Indeterminacy and Endogenous Fluctuations with Arbitrarily Small Liquidity Constraint¤

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June 19, 2001

Abstract

In this paper we study the occurrence of local indeterminacy and endogenous ‡uctuations in a competitive in…nite horizon economy with partial cash-in-advance constraint on consumption expenditures. We observe that, in contrast to many one-sector models, indeterminacy and endogenous ‡uctuations are quite pervasive phenomena: they arise for arbitrarily small amplitudes of the liquidity constraint and for whatever speci…cation for the preferences and the technology. This result seems to question the robustness of the saddle path stability of the Ramsey-Cass-Koopmans model - on the other hand intensively exploited in the RBC literature - with respect to arbitrarily small imperfections in the …nancial market. Such …ndings are con…rmed in a context of sustained growth made possible by the presence of positive externalities in aggregate capital.

Keywords: cash-in-advance, indeterminacy, money demand, growth.
JEL Classi…cation: D90, E32, E41.

¤We are grateful to Guido Cazzavillan, Jean-Michel Grandmont for their very helpful comments and suggestions. Any remaining errors are our own.

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1 Introduction

In this paper we study the occurrence of indeterminacy and endogenous fluctuations in a one-sector infinite horizon economy with capital accumulation and liquidity constraint on consumption expenditures. Yet, unlikely in previous studies (Wilson, 1979; Stockman, 1981; Abel, 1985; Svensson, 1985; Lucas and Stokey, 1987; Coleman, 1987; Cooley and Hansen, 1989), we assume that only a share $q \in (0;1]$ of current consumption purchases must be financed out of money balances held at the beginning of the period.

We observe that, in contrast to many one-sectors models studied in the literature, indeterminacy and endogenous fluctuations are quite pervasive phenomena: they occur for liquidity constraints with arbitrarily small size, and independently upon the specification of the other economic fundamentals. As it seems widely recognized in the literature (Woodford, 1986b; Chiappori and Guesnerie, 1989; Grandmont et al., 1998) which extended the seminal intuition of Shell (1977) and Cass and Shell (1983), indeterminacy in turn opens the door for the existence of sunspot equilibria, i.e. equilibria in which the particular path undertaken by the economic system is correlated to the realizations of some extrinsically stochastic variable - a random variable not affecting fundamentals - and may easily exhibit erratic as well as persistent fluctuations, whose nature on the other hand remains strictly endogenous.

The fact that indeterminacy and endogenous fluctuations occur for arbitrarily small perturbations of financial nature should provide an argument against some criticisms focusing on the size of market imperfections and on the fundamental specifications exploited to generate such phenomena. For instance, the empirical relevance of indeterminacy has been often questioned on the basis that its occurrence relies in many cases either on implausibly high spillover effects in production (see, among the others, Benhabib and Farmer, 1994; Farmer and Guo, 1994), or strong income effects in consumption in response to changes in the interest rate (inter alia, Grandmont 1985; Azariadis and Guesnerie, 1986; Bloise et al., 2000) or, yet, either arbitrarily low elasticities of inputs substitution (e.g., Reichlin, 1986; Woodford, 1986a; Grandmont et al., 1998) or arbitrarily high (Cazzavillan et al., 1998). One interesting theoretical exception is the OLG model presented in Cazzavillan (2001) where indeterminacy and endogenous fluctuations emerge in correspondence to external effect in aggregate capital with arbitrarily small size. Despite the different nature of the two models and of the market imperfection accounted for, the transition from determinacy to indeterminacy in

Cazzavillan (2001) presents some interesting analogies with our own.

The assumption of a partial and variable liquidity constraint, which follows very closely the spirit of Grandmont and Younés (1972), can be easily justi…ed on empirical ground. To see this, it is su¢cient to observe that the amplitude q of the liquidity constraint can be viewed as a proxy of the inverse of the velocity of circulation of money. So, the intensive and almost exclusive assumption in the existing literature of $q = 1$ - i.e. of a constant and unitary velocity of circulation of money - lacks of any empirical support. By allowing q to vary we can, by contrast, account for di¤erent velocities of circulation of money and partially reconcile the theory with the data: the lower q the faster money balances must circulate in order to absorb a given value of consumption (so, indirectly, GDP), since any transaction requires only a limited amount of money. Yet, this is by no means the unique motivation of our assumption. The amplitude of the liquidity constraint can be indeed viewed also as a measure of the degree of imperfection prevailing in the …nancial market, once one observes that the share $1 ¡ q$ …xes the extent to which households have access to credit in order to …nance their expenditures within a given period. Daily experience suggests that the credit system, far from being frictionless and smooth, allows a single household to buy goods before factors income is payed only to a given extent, extent which in our framework is measured by $1 ¡ q$:

By relaxing continuously q from one (Clower constraint) to zero (Ramsey-Cass-Koopmans model), we analyze how the conditions for indeterminacy and endogenous ‡uctuations do vary. We observe that when the amplitude of liquidity constraint, q, is close to one, indeterminacy arises only when consumption is not very substitutable across periods. Conversely, as soon as q is relaxed, the set of utility functions generating indeterminacy becomes larger and larger; below a given threshold it includes the logarithmic case and below a lower one it coincides with the whole space of well behaved utility functions. The model, however, is not continuous at $q = 0$ since it loses one dimension and collapses into the familiar Ramsey-Cass-Koopmans one, which as it is well known, exhibits the saddle path stability. Questioning the robustness of the Ramsey-Cass-Koopmans model with respect to arbitrarily small imperfections in the …nancial market, is not a merely theoretical exercise, once one recognizes for instance that its saddle path stability is an essential feature in its RBC applications.

It is worth here noticing that indeterminacy comes about always through a ‡ip bifurcation, since there is an additional stable root which goes through

¡ 1. This feature, as it is shown, among the others, in Grandmont (1988), implies generically the existence of deterministic cycles close to the unit root …xed points, in correspondence to which all the relevant economic indicators ‡uctuate perpetually.

We …nd analogous results in terms of indeterminacy and endogenous ‡uctuations, also when the economy, in view of the presence of aggregate externalities in capital utilization, displays sustained growth. In this case, indeterminacy consists in a continuum of equilibrium trajectories for the growth rate of the economy, all of them approaching asymptotically the (unique) balanced one, where quantities grow at a common constant rate, while the real and nominal interest rate as well as in‡ation are constant.

We also perform some numerical simulations aiming at testing the sensitivity of the qualitative results obtained with respect the calibration of the structural parameters of the economy. Finally, we simulate some shocks on fundamentals as well on beliefs, and appraise how do they perturb the equilibrium path and how do they are progressively absorbed.

The remainder of the paper is organized as follows. In section 2 we study the benchmark model. In section 3 we extend the analysis to the long-run growth case. Section 4 is devoted to the theoretical aspects of the shocks we consider. In section 5 numerical simulations of the shocks are performed. Section 6 concludes the paper.

2 The model

The economy is populated by a continuum of identical long lived agents and identical …rms with sizes normalized to unity. Time is discrete. Agents are endowed with perfect foresight and maximize the discounted stream of utility functions $\sum_{t=0}^{1} \beta^t u(c_t)$; where $0 < \beta < 1$ is the discount factor and u the per-period utility of consumption which features the following standard properties.

Assumption 1. The per-period utility function $u : R_+ \to R$ is smooth, strictly increasing and strictly concave. Moreover $\lim_{c \to 0^+} u'(c) = +1$:

The boundary condition in Assumption 1 is to ensure an interior solution in agents' maximization problem. In each period $t \geq 0$, the representative household supplies inelastically one unit of labor and chooses how much to

consume, c_t, invest in capital, k_{t+1}; and in money balances, M_t, subject to the budget constraint

$$p_t c_t + p_t [k_{t+1} - (1 - \pm) k_t] + M_{t+1} = p_t r_t k_t + p_t w_t + M_t + ¿_t; \qquad (1)$$

where p is the price of the good, r the real rental price of capital, w the real wage, $\pm \in [0;1]$ the depreciation rate of capital and ¿ the nominal lump-sum transfers issued by the government. A share $q \in (0;1]$ of the purchases of the consumption good requires money balances accumulated in the previous period, i.e. agents are subject to the partial cash-in-advance constraint

$$q p_t c_t \cdot M_t: \qquad (2)$$

The amplitude q of constraint (2) can be viewed as a measure of the degree of imperfection in the …nancial market as well as the inverse of the velocity of circulation of money. Thus, economies featuring a q close to zero are characterized by a very high degree of …nancial intermediation and a very high velocity of circulation of money, whereas a relatively high q re‡ects some important friction or incompleteness in the capital market and a low velocity of circulation of money.

The FOC's of the household maximization problem, in addition to constraints (1) and (2), are

$$\begin{aligned} {}^{-t}u'(c_t) &= ¸_t + q¹_t; && (3)\\ ¸_t &= ¸_{t+1}(1 - \pm + r_{t+1}); && (4)\\ ¸_t p_t^{-1} &= p_{t+1}^{-1} ¸_{t+1} + p_{t+1}^{-1} ¹_{t+1}; && (5) \end{aligned}$$

where ¸ and ¹ are non-negative Lagrange multipliers associated to, respectively, budget constraint and cash-in-advance. Equations (3)-(5) are no-arbitrage conditions. In particular, according to a commonly shared view in monetary economics, (5) establishes that the real value of money at time t, $¸_t p_t^{-1}$, is equal to its (expected) value in the following period (re‡ecting the fact that money is a long-lived asset) plus the (expected) value of the implicit real dividends this asset will pay o¤. Whenever dividends are positive, money is not seen as a speculative bubble. Observe that $¸_t$ can still be interpreted as the marginal indirect utility of real income in period t, but it is not equalized by the individual to the marginal utility of consumption, since part of income cannot be transformed into consumption unless it is …rst used to purchase money. Yet, the marginal utility of consumption is equal

to $\lambda_t + \theta\tau_t$, the term $\theta\tau_t$ representing the value of dividends money pays o¤ and re‡ecting the fact that a share θ of consumption expenditures requires a previous re-investment in money balances.

Constraint (2) binds as long as money is dominated by capital in terms of returns, i.e. when the nominal interest factor $i_t \equiv (1 - \delta + r_t)\,\pi_t$ is greater than one, where $\pi_t \equiv p_t/p_{t-1}$ is the in‡ation factor between periods $t-1$ and t. Under this condition, the Euler equation for the consumer has the form

$$u'(c_t) = \beta u'(c_{t+1}) \frac{\pi_t \theta R_t + (1-\theta)}{\pi_{t+1}\theta + (1-\theta) R_{t+1}^{-1}}; \tag{6}$$

where $R \equiv 1 - \delta + r$ is the gross real interest rate. The arbitrage equation (6) is to be interpreted as follows. Suppose that one wishes to decrease one unit of consumption in period t: then he can dispose of an additional amount of good p_t. A share $1-\theta$ of such an amount was previously held in the form of physical equipment and a share θ in money balances. This implies that in period $t-1$, θp_t units of money can be used to buy a quantity $\theta p_t/p_{t-1}$ of capital, which in period t yields a gross rate of return R_t. The total return of this operation is thus $(\theta p_t/p_{t-1})\,R_t$. Henceforth, the diminution of one unit of consumption in period t makes available in the same period a total amount of resources equal to $\theta\pi_t R_t + 1 - \theta$. Before being convertible into consumption in period $t+1$, part of this amount must be converted into money (which yields the return π_{t+1}^{-1}), while part of it is invested in capital (with a return of R_{t+1}). Let us denote with $\hat{A}$ the share of $\theta\pi_t R_t + 1 - \theta$ to be converted in money. Since it must be $\hat{A}(\theta\pi_t R_t + 1 - \theta)\pi_{t+1}^{-1} = \theta c_{t+1}$ and $(1-\hat{A})(\theta\pi_t R_t + 1 - \theta)\,R_{t+1} = (1-\theta)\,c_{t+1}$, one has $\hat{A} = \theta / \left[\theta + (1-\theta) R_{t+1}^{-1}\pi_{t+1}^{-1}\right]$ and therefore the increase in consumption in period $t+1$ is $\left[\pi_{t+1}\theta + (1-\theta) R_{t+1}^{-1}\right]^{-1}$. In the special case $\theta = 0$, (6) collapses into the Euler equation of the traditional Ramsey-Cass-Koopmans model, $u'(c_t) = \beta u'(c_{t+1}) R_{t+1}$. On the other hand, if we set $\theta = 1$ (Clower constraint) equation (6) reduces to $u'(c_t) = \beta u'(c_{t+1}) R_t \pi_t/\pi_{t+1}$ as in Stockman (1981), Abel (1985) and Bloise et al. (2000).

Optimal plans for the single household must also satisfy the transversality condition

$$\lim_{t \to +\infty} \beta^t u'(c_t)\,[k_{t+1} + \pi_{t+1} m_{t+1}] = 0; \tag{7}$$

where $m_t \equiv M_t/p_t$ are the real balances held by the representative agent at the outset of period t. Firms produce the good by renting capital, k, and labor, l, according to a constant returns to scale production function $F(k; l)$

which, normalizing labor to unity, can be expressed in the intensive form $f(k)$ exhibiting the usual neoclassical features.

Assumption 2. The intensive production function $f : R_+ \to R$ is smooth, strictly increasing and strictly concave. Moreover $f(0) = 0$, $\lim_{k \to 0} f'(k) = +1$ and $\lim_{k \to +1} f'(k) = 0$.

Assumption 2 ensures on the one hand …rms maximization problem to be well de…ned and on the other one the existence of a unique stationary solution. Pro…t maximization of the …rms implies that the real interest rate and the real wage equalize, respectively, the marginal productivity of capital and the marginal productivity of labor:

$$r = f'(k); \tag{8}$$

$$w = f(k) - f'(k)k: \tag{9}$$

Government issues in each period lump-sum transfers ¿ of money balances at the constant rate $\mu - 1$, so that in period t the supply of money, M_t^s, satis…es $M_t^s = \mu^t M_0^s$; where M_0^s is the initial amount of nominal balances, and the nominal transfers satisfy $¿_t = (\mu - 1) M_t^s$:

Equilibrium in labor market is ensured by conditions (8) and (9). Money market equilibrium requires that all nominal balances are held, i.e. $\frac{1}{p_{t+1}} = \mu m_t / m_{t+1}$, and good market clears according to $k_{t+1} + c_t = g(k_t)$, where $g(k) \equiv f(k) + (1 - \delta)k$. Thereby, when constraint (2) binds, intertemporal competitive equilibria are completely described by the dynamic evolution of $(\frac{1}{p_t}; c_t; k_t)$.

De…nition 1 An intertemporal equilibrium with perfect foresight is a sequence $\{\frac{1}{p_t}; c_t; k_t\}_{t=0}^{1}$, $(\frac{1}{p_t}; c_t; k_t) \gg 0$ for $t = 0; 1; ...;$ satisfying for every $t \geq 0$ the following equations:

$$k_{t+1} = g(k_t) - c_t; \tag{10}$$

$$\frac{1}{p_{t+1}} c_{t+1} = \mu c_t; \tag{11}$$

$$-u'(c_{t+1}) \left[\frac{1}{p_{t+1}} q + (1 - q) g'(k_{t+1}) \right]^{-1} = u'(c_t) [\frac{1}{p_t} q g'(k_t) + (1 - q)]^{-1} \tag{12}$$

subject to the initial endowment of capital $k_0 > 0$ and the transversality condition (7).

In particular, identifying x with (π, c, k), we have

$$G_1(x_{t+1}) = G_0(x_t); \tag{13}$$

where G_0 and G_1 are defined, respectively, by the right-hand side and left-hand side of equations (10)-(12). Capital stock, consumption, inflation and real balances are constant at the steady state. Let us denote with a bar over a variable its steady state value. From equation (12), one obtains the stationary level $\bar{k}$ of capital by solving $f'(k) = \beta^{-1} - (1 - \delta)\gamma\mu$ which under Assumption 2 has a unique solution corresponding to the Modified Golden Rule. The stationary consumption $\bar{c}$ is given by output minus investment $f(\bar{k}) - \delta\bar{k}$ and, being constraint (2) binding, one has $\bar{m} = q\bar{c}$. The stationary inflation factor $\bar{\pi}$ is equal to μ: it follows that the steady state price level is constant when the rate of growth $\mu - 1$ of nominal money is zero. Conversely when $\mu > 1$, there is an inflationary stationary equilibrium and when $\mu < 1$ the value of money increases perpetually. The immediate implication of these results is that the amplitude q of the liquidity constraint does not affect the steady state value of any variable other than real balances, while the rate of money growth $\mu - 1$ produces the unique consequence of fixing the stationary inflation rate.[1]

One also readily verifies that constraint (2) binds at the steady state if and only if the discount factor β is lower than the factor μ of money growth. Under this condition, which we will assume to be satisfied throughout the section, system (13) is consistent with intertemporal equilibria remaining in a small neighborhood of the steady state.

2.1 Stability analysis

The steady state of system (13) is said to be locally indeterminate if there exists a continuum of sequences $\{x_t\}_{t=0}^{\infty}$ satisfying system (13) for all $t \geq 0$, subject to the initial stock of capital, k_0, all of which converging to the steady state $\bar{x}$. Following the usual procedure, the study of (local) indeterminacy requires an exam of the linear operator

$$A = [DG_1(\bar{x})]^{-1} DG_0(\bar{x});$$

[1] The effects of inflation on the steady state in the case $q = 1$ have been studied by Stockman (1981) and Abel (1985). It is shown that money is superneutral at the steady state as well as along the transitional dynamics.

which regulates the linear tangent motion to (13) near the steady state.[2] Since in system (13) k_0 is the unique pre-determined variable, indeterminacy occurs if and only if the dimension $n \cdot 3$ of the stable manifold of A is greater than or equal to two, i.e. if and only if A possesses at least two eigenvalues lying inside the unit circle. In this case, there will exist a $n - 1$ dimensional space in which it is possible to place the remaining initial conditions $¼_0$, c_0 in such a way that the equilibrium dynamics converges to the stationary solution and respects the transversality condition (7). In the opposite case, equilibrium of system (13) will be determinate, i.e. there will be only one pair $(¼_0; c_0)$ ensuring the convergence of the system towards its steady state.

To carry out the stability analysis, it is useful to de…ne here the elasticity of intertemporal substitution in consumption, $¾ = -u'(u''c)^{-1}$, the share of pro…t in total income, $s = f'k(f)^{-1}$, the elasticity of the real interest rate, $½ = -(f')^{-1}f''k$, all evaluated at the steady state equilibrium. Taking into account that $\dot{c}=c = \mu s^{-1} - ±$; we obtain the following expression[3] for A:

$$A = \begin{bmatrix} 1 - \frac{1}{\beta} - \frac{1}{\beta}\left(\frac{1}{\beta} - 1\right)\frac{1-\alpha}{\alpha} & \left(\frac{1}{\beta} - 1\right)\left(1 + \frac{1-\alpha}{\alpha}\right) \\ \frac{\delta}{\gamma}\left(\frac{1}{\beta} - 1\right)\frac{1-\alpha}{\alpha} & 1 - \frac{\delta}{\gamma}\left(1 + \frac{1-\alpha}{\alpha}\right) \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1/\beta \end{bmatrix};$$

where

$$\gamma = 1 - ¾ + \left[\frac{1}{\alpha}\frac{1-\alpha}{\alpha}\right]^{-1}. \tag{14}$$

Straightforward although tedious computations show that the characteristic polynomial of A is

$$P(») = »^3 - T»^2 + §» - D; \tag{15}$$

where

$$\begin{aligned} T &= \frac{1+\beta}{\beta} - \left[1 - \frac{1}{\alpha}\frac{1-\alpha}{\alpha}\left(\frac{\mu}{s} - ±\right)\right]; \\ § &= \frac{1}{\beta} - \left[\frac{1+\beta}{\beta} + \frac{1}{\alpha}\left(\frac{\mu}{s} - ±\right)\right]; \\ D &= -1/\beta. \end{aligned}$$

[2] $DG_i(x)$, with $i = 1; 2$, denotes the matrix of the derivatives of G_i with respect to x.

[3] Provided that $¾ \neq 1 + \frac{1-\alpha}{\alpha}$, DG_1 is a linear isomorphism and, so, the formula de…ning A above is meaningful. In addition, we assume that A has no eigenvalues on the unit circle.

The next proposition characterizes the modulus of the eigenvalues of A. It is shown that when the amplitude of liquidity constraint, q, is close to one, indeterminacy comes about only when consumption is not very substitutable across periods (low ¾). Conversely, as soon as q is relaxed, the range for ¾ generating indeterminacy becomes larger and larger; below a given q it includes the logarithmic case ¾ = 1 and below a lower one it includes the whole domain of de…nition of ¾: In other words, when q is low enough, indeterminacy is bound to always prevail and the saddle path stability is de…nitely ruled out.

Proposition 2 The linear operator A has a real eigenvalue $»_1 \in (0;1)$. Moreover, de…ning

$$q^{¤} = \frac{1}{1 + \frac{1}{=} + 4\frac{1+\bar{}}{=}\frac{1}{2½\mu}\left(\frac{\mu}{s} - 1\right)^{3}}; \tag{16}$$

$$q^{¤¤} = \frac{1}{1 + \frac{1}{=} + 2\frac{1+\bar{}}{=}\frac{1}{2½\mu}\left(\frac{\mu}{s} - 1\right)^{3}};$$

$$¾^{¤¤} = 1 + \frac{\bar{}}{1}\frac{1 - q}{q};$$

$$¾^{¤} = \frac{¾^{¤¤}}{2 + \frac{\bar{}2½\mu}{2(1+\bar{})}\left(\frac{\mu}{s} - 1\right)^{3}\left(1 - \frac{\bar{}}{1}\frac{1-q}{q}\right)^{3}};$$

we have:

(i) $0 < q < q^{¤}$. Then there is a root, say $»_2$; belonging to $(-1;0)$ and the steady state is locally indeterminate. In addition, when $¾ < ¾^{¤¤}$, the third root $»_3$ belongs to $(1;+1)$ and when $¾ > ¾^{¤¤}$it belongs to $(-1;-1)$:

(ii) $q^{¤} < q \cdot q^{¤¤}$: When $¾ < ¾^{¤¤}$, then there is a root, say $»_2$; belonging to $(-1;0)$ and the third root $»_3$ is greater than one. The steady state is thus locally indeterminate. When $¾^{¤¤} < ¾ < ¾^{¤}$, then there is a root, say $»_2$; belonging to $(-1;0)$ and the third root $»_3$ is lower than -1: The steady state is thus locally indeterminate. When $¾ > ¾^{¤}$, then there are two roots with modulus greater than one, hence the steady state is locally determinate. When ¾ goes through $¾^{¤}$, the steady state undergoes a ‡ip bifurcation.

(iii) $q^{¤¤} \cdot q \cdot 1$: When $¾ < ¾^{¤}$, then there is a root, say $»_2$; belonging to $(-1;0)$ and the third root $»_3$ belongs to $(1;+1)$: The steady state is thus locally indeterminate. When $¾^{¤} < ¾ < ¾^{¤¤}$, then there is a root, say $»_2$; belonging to $(-1;-1)$ and the third root $»_3$ belongs to $(1;+1)$: The steady

state is thus locally determinate. When $\gamma > \gamma^{\circ\circ}$, then there are two roots with modulus greater than one, hence the steady state is locally determinate. When γ goes through γ°, the steady state undergoes a flip bifurcation.

Proof. The eigenvalues of A correspond to the roots of the characteristic polynomial (15). Performing simple computations we obtain

$$P(-1) = \beta\left[\frac{1+\theta}{\theta} + \frac{1}{2}\left(\frac{\mu}{s} - \varepsilon\right)\left(1 - \frac{1}{\eta}\frac{1-q}{q}\right) - \frac{2}{\gamma}\frac{1+\theta}{\theta}\left(1 + \frac{1}{\eta}\frac{1-q}{q}\right)\right];$$

$$P(0) = \beta/\theta;$$

$$P(1) = -\beta\frac{1}{2}\left(\frac{\mu}{s} - \varepsilon\right)\left(1 + \frac{1}{\eta}\frac{1-q}{q}\right).$$

Observe that $\lim_{\lambda \to +1} P(\lambda) = +1$, $\lim_{\lambda \to -1} P(\lambda) = -1$ and that the polynomial is a continuous function and its domain is connected. One can easily verify that one always has $\text{sign}P(1) = -\text{sign}P(0)$: This implies that there is always a real root, say λ_1, in $(0;1)$: At the same time $\text{sign}P(-1) = -\text{sign}P(0)$ if and only if either

$$0 < q < q^{\circ} \tag{17}$$

or, for $q^{\circ} < q \leq 1$,

$$\gamma < \gamma^{\circ}: \tag{18}$$

Notice that $P(0) > 0$ if and only if $\beta > 0$ i.e. if and only if

$$\gamma < \gamma^{\circ\circ}: \tag{19}$$

It is easy to verify that $q^{\circ} < q^{\circ\circ}$ and that $\gamma^{\circ} > \gamma^{\circ\circ}$ if and only if[4] $q^{\circ} < q < q^{\circ\circ}$:

According to the amplitude of q, three main regimes are possible.

(i) $0 < q < q^{\circ}$. Then $\text{sign}P(-1) = -\text{sign}P(0)$ for whatever γ: It follows that there is a root, say λ_2; belonging to $(-1;0)$ and the steady state is locally indeterminate. Moreover, if $\gamma < \gamma^{\circ\circ}$, then $P(0) > 0$. Then the third root λ_3 belongs to $(1;+1)$: When $\gamma > \gamma^{\circ\circ}$, then $P(0) < 0$. It follows that the third root λ_3 belongs to $(-1;-1)$:

(ii) $q^{\circ} < q \leq q^{\circ\circ}$: Then $\gamma^{\circ} > \gamma^{\circ\circ}$ and three con…gurations are possible. If $\gamma < \gamma^{\circ\circ}$, then $\text{sign}P(-1) = -\text{sign}P(0)$ and $P(0) > 0$: It follows that there

[4]Notice that for $q = q^{\circ\circ}$ one has $\gamma^{\circ} = \gamma^{\circ\circ}$:

is a root, say θ_2; belonging to $(-1;0)$ and that the third root θ_3 is greater than one. The steady state is thus locally indeterminate. If $\gamma^{\circ\circ} < \gamma < \gamma^{\circ}$, then $signP(-1) = -signP(0)$ and $P(0) < 0$: It follows that there is a root, say θ_2; belonging to $(-1;0)$ and that the third root $\theta_3 < -1$: The steady state is thus locally indeterminate. When $\gamma > \gamma^{\circ}$, then $signP(-1) = signP(0)$, $P(0) < 0$ and $D = -P(0) > 1$. By observing that the determinant corresponds to the product of the eigenvalues one may immediately draw the conclusion that there exists at least one root with modulus greater than one. If such a root is real and positive, by the continuity of the polynomial, one has $\theta_i > 1$ for $i = 2;3$. If it is real and negative, one has $\theta_i < -1$ for $i = 2;3$. If such a root is complex, there are two conjugate eigenvalues θ_2, θ_3 such that $|\theta_2| = |\theta_3| > 1$. In either cases, the steady state is locally determinate.

Finally it is immediate to verify that for $\gamma = \gamma^{\circ}$ one eigenvalue goes through -1 and a flip bifurcation generically occurs.

(iii) $q^{\circ\circ} \cdot q \cdot 1$: Then $\gamma^{\circ} < \gamma^{\circ\circ}$ and three con…gurations are possible. If $\gamma < \gamma^{\circ}$, then $signP(-1) = -signP(0)$ and $P(0) > 0$: It follows that there is a root, say θ_2; belonging to $(-1;0)$ and that the third root θ_3 belongs to $(1;+1)$: The steady state is thus locally indeterminate. If $\gamma^{\circ} < \gamma < \gamma^{\circ\circ}$, then $signP(-1) = signP(0)$ and $P(0) > 0$: It follows that there is a root, say θ_2; belonging to $(-1;-1)$ and that the third root θ_3 belongs to $(1;+1)$: The steady state is thus locally determinate. When $\gamma > \gamma^{\circ\circ}$ we get exactly the picture corresponding to $\gamma > \gamma^{\circ}$ in (ii) and in particular the steady state is locally determinate.

Again for $\gamma = \gamma^{\circ}$ one eigenvalue goes through -1 and a flip bifurcation generically occurs. ■

Proposition 2 states that when $0 < q < q^{\circ}$ indeterminacy comes about for all γ's and when $q^{\circ} < q \cdot 1$ for all $\gamma < \gamma^{\circ}$: Conversely, when these conditions are not met, the steady state equilibrium always presents the classical saddle-path stability and, so, there exists a unique convergent path. The elasticity γ re‡ects the degree of intertemporal substitution in consumption; therefore condition $\gamma < \gamma^{\circ}$ for indeterminacy requires moderate income e¤ects. Yet, γ° can be view as a decreasing function of q in $(q^{\circ};1)$: It follows that as soon as the amplitude q of the liquidity constraint is relaxed the interval of $\gamma^{0}s$ giving raise to indeterminacy widens continuously. Finally, when q becomes lower than q° (and γ° is negative) the steady state of system (13) becomes locally indeterminate for whatever elasticity of intertemporal substitution in consumption γ.

The result that indeterminacy occurs for whatever ¾ when 0 < q < q¤ is quite interesting if not even surprising: it seems to suggest that such a phenomenon - very far from being a kind of "exotic" or "pathological" feature in macroeconomic models - is very pervasive. It is su¢cient indeed to introduce an arbitrarily small imperfection of …nancial nature in the traditional Ramsey-Cass-Koopmans model in order to rule out the saddle path stability and to obtain multiple deterministic as well as stochastic equilibria. The fact that q¤ < 1 together with the expression of ¾¤ also explain the reason why in the case q = 1 indeterminacy arises only in correspondence to ¾'s su¢ciently low, namely lower than

$$\tfrac{3}{4}^{\text{¤}}(1) = 1 = \text{"}\ 2 + 2^{\text{¡}1}(1 + \text{¯})^{\text{¡}1-2}\tfrac{1}{2}\mu\ \tilde{A}\ \frac{\mu}{s}\ \text{¡}\ \pm\ !\#$$

as it is shown in Bloise et al. (2000).

Proposition 2 also shows that the change in stability which leads to indeterminacy occurs always through a ‡ip bifurcation: for a given q > q¤, when ¾ = ¾¤; there is one characteristic root that goes through ¡1. This implies (see, e.g., Grandmont, 1988) that when ¾ is arbitrarily close to ¾¤, there will generically emerge, according to the direction of the bifurcation, a stable or unstable two-periods cycle, in which in‡ation and quantities ‡uctuate perpetually. In an analogous way, …xing ¾, one can study the family of maps indexed by the amplitude of the liquidity constraint q and verify that when q goes trough q¤, where q¤ is such that ¾ = ¾¤, the steady state still undergoes a ‡ip bifurcation. Let us stress that system (13) is not continuous at q = 0, since it loses one dimension and collapses into the standard Ramsey-Cass-Koopmans model which, as it is well known, is bi-dimensional and its unique steady state always determinate

Fig. 1 shows the local dynamics generated by the combinations of ¾ and q: In particular, the shaded area corresponds to the indeterminacy region, and its boundary de…nes the values q¤ and ¾¤ undergoing ‡ip bifurcations. We have assumed the period to be short, of about one month and a half (see Jones and Manuelli, 1993). As a consequence, we focus on the following calibration for the structural parameters: ¯ = $(0{:}98)^{1=8}$, ± = 1 ¡ $(0{:}9)^{1=8}$, ¹ = $(1{:}02)^{1=8}$, s = 0:33333: In addition, we retain a Cobb-Douglas production function, and

so ½ = 1 ¡ s = 0:66667:

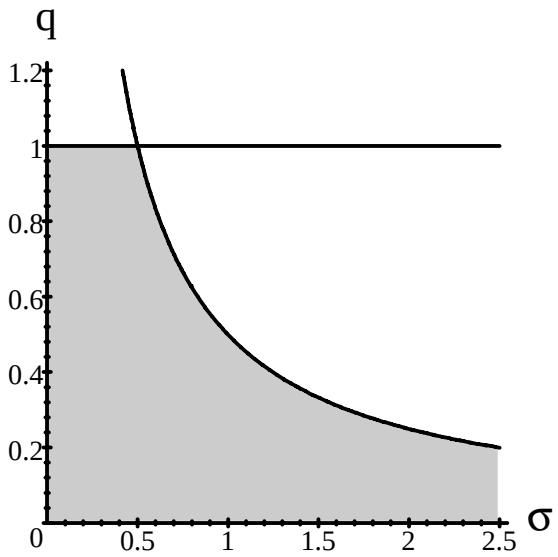

Figure 1: Indeterminacy region.

As it emerges in Fig. 1, for ¾ < 1=2, indeterminacy occurs for all q · 1. When we increase ¾ from 1=2, the upper bound of the q's generating indeterminacy decreases. It is in particular about 0:5 for ¾ = 1 (logarithmic utility) and then converges to a some value close to 0:2 when ¾ tends to in…nite.

In the next section we extend the analysis of the e¤ects of the cash-in-advance constraint to the case in which, in view of the presence of externalities from physical capital, the economy exhibits long-run growth. We show that in close analogy to the case analyzed in this section, multiple equilibria emerge for arbitrarily small amplitudes of the liquidity constraint and that changes in stability occur through a ‡ip bifurcation.

3 Liquidity constraint and long-run growth

One popular speci…cation for the technology which allows for sustained growth is that of Romer (1986) which assumes an externality from physical capital and is related to some of the previous work on learning-by-doing (Arrow, 1962; Shell, 1967). Following this approach, we now assume that each …rm produces the good according to the production function $A\hat{k}^{1-s}k^{s}l^{1-s}$, where $\hat{k}$

is the average stock of capital available in the economy, $s \in (0,1)$ the share of profit in total income and $A > 0$ a productivity parameter. So $\hat{k}^{1-s}$ is the externality contribution to production. Since firms are identical at equilibrium one has $\hat{k} = k$. Normalizing to one the labor supply, aggregate production is given by Ak, while the return on investment r is constant and equal to sA. Firms ignore indeed the impact of their own investment decisions on the productivity of other firms through their effect on $\hat{k}^{1-s}$. In order to obtain a balanced consumption growth rate, we chose preferences to be of the CES form

$$u(c) = \begin{cases} \left(c^{1-1/\gamma} - 1\right)/(1-1/\gamma) & \text{if } \gamma > 0,\ \gamma \neq 1; \\ \ln c & \text{if } \gamma = 1; \end{cases} \tag{20}$$

where, as usual, γ is the elasticity of intertemporal substitution in consumption. Assuming constraint (2) to be binding, setting $z_t \equiv c_t/k_t$ the consumption-capital ratio and $R \equiv 1 - \delta + sA$ the (constant) real gross interest factor, and keeping in mind the specification in (20) for the per-period utility function, we can exploit equations (10)-(12) in order to describe, after opportune rearrangements, intertemporal equilibria in terms of sequences of (φ, z).

Definition 3 An intertemporal equilibrium with perfect foresight is a sequence $\{\varphi_t, z_t\}_{t=0}^{1}$, $(\varphi_t, z_t) \gg 0$ for $t = 0, 1, \ldots$, satisfying for every $t \geq 0$ equations

$$1 + \frac{1}{\varphi_{t+1} z_{t+1}} = \frac{1 + A - \delta}{z_t}; \tag{21}$$

$$\left(\frac{1}{\varphi_{t+1}}\right)^{1/\gamma} (1 - q + qR\varphi_{t+1}) = \beta R (1 - q + qR\varphi_t) \tag{22}$$

and such that the level variables φ, c, k, m satisfy the transversality condition (7).

In particular, identifying x with (φ, z), we have

$$H_1(x_{t+1}) = H_0(x_t); \tag{23}$$

where H_0 and H_1 are defined, respectively, by the right-hand side and left-hand side of equations (21) and (22). It is easy to verify that system (23) has a unique stationary solution given by $\bar{\varphi} = 1/(\beta R)^{\gamma}$ and $\bar{z} = 1 + A - \delta - (\beta R)^{\gamma}$.

In correspondence to γ_4 and $\hat{z}$, one has that c; k and m growth at the balanced rate

$$\varrho = (\bar{R})^{\frac{3}{4}} - 1:$$

Therefore, strictly positive growth requires $\bar{R} > 1$. Under the domain of this inequality, the transversality conditions (7) is satis…ed along the balanced growth path if and only if the degree of intertemporal substitution in consumption is not very high, i.e. if and only if

$$\frac{3}{4} < \ln R / \ln(\bar{R}): \tag{24}$$

Notice that condition (24) ensures also $\hat{z} > 0$. At the same time the liquidity constraint (2) is binding in a small neighborhood of the steady state of system (23) if and only if

$$\frac{3}{4} < \ln(\iota R) / \ln(\bar{R}): \tag{25}$$

Conditions (24) and (25) are not independent: actually, when $\iota > 1$ (24) implies (25) and conversely (25) implies (24) when $\iota < 1$. Finally, in the special case $\iota = 1$ the two conditions merge together.

3.1 Local dynamics

The dynamics of system (23) around its stationary solution is topologically equivalent to that induced by the linear operator

$$B = [DH_1(\hat{x})]^{-1} DH_0(\hat{x}):$$

Neither variables γ_4 nor z in system (23) are pre-determined. The stationary solution of (23) will then be locally determinate if and only if it is asymptotically unstable, i.e. a source, condition requiring both eigenvalues of B to lie outside the unit disk. Under this con…guration the unique equilibrium path for the associated level variables corresponds to the balanced growth one. On the other hand, the steady state of (23) is indeterminate if and only if the dimension $n \cdot 2$ of the stable manifold of B is greater or equal to one, i.e. if and only if there is at least one eigenvalue belonging to the interior of the unit disk. In this case there will be in fact a n-dimensional non-linear manifold in which it is possible to place the initial condition $(\gamma_{40}; z_0)$: Performing some computations, we obtain that B coincides with the following

triangular matrix:[5]

$$B = \begin{bmatrix} \dfrac{1}{1 - \frac{1}{\gamma}\left[1 + \frac{1-q}{q}\frac{(\bar{R})^{\gamma}}{{}^{1}R}\right]} & 0 \\ -\dfrac{\frac{1}{\gamma}-1}{1 - \frac{1}{\gamma}\left[1 + \frac{1-q}{q}\frac{(\bar{R})^{\gamma}}{{}^{1}R}\right]} & \dfrac{1+A-\pm}{(\bar{R})^{\gamma}} \end{bmatrix}.$$

One immediately verify that the two eigenvalues of B are

$$\lambda_1 = \frac{1}{1 - \frac{1}{\gamma}\left[1 + \frac{1-q}{q}\frac{(\bar{R})^{\gamma}}{{}^{1}R}\right]};$$

$$\lambda_2 = \frac{1 + A - \pm}{(\bar{R})^{\gamma}}.$$

In view of inequality (24), λ_2 is bound to be always greater than one. By contrast, the stability of λ_1 depends crucially upon the magnitudes of γ and q. Indeed λ_1 is greater than one when its denominator is positive, i.e. when

$$\gamma > 1 + \frac{1-q}{q}\frac{(\bar{R})^{\gamma}}{{}^{1}R}; \tag{26}$$

and is negative otherwise. But what is more instructive for the purpose of our analysis is that λ_1 belongs to $(-1;0)$ if and only if

$$\gamma < \frac{1}{2}\left[1 + \frac{1-q}{q}\frac{(\bar{R})^{\gamma}}{{}^{1}R}\right]. \tag{27}$$

All this proves the following proposition which fully characterizes the local stability of the stationary solution of system (23).

Proposition 4 The eigenvalue λ_2 of B is real and greater than one. In addition:

(i) If inequality (26) does hold, then the (real) eigenvalue λ_1 is greater than one. The steady state of system (23) is a source and therefore locally determinate;

(ii) If inequality (27) does hold, then λ_1 belongs to $(-1;0)$: The steady state of system (23) is a saddle and therefore locally indeterminate;

(iii) If neither inequality (27) nor inequality do not hold, then $\lambda_1 < -1$. The steady state of system (23) is a source and therefore locally determinate.

[5] The matrix B is de…ned provided $\gamma \neq 1 + \frac{1-q}{q}\frac{(\bar{R})^{\gamma}}{{}^{1}R}$: We assume in addition that B has no eigenvalues on the unit circle.

Replacing the inequality sign in (27) with an equality sign, we can implicitly de…ne, when it exists, a critical elasticity in intertemporal substitution in consumption as a function of q, $\sigma^{\alpha 0}(q)$, compatible with conditions (24) and (25) and such that indeterminacy occurs if and only if $\sigma < \sigma^{\alpha 0}(q)$. If such $\sigma^{\alpha 0}(q)$ did not exist, indeterminacy would be bound to prevail under whatever σ since inequality (27) would be always satis…ed. It is easy to prove that $\sigma^{\alpha 0}(q)$ is de…ned for a whole range of high q's, that is decreasing in q and that …nally, for q low enough, it is no more de…ned. In the particular case in which $q = 1$; condition (27) reduces to $\sigma < 1=2$ as in Bloise et al. (2000). Analogously as in the model introduced in Section 2, there will then exist a critical amplitude of the liquidity constraint $q^{\alpha 0}$ such that for all q in $(0; q^{\alpha 0})$ indeterminacy arises for all σ satisfying conditions (24) and (25). For $\sigma = \sigma^{\alpha 0}(q)$ one eigenvalue goes through $¡ 1$ and undergoes a ‡ip bifurcation: accordingly, a two-period cycle will emerge in correspondence to which in‡ation and consumption to capital ratio ‡uctuate forever. Let us also observe that inequality (27) is perfectly consistent with its analogous in (18): indeed, the elasticity of the interest rate comprehensive of the externalities, under the Cobb-Douglas speci…cation of the technology here assumed, is equal to zero and so inequality (18) reduces to (27).

Condition (27) can be also be inverted and rewritten as

$$q < \frac{1}{1 + \frac{1-R}{(\bar{} R)^{\sigma}}(2\sigma ¡ 1)} \acute{} q^{\alpha 0}(\sigma); \tag{28}$$

where $q^{\alpha 0}(\sigma)$ is the bifurcation value of the amplitude of the liquidity constraint. One immediately veri…es that $q^{\alpha 0}(\sigma)$ is decreasing for σ satisfying conditions (24) and (25). In view of these considerations, we can immediately draw the conclusion that even when one accounts for exogenous growth, indeterminacy is more likely to occur when the amplitude of the liquidity constraint and the elasticity of intertemporal substitution in consumption are both relatively mild. When $\sigma = 1$ condition (28) reduces to $q < \bar{}=(\bar{} + 1)$; which as one could verify by a simple inspection of equation (16), is the same holding in the model without externalities.

Under the previous calibration and s = 1=3 and A = 0:055385; the indeterminacy region correspond to the shaded area depicted in Fig. 2:

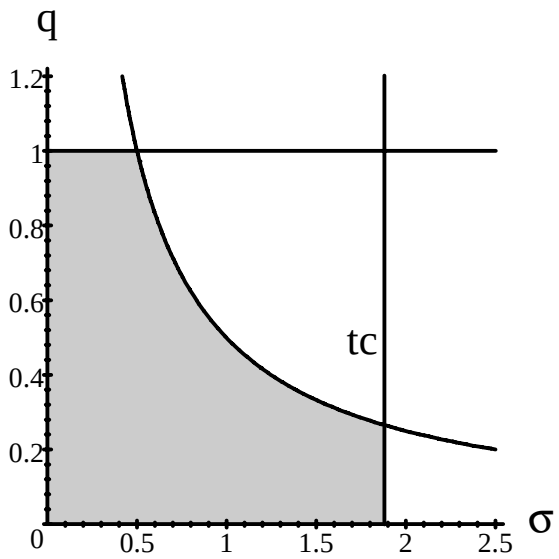

Figure 2: Indeterminacy region.

One immediately sees that for q close to one indeterminacy occurs for ¾ low enough, namely close to 0:5. When q is set lower, the range for ¾ increases. As an example, for q equal to 0:27 indeterminacy arises for all ¾ < 1:8885: Notice that ¾ can not be greater than 1:8885 otherwise the tranversality condition (24) would be violated (see the vertical line tc).

4 Shocks and propagation mechanisms

In the past the occurrence of indeterminacy has been considered to be a theoretical weakness, while a recent literature admits that the theory of indeterminacy can partially explain economic ‡uctuations. The traditional approach highlights the existence of shocks on the fundamentals such as technology and tastes, and simulates the propagation mechanism of these shocks. Indeterminacy may provide a rich source of propagation dynamics to an equilibrium model and sunspot may provide an alternative shock to technology or taste changes.

In the following we focus on both the classes of shocks: on beliefs to generate endogenous real business cycles, on fundamentals to obtain exogenous

business cycles. By simplicity we shall take into account a temporary shock on the beliefs about the in‡ation factor ¼; and a permanent shock on the technology parameter A:

4.1 Exogenous growth

Dynamic system (13) is three-dimensional with two non-predetermined variables ¼ and c: We have seen that indeterminacy arises when the stable manifold is two-dimensional, while equilibrium is determinate if and only if the stable manifold is one-dimensional. In what follows we shall consider at …rst the indeterminacy case which allows shocks on the beliefs and fundamentals. For simplicity we shall study only a …rst kind temporary shock on in‡ation forecasting. Second, we shall assume equilibrium determinacy and simulate a permanent technological shock.

Dynamics (13) are linearized around the steady state.

$$(x_t - \bar{x}) = A^t (x_0 - \bar{x}) = V \Box^t V^{-1} (x_0 - \bar{x});$$

where $x_t \equiv (\pi_t; c_t; k_t)^T$; $\Box$ is the diagonal matrix of possibly complex eigenvalues and V the transformation matrix of possibly complex eigenvectors[6].

Now we follow Farmer and Guo (1994) and Benhabib and Farmer (1999). We refer to linearized dynamics and we assume that the realization of the vector at time t depends on the expectation formed at time t of the vector at time $t + 1$:

$$\begin{bmatrix} \pi_t - \bar{\pi} \\ c_t - \bar{c} \\ k_t - \bar{k} \end{bmatrix} = M E_t \begin{bmatrix} \pi_{t+1} - \bar{\pi} \\ c_{t+1} - \bar{c} \\ k_{t+1} - \bar{k} \end{bmatrix}; \tag{29}$$

where M is the matrix describing the dynamics in forward looking and E is the usual expectation operator. If M is invertible equation (29) is reset as follows:

$$\begin{bmatrix} \pi_{t+1} - \bar{\pi} \\ c_{t+1} - \bar{c} \\ k_{t+1} - \bar{k} \end{bmatrix} = A \begin{bmatrix} \pi_t - \bar{\pi} \\ c_t - \bar{c} \\ k_t - \bar{k} \end{bmatrix} + \begin{bmatrix} \varepsilon_{\pi;t+1} \\ \varepsilon_{c;t+1} \\ \varepsilon_{k;t+1} \end{bmatrix};$$

[6] We deal for simplicity with a possibly complex diagonal $\Box$ instead of a real Jordan canonical form.

where $A = M^{-1}$ is the Jacobian matrix (37) and

$$\begin{bmatrix} \varepsilon_{\pi;t+1} \\ \varepsilon_{c;t+1} \\ \varepsilon_{k;t+1} \end{bmatrix} \equiv \begin{bmatrix} \pi_{t+1} - E_t \pi_{t+1} \\ c_{t+1} - E_t c_{t+1} \\ k_{t+1} - E_t k_{t+1} \end{bmatrix}.$$

The deterministic system has been augmented by taking into account the vector of forecasting errors at time $t+1$.

4.1.1 Shocks on the beliefs

In our dynamic system there is one pre-determined variable. Let the economy be at the steady state $(\bar{\pi}, \bar{c}, \bar{k})$ before the shock. To make things easy we assume that $E_t \pi_{t+1} = \bar{\pi}$, $E_t c_{t+1} = \bar{c}$, $k_{t+1} = \bar{k}$; i.e. before the shock on beliefs agents expect the inflation factor and the consumption at time $t+1$ to get their long-run value.

To simplify the notation we set $t+1 = 0$. The shock occurs in period 0 on the non-predetermined variable π_0. The shock is temporary. The steady state is not affected by the shock because it depends only on the fundamentals. We still assume that the non-predetermined initial condition (π_0, c_0) lies in a sufficiently small neighborhood of $(\bar{\pi}, \bar{c})$. The capital stock k_0 is determined prior to the shock. Hence the shock vector on the beliefs becomes:

$$\begin{bmatrix} \varepsilon_{\pi;0} \\ \varepsilon_{c;0} \\ \varepsilon_{k;0} \end{bmatrix} = \begin{bmatrix} \pi_0 - \bar{\pi} \\ c_0 - \bar{c} \\ 0 \end{bmatrix}.$$

The stable manifold is two-dimensional by assumption of indeterminacy (there are two non-predetermined variables). The starting point of our linearized dynamics must belong to the intersection between the horizontal plane $\{k_0 = k\}$ and the tangent plane to the stable manifold in the steady state. The equation of the tangent plane is obtained by ruling out the explosive action of the unique unstable eigenvalue. More precisely we set:

$$\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}^t W (x_0 - x) = \begin{bmatrix} \lambda_1^t & 0 & 0 \\ 0 & \lambda_2^t & 0 \\ 0 & 0 & \lambda_3^t \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix} \begin{bmatrix} \pi_0 - \bar{\pi} \\ c_0 - \bar{c} \\ k_0 - \bar{k} \end{bmatrix} \quad (30)$$

(where $W \equiv V^{-1}$; λ_1 and λ_2 are the possibly complex stable eigenvalues, while λ_3 is the real unstable one) and we require the third component of the

vector $W(x_0 - \bar{x})$ relative to the explosive λ_3 in the product, to be zero. The stable manifold is approximated by

$$w_{31}(\pi_0 - \bar{\pi}) + w_{32}(c_0 - \bar{c}) + w_{33}(k_0 - \bar{k}) = 0. \tag{31}$$

The intersection with the plane of the predetermined variable $k_0 = \bar{k}$ defines the locus of rational initial conditions for the trajectory generated by the shock. Therefore[7]

$$\varepsilon_{c;0} = -\frac{w_{31}}{w_{32}}\varepsilon_{\pi;0}. \tag{32}$$

The temporary shock on the non-predetermined variable π_0 determines the shock on c_0 according to (32), and occurs between -1 and 0 after a sunspot signal. Agents coordinate on the new point $(\pi_0; c_0; k_0)$ which belongs to the stable manifold. This freedom of choice is compatible with a rational behavior because there is indeterminacy and each alternative equilibrium is compatible with the optimizing individual behavior.

Eventually dynamics are provided by

$$\begin{bmatrix} \pi_{t} - \bar{\pi} \\ c_{t} - \bar{c} \\ k_{t} - \bar{k} \end{bmatrix} = V \Lambda^{t} V^{-1} \begin{bmatrix} 1 \\ -w_{31}/w_{32} \\ 0 \end{bmatrix} \varepsilon_{\pi;0}, \tag{33}$$

where we stress the expectation error about inflation $\varepsilon_{\pi;0}$ as a shocks on the beliefs.

4.1.2 Shocks on the fundamentals

Each fundamental parameter of the model can be shocked. We consider a technological shock as it is usually done in the RBC literature. For the sake of simplicity we deal with a pure shock on this fundamental, without mixing it with any shock on the beliefs. To rule out this possibility, we treat the determinacy case. We still have two non-predetermined variables in a three-dimensional system. A one-dimensional stable manifold is required to observe determinacy. Therefore we assume two eigenvalues, say λ_2 and λ_3; to be unstable; λ_1 being stable. The production function is specified as a reduced form of a Cobb-Douglas: $f(k) = Ak^{s}$:

[7] The ratio w_{31}/w_{32} is set to be real, by possibly normalizing the eigenvector of the unstable eigenvalue by an opportune complex number.

As above we require the convergence from the vector $W(x_0 - \bar{x})$; where now the initial condition x_0 depends on A_0; the technological parameter before the permanent shock $A - A_0$; and the new steady state $\bar{x}$ depends on A: The explosive action of λ_2 and λ_3 is sterilized by setting as usual the second and third component of $W(x_0 - \bar{x})$ equal to zero. We obtain the one-dimensional linearized stable manifold as intersection of the following two planes.

$$w_{21}(k_0 - \bar{k}) + w_{22}(c_0 - \bar{c}) + w_{23}\left(\frac{k_0}{3} - \frac{\bar{k}}{3}\right) = 0; \tag{34}$$

$$w_{31}(k_0 - \bar{k}) + w_{32}(c_0 - \bar{c}) + w_{33}\left(\frac{k_0}{3} - \frac{\bar{k}}{3}\right) = 0; \tag{35}$$

where as above w_{ij}'s are the ordered components of the inverse W of the transformation matrix V:

Under the transversality condition agents are required to start on the stable manifold, i.e. to coordinate on the intersection between the horizontal plane through the predetermined variable k_0; and the approximated stable manifold (34-35). Given the gap $k_0 - \bar{k}$; they choose the non predetermined variables in consequence:[8]

$$\begin{bmatrix} k_0 - \bar{k} \\ c_0 - \bar{c} \end{bmatrix} = -\frac{k_0 - \bar{k}}{3} \begin{bmatrix} w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix}^{-1} \begin{bmatrix} w_{23} \\ w_{33} \end{bmatrix}.$$

Linearized dynamics become

$$\begin{bmatrix} k_{t+1} - \bar{k} \\ c_{t+1} - \bar{c} \\ k_t - \bar{k} \end{bmatrix} = V \Lambda^t W \begin{bmatrix} -\frac{k_0 - \bar{k}}{3} \begin{bmatrix} w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix}^{-1} \begin{bmatrix} w_{23} \\ w_{33} \end{bmatrix} \\ k_0 - \bar{k} \end{bmatrix}$$

$$= \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix} \begin{bmatrix} w_{13} - (w_{11}; w_{12}) \begin{bmatrix} w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix}^{-1} \begin{bmatrix} w_{23} \\ w_{33} \end{bmatrix} \end{bmatrix} (A_0 - A) \frac{dk}{dA} \lambda_1^t; \tag{36}$$

[8] The two components of the vector

$$\begin{bmatrix} w_{21} & w_{22} \\ w_{31} & w_{32} \end{bmatrix}^{-1} \begin{bmatrix} w_{23} \\ w_{33} \end{bmatrix}$$

are set to be real, by possibly normalizing the second and the third eigenvector of the transformation matrix V by an opportune complex number.

where $(A_0 - A)\,dk/dA$ is the approximated impact of the productivity parameter A on the stationary level of capital and $(v_{11}; v_{12}; v_{13})$ is the eigenvector associated to the stable root.

4.2 Endogenous growth

4.2.1 Shocks on the beliefs

We consider the Jacobian matrix B evaluated at the steady state. In our special case the Jacobian matrix is triangular and by notational simplicity we call its components as follows:

$$B = \begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix}. \tag{37}$$

We observe that b_{11} is the stable eigenvalue, while b_{22} is the unstable one. The associated eigenvectors are organized in the following transformation matrix:

$$V = \begin{bmatrix} b_{11} - b_{22} & 0 \\ b_{21} & 1 \end{bmatrix}.$$

Convergence requires that the initial condition $x_0 = (\gamma_0; z_0)^T$ belongs to the stable manifold. Moreover x_0 must lie in a sufficiently small neighborhood of $(\gamma^*; z^*)$. To compute the linearized saddle path we require the second component of $V^{-1}(x_0 - x^*)$ to be zero, i.e.

$$z_0 - z^* = \frac{b_{21}}{b_{11} - b_{22}} (\gamma_0 - \gamma^*). \tag{38}$$

At the period t we have

$$z_t - z^* = \frac{b_{21}}{b_{11} - b_{22}} (\gamma_t - \gamma^*). \tag{39}$$

Directly from the Jacobian we get the inflation trajectory, i.e. the first component of $B^t(x_0 - x^*)$:

$$\gamma_t - \gamma^* = (\gamma_0 - \gamma^*)\, b_{11}^t \tag{40}$$

and from (39) and (40) the motion of the consumption-capital ratio.

$$z_t - z^* = \frac{b_{21}}{b_{11} - b_{22}} (\gamma_0 - \gamma^*)\, b_{11}^t. \tag{41}$$

As in the previous section we refer to linearized dynamics and we assume that the realization of the vector at time ¿ depends on the expectation formed at time ¿ of the vector at time ¿ + 1 :

$$\begin{bmatrix} \frac{1}{4}_{¿} \text{ ¡ } \frac{1}{4} \\ z_{¿} \text{ ¡ } \hat{z} \end{bmatrix} = M E_{¿} \begin{bmatrix} \frac{1}{4}_{¿+1} \text{ ¡ } \frac{1}{4} \\ z_{¿+1} \text{ ¡ } \hat{z} \end{bmatrix} ; \tag{42}$$

where M is the matrix describing the dynamics in forward looking and E is the usual expectation operator. As above, equation (42) is reset as follows.

$$\begin{bmatrix} \frac{1}{4}_{¿+1} \text{ ¡ } \frac{1}{4} \\ z_{¿+1} \text{ ¡ } \hat{z} \end{bmatrix} = B \begin{bmatrix} \frac{1}{4}_{¿} \text{ ¡ } \frac{1}{4} \\ z_{¿} \text{ ¡ } \hat{z} \end{bmatrix} + \begin{bmatrix} \frac{1}{4};t+1 \\ z;t+1 \end{bmatrix} ;$$

where $B = M^{\text{¡}1}$ is the invertible Jacobian matrix (37) and

$$\begin{bmatrix} \frac{1}{4};t+1 \\ z;t+1 \end{bmatrix} , \begin{bmatrix} \frac{1}{4}_{¿+1} \text{ ¡ } E_{¿}\frac{1}{4}_{¿+1} \\ z_{¿+1} \text{ ¡ } E_{¿} z_{¿+1} \end{bmatrix} :$$

The deterministic system has still been augmented by taking into account the vector of forecasting errors at time ¿ + 1:

In our dynamic system there are no pre-determined variables. By simplicity we assume that $E_{¿}\frac{1}{4}_{¿+1} = \frac{1}{4}$; $E_{¿} z_{¿+1} = \hat{z}$; i.e. before the shock on the beliefs agents expect the in‡ation factor and the consumption-capital ratio at time ¿ + 1 to get their long-run value.

Moreover the two errors are not independent. Agents' coordination rules out the paths outside the stable manifold and the approximated relation (38) must hold, where b_{11} is the stable eigenvalue and b_{22} is the explosive one. The temporary shock on the non-predetermined variable $\frac{1}{4}_{¿+1}$ determines the shock on z_{t+1} according to (38), and occurs between ¿ and ¿ + 1 after a sunspot signal. Agents coordinate on the new point $(\frac{1}{4}_{t+1}; z_{¿+1})$ which belongs to the stable manifold. Summing up, the vector of forecasting errors becomes

$$\begin{bmatrix} \frac{1}{4};t+1 \\ z;t+1 \end{bmatrix} , \begin{bmatrix} \frac{1}{4}_{¿+1} \text{ ¡ } \frac{1}{4} \\ \frac{b_{21}}{b_{11} \text{ ¡ } b_{22}} (\frac{1}{4}_{t+1} \text{ ¡ } \frac{1}{4}) \end{bmatrix} :$$

To simplify the notation we set ¿ + 1 = 0: The shock occurs in 0 on the non-predetermined variable $\frac{1}{4}_{0}$: Before the shock the economy is at the steady state $(\frac{1}{4}; \hat{z})$: There are no pre-determined variables. The shock is temporary. The steady state is not a¤ected by the shock because it depends only on the fundamentals. We still assume that the non-predetermined initial

condition $(\varphi_0; z_0)$ lies in a sufficiently small neighborhood of $(\bar{\varphi}; \bar{z})$: In the saddle case the eigenvalues are real, one inside and one outside the unit circle. Agents have rational expectations and respect the transversality condition. As seen above, we require $(\varphi_0; z_0)$ belongs to the saddle path according to the transversality condition. As stressed above we consider an approximated z_0 because we compute a linearized saddle path around the steady state. The linearized dynamics, where the initial condition are interpreted as shocks on the beliefs, are still provided by system (40-41).

4.2.2 Shocks on fundamentals

The case is trivial under the usual condition of equilibrium determinacy. The two-dimensional steady state is now a source and rational agents directly coordinate on it to respect the transversality condition. The steady state before the shock, that we assume to be permanent, still plays the role of initial condition, but now there is no longer transition and the economy literally jumps on the new stationary point thanks to the double freedom degree in $(\varphi_t; z_t)$:

5 Numerical computation

5.1 Exogenous growth

5.1.1 Shocks on the beliefs

We set the parameter values as above (the length of the period is 1:5 months): $\beta = 0{:}33333$; $\bar{\gamma} = 0{:}99748$; $\pm = 0{:}013084$; $\gamma^{-1} = 1{:}0025$; $\rho = 0{:}66667$; $s = 0{:}33333$; $\mu = 0{:}01561$ We fix A to have the same gross interest rate of the endogenous growth setup and to normalize the capital of modified golden rule to one: $A = 0{:}046832$: Moreover we set φ and q to obtain indeterminacy: $\varphi = 1{:}3$; $q = 0{:}3$: The steady state is: $\bar{\varphi} = 1{:}0025$; $\bar{c} = 0{:}033746$; $\bar{k} = 1$: After computing the Jacobian matrix, we have two stable eigenvalues and one unstable $(-0{:}64301; 0{:}98009; 1{:}0229)$. A converging trajectory along the linearized stable manifold is eventually obtained after the shock $\varepsilon_{\varphi;0} = 0{:}001$ on the expected inflation factor according to system (33).

$$\varphi_t = 1{:}0025 + 9{:}9991 \times 10^{-4} (-0{:}64301)^t + 1{:}0826 \times 10^{-7} \times 0{:}98009^t;$$
$$c_t = 0{:}033746 - 1{:}3173 \times 10^{-5} (-0{:}64301)^t + 1{:}7963 \times 10^{-7} \times 0{:}98009^t;$$

$$k_t = 1 - 8{:}0054 \times 10^{-6} (-0{:}64301)^t + 8{:}0171 \times 10^{-6} \times 0{:}98009^t.$$

Dynamics are represented in Fig. 3:

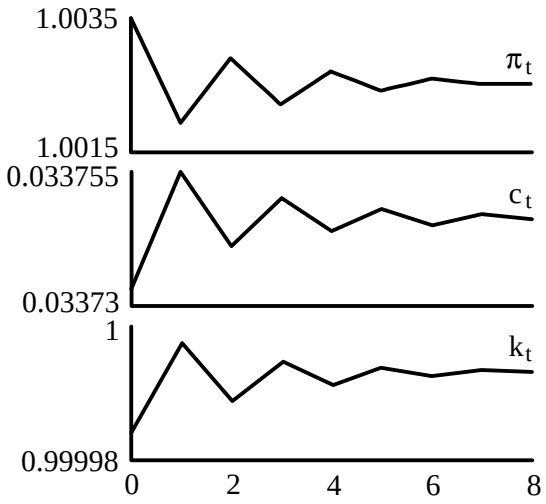

Figure 3: Shock on the expected in‡ation.

5.1.2 Shocks on the fundamentals

We calibrate the model in order to obtain a determinate stationary solution. More precisely we keep the calibration for the parameter values introduced in the previous sub-section and we increase q up to 0:7 in order to get determinacy. If we specify the production function as Ak^{α} we …nd the following stationary values of the variables appearing in system (13): $\pi = 1{:}0025$; $c = 0{:}033746$; $k = 1$: The corresponding eigenvalues are $\lambda_1 = 0{:}98011$; $\lambda_2 = 1{:}0229$; $\lambda_3 = -10{:}282$: We consider as usual a shock on the productivity parameter A: The impact of the shock on the stationary capital is given by $dk/dA = [(1 - \alpha) A]^{-1}$: The technological shock is set to be positive and equal to $A - A_0 = 0{:}01$: Thereby the linearized dynamics are

$$\begin{aligned} \pi_t &= 1{:}0025 - 0{:}0066085 \times 0{:}98011^t; \\ c_t &= 0{:}033746 - 0{:}010984 \times 0{:}98011^t; \\ k_t &= 1 - 0{:}49057 \times 0{:}98011^t. \end{aligned}$$

Dynamics are represented in Fig. 4:

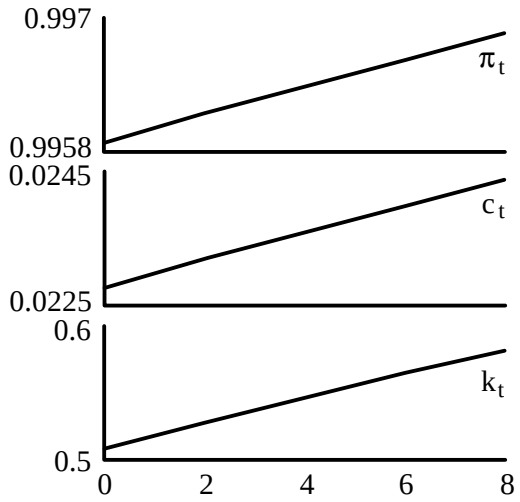

Figure 4: Technological shock.

5.2 Endogenous growth

5.2.1 Shocks on the beliefs

We set the parameters as above ($\bar{} = 0:99748$; $\pm = 0:013084$; $^1 = 1:0025$; $® = 0:33333$; $A = 0:055385$). Moreover we ...x $¾ = 1:3$ and $q = 0:3$: The stationary state gets the following values: $¼ = 0:99881$; $ź = 0:038602$; and the triangular Jacobian matrix becomes:

$$B = \begin{bmatrix} ¡\,0:64241 & 0 \\ 0:024828 & 1:0385 \end{bmatrix}$$

with the two real eigenvalues of the saddle con...guration on the main diagonal. The oscillating linearized dynamics after a shock of 0:001 on the belief about the in‡ation factor $¼_0$ are the following:

$$\begin{aligned} ¼_t &= 0:99881 + 0:001\,(¡\,0:64241)^t; \\ z_t &= 0:038602 ¡ 1:4771 £ 10^{¡5}\,(¡\,0:64241)^t; \end{aligned}$$

and are represented in Fig. 5:

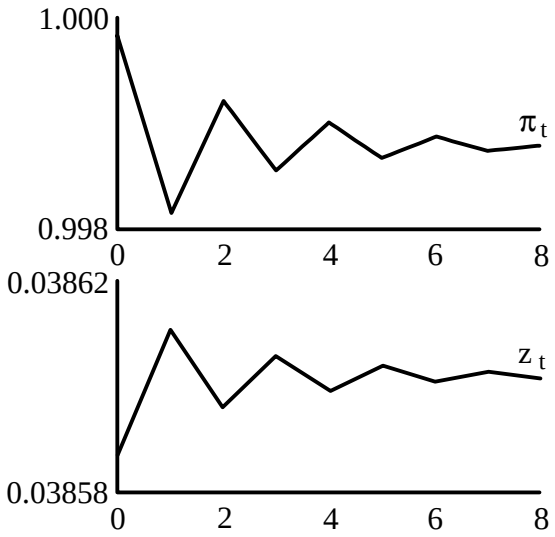

Figure 5: Shock on the expected in‡ation.

6 Conclusion

In this paper we have presented a one-sector productive in…nite horizon economy with partial cash-in-advance constraint on consumption purchases and studied the occurrence of indeterminacy and endogenous ‡uctuations. We have focused on the stationary economy as well as on the economy displaying long-run growth. We have shown that a relaxation of the liquidity constraint makes indeterminacy more and more likely to occur. Yet, for an amplitude of the liquidity constraint su¢ciently low, the unique steady state as well as the unique balanced growth rate are bound to be indeterminate for whatever fundamentals speci…cation.

The results obtained provide additional support to the increasingly shared view that the existence of multiple equilibria and deterministic as well as stochastic ‡uctuations in competitive economies has by no means any "pathological" or "exotic" character. It is conversely the robustness of the saddle path stability of the traditional Ramsey-Cass-Koopmans model with respect arbitrarily small imperfection in …nancial market to be questioned.

A natural and we think useful extension of the model should account for an endogenous amplitude of liquidity constraint, e.g. by shaping q either as a

function of the average level of capital and wealth available in the economy, or of consumption, or of capital, or of real balances, or of consumption to capital ratio. This would allow to emphasize more its endogenous and unstable nature and give raise to more complex and richer dynamics.

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