 and 3, only $k<\frac{K+1}{2}$ is k = 1. From (12), we have

[10] Since n_1 is cancelled out in $N-2M-n_1$, the proof holds true for any n_1. That is, as far as the size of n_1 is concerned, Assumption 1 should not be restrictive. However, Theorem 1 is not just for merging between 0 and 1. So, we state Assumption 1 in general terms.

$$Sh_0 + Sh_1 - Sh_{0+1} = \frac{1}{6}[\{\pi(0) + \pi(n_1 + n_2)\} - \{\pi(n_1) + \pi(n_2)\}] \qquad \text{for } K = 2;$$

$$Sh_0 + Sh_1 - Sh_{0+1} = \frac{1}{6}[\{\pi(0) + \pi(n_1 + n_2 + n_3)\} - \{\pi(n_1) + \pi(n_2 + n_3)\}] \qquad \text{for } K = 3.$$

From (14), we know that they are greater than zero if $\pi''(n) > 0$, and vice versa. It does not depend upon Assumption 1. Q.E.D.

Even for large $K \geq 4$, Assumption 1 may not be necessary for Theorem 1. For example, consider the case of $K = 4$. Then, (12) implies:

$$\begin{aligned} Sh_0 + Sh_1 - Sh_{0+1} = &\frac{3}{20}[\{\pi(0) + \pi(n_1 + n_2 + n_3 + n_4)\} - \{\pi(n_1) + \pi(n_2 + n_3 + n_4)\}] \\ &+ \frac{1}{60}[\{\pi(n_2) + \pi(n_1 + n_3 + n_4)\} - \{\pi(n_1 + n_2) + \pi(n_3 + n_4)\}] \\ &+ \frac{1}{60}[\{\pi(n_3) + \pi(n_1 + n_2 + n_4)\} - \{\pi(n_1 + n_3) + \pi(n_2 + n_4)\}] \qquad \text{-- (15)} \\ &+ \frac{1}{60}[\{\pi(n_4) + \pi(n_1 + n_2 + n_3)\} - \{\pi(n_1 + n_4) + \pi(n_2 + n_3)\}] \ . \end{aligned}$$

Suppose $n_4 > n_2 + n_3$, which violates Assumption 1. If $\pi''(n) > 0$, then the first three terms in the above are positive, while the last term is negative. If the positive terms are dominant, we still have $Sh_0 + Sh_1 > Sh_{0+1}$.

To show the plausibility of the dominance of positive terms in (15),[11] specify $\pi(n) = n^{\alpha}$ ($\alpha > 0$). Consider the case with $n_1 = n_2 = n_3 = 1$ and $n_4 = 3$; viz., 4 is a dominant player. Then, we have:

$$60(Sh_0 + Sh_1 - Sh_{0+1}) \equiv \Delta(\alpha) = 9 \cdot 6^{\alpha} + 2 \cdot 3^{\alpha} - 7 \cdot 5^{\alpha} - 3 \cdot 4^{\alpha} - 3 \cdot 2^{\alpha} - 7 .$$

Figure 1 shows that $\Delta(\alpha) > 0 \Leftrightarrow \alpha > 1$. That is, with the specific example, it holds true that $Sh_0 + Sh_1 > Sh_{0+1} \Leftrightarrow \pi''(n) > 0$.

Insert <Figure 1> here.

This example is to emphasize the fact that Assumption 1 is just a sufficient condition for Theorem, and the plausibility that Theorem 1 holds true in many circumstances where Assumption 1 is violated.

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[11] The dominance of positive terms seems plausible. But, we cannot prove it algebraically.

3. Horizontal Integration

Consider whether two downstream operators, say 1 and 2, can benefit from horizontal integration. Let 1+2 denote the integrated firm. Then, we can modify (7) to obtain:

$$\begin{aligned}
Sh_{1+2} = \; & \frac{(K-2)!}{K!}\pi(n_1+n_2) \\
& + \sum_{i\in I/\{0,1,2\}} \frac{2!(K-3)!}{(K+1)!}\{\pi(n_1+n_2+n_i)-\pi(n_i)\} \\
& + \ldots\ldots\ldots\ldots \\
& + \sum_{\substack{i_1<\ldots<i_k \\ \{i_1,\ldots,i_k\}\subseteq I/\{0,1,2\}}} \frac{(k+1)!(K-k-2)!}{K!}\{\pi(n_1+n_2+n_{i_1}+\ldots.+n_{i_k})-\pi(n_{i_1}+\ldots.+n_{i_k})\}^{12} \\
& + \ldots\ldots\ldots\ldots \qquad\qquad \text{--- (16)} \\
& + \sum_{i\in I/\{0,1,2\}} \frac{(K-2)!}{K!}\{\pi(N-n_i)-\pi(N-n_1-n_2-n_i)\} \\
& + \frac{(K-1)!}{K!}\{\pi(N)-\pi(N-n_1-n_2)\}
\end{aligned}$$

The two operators can benefit from horizontal integration if and only if Sh_{1+2} > Sh_1+Sh_2.

Theorem 3. For K = 2 and 3, we have:

$$Sh_1 + Sh_2 > Sh_{1+2} \quad \Leftrightarrow \quad \pi''(n) > 0 .$$

Proof. For K =2, from (7) and (16), we have:

$$Sh_1 + Sh_2 - Sh_{1+2} = \frac{1}{6}[\{\pi(0)+\pi(n_1+n_2)\}-\{\pi(n_1)+\pi(n_2)\}],$$

which is greater than zero if $\pi''(n) > 0$, and vice versa, as explained before. For K =3, we have:

$$\begin{aligned}
Sh_1 + Sh_2 - Sh_{1+2} &= \frac{1}{6}[\{\pi(n_3)+\pi(n_1+n_2+n_3)\}-\{\pi(n_1+n_3)+\pi(n_2+n_3)\}] \\
&= \frac{1}{6}\int_0^{n_2}\int_0^{n_1}\pi''(n_3+x+y)dxdy,
\end{aligned}$$

[12] To interpret this in terms of (5), consider i = 1+2 and $S = \{0,1+2,i_1,\ldots\ldots,i_k\}$.

which implies what we want to show. Q.E.D.

Theorem 2 and 3 show that we have the same necessary and sufficient condition for two operators to benefit from a merger, whether it is vertical or horizontal, for K = 2 and 3. How about K > 3? Can we show the result similar to Theorem 1? The similarity does not carry to the case with K > 3 though. We will show that it is impossible to establish the result comparable to Theorem 1 for horizontal integration by constructing a counterexample. Suppose all downstream firms have the same size with $n_i = n \ (i = 1,....,K)$, and hence Assumption 1 is satisfied. Notice that in this symmetric case, the general (k+1)-th term in (7) is:

$$\sum_{\substack{i_1<...<i_k \\ \{i_1,...,i_k\}\subseteq I/\{0,1\}}} \frac{(k+1)!(K-k-1)!}{(K+1)!}\{\pi(n_1+n_{i_1}+....+n_{i_k})-\pi(n_{i_1}+....+n_{i_k})\}$$

$$= \ {}_{K-1}C_k \ \frac{(k+1)!(K-k-1)!}{(K+1)!}\{\pi((k+1)n)-\pi(kn)\}$$

$$= \frac{k+1}{(K+1)K}\{\pi((k+1)n)-\pi(kn)\}, \qquad \text{-- (17)}$$

where ${}_{K-1}C_k = \frac{(K-1)!}{(K-k-1)!k!}$ represents the number of picking up k different numbers from the set of $I/\{0,1\}$. Plugging (17) into (7), and using symmetry, we have:

$$Sh_1 + Sh_2 = \frac{2}{(K+1)K}\sum_{k=0}^{K-1}\{(k+1)\pi((k+1)n)-(k+1)\pi(kn)\}$$

$$= \frac{2}{(K+1)K}\left\{\sum_{k=1}^{K} k\pi(kn) - \sum_{k=0}^{K-1}(k+1)\pi(kn)\right\}$$

$$= \frac{2}{(K+1)K}\left\{K\pi(Kn) - \sum_{k=1}^{K-1}\pi(kn)\right\} \qquad \text{-- (18)}$$

Similarly,

$$Sh_{1+2} = \frac{1}{K(K-1)}\sum_{k=0}^{K-2}\{(k+1)\pi((k+2)n)-(k+1)\pi(kn)\}$$

$$= \frac{1}{K(K-1)}\left\{\sum_{k=2}^{K}(k-1)\pi(kn) - \sum_{k=0}^{K-2}(k+1)\pi(kn)\right\}$$

$$= \frac{1}{K(K-1)}\left\{(K-1)\pi(Kn) + (K-2)\pi((K-1)n) - 2\sum_{k=1}^{K-2}\pi(kn)\right\}$$

$$= \frac{1}{K(K-1)}\left\{(K-1)\pi(Kn) + K\pi((K-1)n) - 2\sum_{k=1}^{K-1}\pi(kn)\right\} \qquad \text{-- (19)}$$

From (18) and (19), we have:

$$(Sh_1 + Sh_2) - Sh_{1+2} \; = \; \frac{1}{(K+1)K(K-1)}\left\{(K-1)^2\pi(Kn) - K(K+1)\pi((K-1)n) + 4\sum_{k=1}^{K-1}\pi(kn)\right\}. \qquad \text{-- (20)}$$

To find a counterexample which shows that Theorem 3 does not extend to $K > 3$, consider $\pi(n) = n^{\alpha}$. Denote $(Sh_1 + Sh_2) - Sh_{1+2} \equiv \Delta(K,n,\alpha)$. Given $\pi(kn) = (kn)^0 = 1$ for any k and n, we have:

$$\Delta(K,n,0) \; = \; \frac{1}{(K+1)K(K-1)}\left\{(K-1)^2 - K(K+1) + 4(K-1)\right\}$$

$$= \; \frac{K-3}{(K+1)K(K-1)} \; > 0 \quad \text{for } K > 3.$$

On the other hand, with $\pi(kn) = (kn)^1 = kn$, we have:

$$\Delta(K,n,1) \; = \; \frac{1}{(K+1)K(K-1)}\left\{(K-1)^2 K - K(K+1)(K-1) + 4\sum_{k=1}^{K-1}k\right\}n \; = \; 0.$$

Since Δ is continuous in α, we know that there exists θ such that $0 < \theta < 1$ and $\Delta(K,n,\alpha) > 0$ if $0 < \alpha < \theta$ and $K > 3$. This shows the case in which $Sh_1 + Sh_2 > Sh_{1+2}$ even if $\pi''(n) < 0$.

Insert <Figure 2> here.

Figure 2 depicts $\Delta(K,n,\alpha)$ for $K = 4$ and $n = 1$, where

$$\Delta(4,1,\alpha) \; = \; \frac{1}{60}(9 \cdot 4^{\alpha} - 16 \cdot 3^{\alpha} + 4 \cdot 2^{\alpha} + 4).$$ [13]

The bottom line is, Theorem 3 does not extend to the cases with $K > 3$.

Reference

Owen, G. (1982), *Game Theory*, New York: Academic Press.

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[13] Incidentally, this example shows that $\Delta > 0$ when $\alpha > 0$; i.e., $Sh_1 + Sh_2 > Sh_{1+2}$ if $\pi''(n) > 0$, but not vice versa.