DERIVING TESTS OF THE REGRESSION MODEL USING THE DENSITY FUNCTION OF A MAXIMAL INVARIANT

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Abstract

In the context of the linear regression model in which some regression coefficients are of interest and others are purely nuisance parameters, we derive the density function of a maximal invariant statistic. This allows the construction of a range of optimal test statistics including the locally best invariant test which is equivalent to the well-known one-sided t-test. The approach is also extended to the general linear regression model in which the covariance matrix is nonscalar and to non-linear regression models.

Key words: Invariance; linear regression model; locally best invariant test; non-linear regression model; nuisance parameters; t-test.

JEL classification: C2, C12.

1. Introduction

This paper is concerned with the problem of testing the null hypothesis that one regressor coefficient is zero, against the alternative that it is non-negative, in the context of linear and non-linear regression models. Statistical models and particularly those used by econometricians, involve a large number of influences. These kinds of models contain two types of parameters, those of interest and those not of immediate interest that are known as nuisance parameters. Their presence causes unexpected complications in statistical inference. Kalbfleisch and Sprott (1970) discussed methods of eliminating nuisance parameters from the likelihood function so that inference can be made about the parameters of interest.

In practice, many statistical problems including testing of hypothesis, display symmetries, which impose additional restrictions for the choice of proper statistical tests. In the statistical testing literature, the idea of invariance dates back half a century. The unpublished work of Hunt and Stein (1946) (see Lehmann, 1959 and 1986) introduced the principle and demonstrated its applicability and meaningfulness in the framework of hypothesis testing. According to Lehmann (1950), the notion of invariance was introduced into the statistical literature in the writings of R. A. Fisher, Hotelling (1933, 1936), Pitman (1939), Stein (1948) and others, in connection with various special problems. Among others Lehmann (1959, 1986), King (1979, 1980, 1983a), King and Smith (1986), Ara and King (1993), and Ara (1995) suggested the use of invariance arguments to overcome the problem of nuisance parameters. It is a generally accepted principle that if a problem with a unique solution is invariant under a certain group of transformations, then the solution should be invariant under that transformations. A hypothesis testing problem is invariant under a group of transformations acting on the sample space of the observed data vector if for any transformation the probability distribution of transformed data vector belongs to the same set (null or alternative hypothesis) as the original data vector. The idea behind invariance is that if the hypothesis testing problem under consideration has a particular invariance property, then we should restrict attention to only those tests that share this invariance property. The class of all invariant functions can be obtained as the totality of functions of a maximal invariant. A maximal invariant is a statistic which takes the same value for the observed data vectors that are connected by transformations and different values for those data vectors that are not connect by transformations. Consequently any invariant test statistic can be written as a function of the maximal invariant. This means, we can treat the maximal invariant as the observed data, find its density and then construct appropriate tests based on this density.

The performance of a statistical test is assessed by its size and power properties. Econometricians are always interested in optimality of power and for any testing problem they would like to use a uniformly most powerful (UMP) test. Many testing problems involving the linear regression model can be reduced, either by conditioning on sufficient statistics or by invariance arguments to testing a simple null hypothesis against one sided alternatives. Ideally we would then like to use a UMP test but unfortunately it is rarely possible to find a UMP test when the alternative hypothesis is composite and/or in the presence of nuisance parameters. Cox and Hinkley (1974, p.102) discuss three approaches for constructing tests of simple hypotheses against composite alternative hypotheses when no UMP test exits. They involve choosing a point at which the power is optimised:

(i) “to pick, somewhat arbitrarily, a ‘typical’ point” in the alternative parameter space and use it in test construction in order to find a test that at least has optimal power for the chosen point;

(ii) removing this arbitrariness by choosing a point to be close to the null hypothesis which leads to the locally best (LB) test, i.e. to maximize the power locally near the null hypothesis;

(iii) choosing a test which maximizes some weighted average of the powers over the alternative parameter space.

Option (i), labelled by King (1987b) as the point optimal (PO) approach, uses the most powerful test against a specific alternative solution. Option (ii) is the most popular for a single parameter and leads us to a LB or a locally most powerful test. These tests are constructed by maximizing power locally at the null hypothesis. The LB test is also optimal in the sense that its power curve has the steepest slope, at the null hypothesis, of all power curves from tests with the same size. Following Neyman and Pearson (1936) a number of authors, notably Ferguson (1967), Efron (1975), King (1981, 1984, 1987a), King and Hillier (1985), Sengupta (1987) and Wu and King (1994) among others, have recommended the use of LB tests.

Our interest in this article is to derive the density function of the maximal invariant statistic in the context of regression model and invariant transformation and then construct a LBI test for linear and non-linear regressors, which is called the Locally Best Invariant (LBI) test. Also we show that this LBI test is equivalent to the one-sided t-test in the case of testing regression coefficients.

The plan of this paper is as follows. First of all we derive the density function of the maximal invariant in section 2 and in section 3 we construct the LBI test statistic which is shown to also be equivalent to the one-sided t-test. In section 4, we derive the density function of the maximal invariant for a non-linear regression function and find the LBI test statistic. Finally, some concluding remarks are made in section 5.

2. Derivation of Density Function

Let us consider the linear model,

$$y = X_1\beta_1 + X_2\beta_2 + u \tag{2.1}$$

where y is $n \times 1$, X_1 is an $n \times q$ nonstochastic matrix, X_2 is an $n \times p$ nonstochastic matrix, β_1 is a $q \times 1$ vector and β_2 is $p \times 1$ vector. Here $[X_1 \colon X_2]$ is full column rank.

Considering first the case of $p = 1$, we are interested in testing $H_0 \colon \beta_2 = 0$ against $H_a \colon \beta_2 > 0$ in the context of the above linear regression model. It is assumed that $u \sim N(0, \sigma^2 I_n)$ where σ^2 is unknown. This problem is invariant under the class of transformations

$$y \to \gamma_0 y + X_1\gamma \tag{2.2}$$

where γ_0 is a positive scalar and γ is a $q \times 1$ vector.

Let $M_1 = I - X_1(X_1'X_1)^{-1}X_1'$.

Then

$$\begin{aligned} M_1 y &= M_1X_1\beta_1 + M_1X_2\beta_2 + M_1u \\ &= M_1X_2\beta_2 + M_1u. \end{aligned} \tag{2.3}$$

Multiplying both sides of (2.2) by P where P is an $m \times n$ matrix such that $PP' = I_m$, $P'P = M_1$ and $m = n - q$, we get

$$Py = PX_2\beta_2 + Pu.$$

Note that $PM_1y = Py$ and $PM_1 = P$. Thus $Py \sim N(PX_2\beta_2, \sigma^2 I_m)$. Let $z = Py$. Then the joint density function of z is

$$f(z) = (2\pi\sigma^2)^{-m/2} \exp\{-\frac{1}{2\sigma^2}(z - PX_2\beta_2)'(z - PX_2\beta_2)\}\,. \qquad (2.4)$$

Let $r^2 = z'z$ be the usual squared distance of z from the origin. Now, we change z to the *m*-dimensional polar co-ordinates $(r, \theta_1, \theta_2, ..., \theta_{m-1})$ as follows:

$$\begin{aligned} z_1 &= r\cos\theta_1 \\ z_j &= r(\prod_{k=1}^{j-1}\sin\theta_k)\cos\theta_j;\ \text{for } 2 \le j \le m-1, \qquad (2.5) \\ z_m &= r\prod_{k=1}^{m-1}\sin\theta_k \end{aligned}$$

where $0 \le r \le \infty$, $0 \le \theta_k \le \pi$, for $k = 1,2,...,(m-2)$
and $0 \le \theta_{m-1} \le 2\pi$.

The Jacobian of the transformation is

$$J_m(r, \theta_1, \theta_2, ..., \theta_{m-1}) = \left|\frac{\partial(z_1, z_2, ..., z_m)}{\partial(r, \theta_1, \theta_2, ..., \theta_{m-1})}\right| = r^{m-1}\prod_{k=1}^{m-2}\sin\theta_k^{\ m-1-k}$$

(Miller, 1964, p.13). To construct the LBI test, we have to find the density function of the maximal invariant statistic. So we want to find the distribution of the maximal invariant

$$w = z/(z'z)^{1/2} = z/r\,.$$

Note that

$$z = rw\,.$$

Now the joint density function of z becomes, after the above change of variables,

$$f(r, \theta_1, \theta_2, ..., \theta_{m-1}) = (2\pi\sigma^2)^{-m/2}\exp\{-\frac{1}{2\sigma^2}(rw - PX_2\beta_2)'(rw - PX_2\beta_2)\}r^{m-1}\prod_{k=1}^{m-2}\sin\theta_k^{\ m-1-k}$$

$$= (2\pi\sigma^2)^{-m/2} \exp\{-\frac{1}{2\sigma^2}(r^2 w'w - 2w'rPX_2\beta_2 + \beta_2{}' X_2{}' P'PX_2\beta_2)\}$$

$$r^{m-1}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k} \,. \tag{2.6}$$

To find the density function of w from the above joint density function of z, first we have to find the marginal density function of $(\theta_1, \theta_2, ..., \theta_{m-1})$. The components of w are $w_1(\theta_1, \theta_2, ..., \theta_{m-1}), \ldots, w_m(\theta_1, \theta_2, ..., \theta_{m-1})$ and they are defined by (2.5). Therefore the marginal density function of $(\theta_1, \theta_2, ..., \theta_{m-1})$ can be obtained by integrating out r of (2.6),

$$f(\theta_1, \theta_2, ..., \theta_{m-1}) = (2\pi\sigma^2)^{-m/2}\int_0^\infty \exp\{-\frac{1}{2\sigma^2}(r^2 w'w - 2w'rPX_2\beta_2 + \beta_2{}' X_2{}' P'PX_2\beta_2)\}$$

$$r^{m-1}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}\, dr$$

$$= (2\pi\sigma^2)^{-m/2}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}\int_0^\infty \exp\{-\frac{1}{2\sigma^2}(r^2 w'w - 2w'rPX_2\beta_2 + \beta_2{}' X_2{}' P'PX_2\beta_2)\}r^{m-1}dr$$

$$= (2\pi\sigma^2)^{-m/2}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}\int_0^\infty \exp\{-\frac{1}{2}(\frac{r^2}{\sigma^2}w'w - 2w'\frac{r}{\sigma}PX_2\frac{\beta_2}{\sigma} + \frac{\beta_2{}'}{\sigma}X_2{}'P'PX_2\frac{\beta_2}{\sigma})\}r^{m-1}dr$$

$$= (2\pi\sigma^2)^{-m/2}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}\int_0^\infty \exp\{-\frac{1}{2}(\lambda^2 - 2\lambda w'PX_2\beta^* + \beta^{*\prime} X_2{}'P'PX_2\beta^*)\}\lambda^{m-1}\sigma^{m-1}\sigma\, d\lambda;$$

where $w'w = 1, \beta^* = \frac{\beta_2}{\sigma}$, $\lambda = \frac{r}{\sigma}$, and $dr = \sigma d\lambda$. If we set

$$a(w, \beta^*) = w'PX_2\beta^* \,, \tag{2.7}$$

$$b(w, \beta^*) = \frac{1}{2}(\beta^{*\prime} X_2{}' P'ww'PX_2\beta^* - \beta^{*\prime} X_2{}' P'PX_2\beta^*)$$

$$= -\frac{1}{2}\beta^{*\prime} X_2{}' P'M_w PX_2\beta^* \tag{2.8}$$

and $M_w = I - ww' = I - w(w'w)^{-1}w'$, then $b(w,\beta^*)$ is sum of squared errors of the OLS regression of $PX_2\beta^*$ on w and

$$f(\theta_1,\theta_2,...,\theta_{m-1}) = (2\pi)^{-m/2}\prod_{k=1}^{m-2}\sin\theta_k^{\,m-1-k}\int_0^\infty \exp\{-\frac{1}{2}(\lambda - a(w,\beta^*))^2 + b(w,\beta^*)\}\lambda^{m-1}d\lambda;$$

or

$$\begin{aligned}
&f(\theta_1,\theta_2,...,\theta_{m-1})\\
&= (2\pi)^{-m/2}\prod_{k=1}^{m-2}\sin\theta_k^{\,m-1-k}\exp(b(w,\beta^*))\int_0^\infty \exp\{-\frac{1}{2}(\lambda - a(w,\beta^*))^2\}\lambda^{m-1}d\lambda\\
&\qquad = \prod_{k=1}^{m-2}\sin\theta_k^{\,m-1-k}\exp(b(w,\beta^*))\int_0^\infty \lambda^{m-1}\varphi\{(\lambda - a(w,\beta^*))^2\}d\lambda\\
&\qquad = c(w,\beta^*)\int_0^\infty (\eta + a(w,\beta^*))^{m-1}\varphi(\eta^2)d\eta \ ,
\end{aligned}$$

where $\varphi\{(\lambda - a(w,\beta^*))^2\} = (2\pi)^{-m/2}\exp\{-\frac{1}{2}(\lambda - a(w,\beta^*))^2\}$,

(King, 1979, chapter 3),

$$c(w,\beta^*) = \exp(b(w,\beta^*))\prod_{k=1}^{m-2}\sin\theta_k^{\,m-1-k}, \tag{2.9}$$

$\eta = \lambda - a(w,\beta^*)$, $\lambda = \eta + a(w,\beta^*)$ and $d\lambda = d\eta$.

We can further write

$$\begin{aligned}
f(\theta_1,\theta_2,...,\theta_{m-1}) = c\int_0^\infty \{\eta^{m-1} + (m-1)\eta^{m-2}a(w,\beta^*) + {}\\
\frac{(m-1)(m-2)}{2}\eta^{m-3}a^2(w,\beta^*) + ... + a^{m-1}(w,\beta^*)\}\varphi(\eta^2)d\eta
\end{aligned}$$

$$\begin{aligned}
&= c(w,\beta^*)\{\int_0^\infty \eta^{m-1}\varphi(\eta^2)d\eta + a(w,\beta)(m-1)\int_0^\infty \eta^{m-2}\varphi(\eta^2)d\eta + {}\\
&a^2(w,\beta^*)\frac{(m-1)(m-2)}{2}\int_0^\infty \eta^{m-3}\varphi(\eta^2)d\eta + ... + a^{m-1}(w,\beta^*)\int_0^\infty \varphi(\eta^2)d\eta\}
\end{aligned}$$

$$\begin{aligned}
&= c(w,\beta^*)\{\frac{1}{2}\Gamma(m/2)\pi^{-m/2} + \frac{1}{2}a(w,\beta^*)(m-1)\Gamma(\frac{m-1}{2})\pi^{-\frac{m-1}{2}} + {}\\
&a^2(w,\beta^*)\frac{(m-1)(m-2)}{4}\Gamma(\frac{m-2}{2})\pi^{-\frac{m-2}{2}} + ... + a^{m-1}(w,\beta^*)(2\pi)^{-m/2}2^{-3/2}\Gamma(1/2)\};
\end{aligned}$$

where $\int_0^\infty \eta^{m-1}\varphi(\eta^2)d\eta = \frac{1}{2}\Gamma(m/2)\pi^{-m/2}$, $\int_0^\infty \varphi(\eta^2)d\eta = (2\pi)^{-m/2}2^{-3/2}\Gamma(1/2)$ and $\Gamma(1/2) = \sqrt{\pi}$.

Therefore the marginal density function of $\theta_1, \theta_2, ..., \theta_{m-1}$ is,

$$f(\theta_1, \theta_2, ..., \theta_{m-1}) = c(w, \beta^*)\{\frac{1}{2}\Gamma(m/2)\pi^{-m/2} + \frac{1}{2}a(w, \beta^*)(m-1)\Gamma(\frac{m-1}{2})\pi^{-\frac{m-1}{2}}$$

$$+\frac{1}{4}a^2(w, \beta^*)(m-1)(m-2)\Gamma(\frac{m-2}{2})\pi^{-(m-2/2)} + ... + a^{m-1}(w, \beta^*)(2\pi)^{-m/2}2^{-3/2}\pi^{1/2}\};$$

or

$$f(\theta_1, \theta_2, ..., \theta_{m-1}) = \exp(b(w, \beta^*))\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}\{\frac{1}{2}\Gamma(m/2)\pi^{-m/2} +$$

$$\frac{1}{2}a(w, \beta^*)(m-1)\Gamma(\frac{m-1}{2})\pi^{-\frac{m-1}{2}} + \frac{1}{4}a^2(w, \beta^*)(m-1)(m-2)\Gamma(\frac{m-2}{2})\pi^{-\frac{m-2}{2}} +$$

$$... + \frac{1}{2}a^{m-1}(w, \beta^*)(2\pi)^{-m/2}2^{-1/2}\pi^{1/2}\}. \qquad (2.10)$$

Now the transformation from $(\theta_1, \theta_2, ..., \theta_{m-1})$ to $w = z/(z'z)^{1/2}$ is straightforward since the components of w are defined by (2.5).

Therefore the density function of w is,

$$f(w) = \exp(b(w, \beta^*))\{\frac{1}{2}\Gamma(m/2)\pi^{-m/2} + \frac{1}{2}a(w, \beta^*)(m-1)\Gamma(\frac{m-1}{2})\pi^{-\frac{m-1}{2}}$$

$$+\frac{1}{4}a^2(w, \beta^*)(m-1)(m-2)\Gamma(\frac{m-2}{2})\pi^{-\frac{m-2}{2}} + ... + \frac{1}{2}a^{m-1}(w, \beta^*)(2\pi)^{-m/2}2^{-1/2}\pi^{1/2}\}$$

or

$$f(w) = \frac{1}{2}\pi^{-m/2}\exp(b(w, \beta^*))\{\Gamma(m/2) + a(w, \beta^*)(m-1)\Gamma(\frac{m-1}{2})\pi^{1/2}$$

$$+\frac{1}{2}a^2(w, \beta^*)(m-1)(m-2)\Gamma(\frac{m-2}{2})\pi + \frac{1}{6}a^3(w, \beta^*)(m-1)(m-2)(m-3)$$

$$\Gamma(\frac{m-3}{2})\pi^{3/2} + ... + a^{m-1}(w, \beta^*)2^{-\frac{m+1}{2}}\pi^{1/2}\} \qquad (2.11)$$

where $a(w,\beta^*)$ and $b(w,\beta^*)$ are defined by (2.7) and (2.8). Here (2.11) is the density function of the maximal invariant w and using this density function we can construct the LBI test statistic.

3.Tests of a Linear Regressor

We are interested in testing the hypothesis $H_0{:}\beta_2 = 0$ against $H_a{:}\beta_2 > 0$ in the context of the linear regression model (2.1). Let us first consider $p=1$, i.e. β_2 is a scalar. A LBI test of H_0 against H_a is that with critical region of the form

$$\left.\frac{\partial \log f(w)}{\partial \beta_2}\right|_{\beta_2=0} \geq c_\alpha . \qquad (3.1)$$

In the previous section we have derived $f(w)$.

3.1 Construction of the LBI test statistic

The density function of the maximal invariant w is given by (2.11). Taking logs on both sides of (2.11) we get

$$\log\{f(w)\} = -\log 2 - \frac{m}{2}\log\pi + b(w,\beta^*) + \log\{\Gamma(m/2) + a(w,\beta^*)(m-1)\Gamma(\frac{m-1}{2})\pi^{1/2}$$

$$+\frac{1}{2}a^2(w,\beta^*)(m-1)(m-2)\Gamma(\frac{m-2}{2})\pi + ... + a^{m-1}(w,\beta^*)2^{-\frac{m+1}{2}}\pi^{1/2}\}$$

$$= -\log 2 - \frac{m}{2}\log\pi + \frac{1}{2}(\beta^{*\prime} X_2{}' P'ww'PX_2\beta^* - \beta^{*\prime} X_2{}' P'PX_2\beta^*) + \log\{\Gamma(m/2)$$

$$+a(m-1)\Gamma(\frac{m-1}{2})\pi^{1/2} + \frac{1}{2}a^2(m-1)(m-2)\Gamma(\frac{m-2}{2})\pi + ... + a^{m-1}2^{-\frac{m+1}{2}}\pi^{1/2}\}$$

$$\frac{\partial \log f(w)}{\partial \beta^*} = (X_2{}' P'ww'PX_2\beta^* - X_2{}' P'PX_2\beta^*) + \{\Gamma(m/2) + a(w,\beta^*)(m-1)\Gamma(\frac{m-1}{2})\pi^{1/2}$$

$$+\frac{1}{2}a^2(w,\beta^*)(m-1)(m-2)\Gamma(\frac{m-2}{2})\pi + ... + a^{m-1}(w,\beta^*)2^{-\frac{m+1}{2}}\pi^{1/2}\}^{-1}$$

$$\frac{\partial}{\partial \beta^*}\{\Gamma(m/2) + a(w,\beta^*)(m-1)\Gamma(\frac{m-1}{2})\pi^{1/2} + \frac{1}{2}a^2(w,\beta^*)(m-1)(m-2)\Gamma(\frac{m-2}{2})\pi +$$

$$... + a^{m-1}(w,\beta^*)2^{-\frac{m+1}{2}}\pi^{1/2}\}$$

$$= (X_2{}' P' w w' P X_2 \beta^* - X_2{}' P' P X_2 \beta^*) + \{\Gamma(m/2) + a(w,\beta^*)(m-1)\Gamma(\frac{m-1}{2})\pi^{1/2}$$

$$+\frac{1}{2} a^2(w,\beta^*)(m-1)(m-2)\Gamma(\frac{m-2}{2})\pi + ... + a^{m-1}(w,\beta^*) 2^{-\frac{m+1}{2}} \pi^{1/2}\}^{-1}\{(m-1)\Gamma(\frac{m-1}{2})\pi^{1/2}$$

$$(w' P X_2)' + (m-1)(m-2)\Gamma(\frac{m-2}{2})\pi(w' P X_2 \beta^*)(w' P X_2)' + ...$$

$$+2^{-\frac{m+1}{2}}\pi^{1/2}(m-1)(w' P X_2 \beta^*)^{m-2}(w' P X_2)'\};$$

$$\left.\frac{\partial \log f(w)}{\partial \beta^*}\right|_{\beta^*=0} = \{\Gamma(m/2)\}^{-1}(m-1)\Gamma(\frac{m-1}{2})\pi^{1/2}(w' P X_2)'$$

$$= \frac{1}{\Gamma(m/2)}\pi^{1/2}(m-1)\Gamma(\frac{m-1}{2})(w' P X_2)' .$$

Therefore,

$$\left.\frac{\partial \log f(w)}{\partial \beta^*}\right|_{\beta^*=0} = (m-1)\frac{\Gamma(\frac{m-1}{2})\pi^{1/2}}{\Gamma(m/2)}(w' P X_2)'.$$

Hence the LBI test rejects H_0 for

$$\left.\frac{\partial \log f(w)}{\partial \beta^*}\right|_{\beta^*=0} = (m-1)\frac{\Gamma(\frac{m-1}{2})\pi^{1/2}}{\Gamma(m/2)}(w' P X_2)' \geq c_\alpha \qquad (3.2)$$

or $(w' P X_2)' \geq d_\alpha$,

or $$\frac{X_2{}' P' P y}{(y' P' P y)^{1/2}} = \frac{X_2{}' M_1 y}{(y' M_1 y)^{1/2}} \geq d_\alpha , \qquad (3.3)$$

where d_α is an appropriate critical value.

Thus $s = \frac{X_2{}' M_1 y}{(y' M_1 y)^{1/2}}$ is the LBI test statistic.

3.2 Relationship between the t-test and the required LBI test

From the above discussion we can say that the LBI test rejects H_0 for large values of

$$s = \frac{X_2{}' M_1 y}{(y' M_1 y)^{1/2}} \tag{3.4}$$

for testing $H_0: \beta_2 = 0$ against $H_a: \beta_2 > 0$ in the linear model (2.1) if X_2 is a vector , i.e. $p = 1$.

The ordinary least squares (OLS) estimator of β_2 in (2.1) is

$$\hat{\beta}_2 = (X_2{}' M_1 X_2)^{-1} X_2{}' M_1 y \,,$$

and the unbiased OLS estimator of the error variance is

$$\hat{\sigma}^2 = y'(M_1 - M_1 X_2 (X_2{}' M_1 X_2)^{-1} X_2{}' M_1) y / (m-1) .$$

Thus the t test statistic is

$$t = \hat{\beta}_2 / \{\hat{\sigma}(X_2{}' M_1 X_2)^{-1/2}\} ;$$

$$= (m-1)^{-1/2} (X_2{}' M_1 X_2)^{-1} X_2{}' M_1 y / \{y'(M_1 - M_1 X_2 (X_2{}' M_1 X_2)^{-1} X_2{}' M_1) y\}^{1/2} (X_2{}' M_1 X_2)^{-1/2}$$

or

$$t = (m-1)^{1/2} (X_2{}' M_1 X_2)^{-1/2} s / \{1 - (X_2{}' M_1 X_2)^{-1} s^2\} ; \tag{3.5}$$

where $X_2{}' M_1 X_2$ is a positive scalar. So, clearly (3.5) is a monotonic increasing function of the test statistic s. Thus we may conclude that our LBI test is equivalent to the t-test.

3.3 Whether the test statistic s is UMPI or not

We have

$$w = z / (z'z)^{1/2} = \frac{Py}{(y'P'Py)^{1/2}} = \frac{Py}{(y'M_1 y)^{1/2}} . \tag{3.6}$$

Replacing (3.6) and (3.4) in (2.11) we get,

$$f(w)=\frac{1}{2}\pi^{-m/2}\exp(\frac{1}{2}s^2\beta^{*2}-X_2{}'P'PX_2\beta^{*2})\{\Gamma(\frac{m}{2})+s\beta^{*}(m-1)\Gamma(\frac{m-1}{2})\pi^{1/2}$$

$$+\frac{1}{2}s^2\beta^{*2}(m-1)(m-2)\Gamma(\frac{m-2}{2})\pi+\frac{1}{6}s^3\beta^{*3}(m-1)(m-2)(m-3)\Gamma(\frac{m-3}{2})\pi^{3/2}+$$

$$...+(s\beta^{*})^{m-1}2^{-\frac{m+1}{2}}\pi^{1/2}\}.$$

Thus

$$f(w)=\frac{1}{2}\pi^{-m/2}\exp\{\frac{1}{2}(s^2\beta^{*2}-X_2{}'P'PX_2\beta^{*2})\}\{\Gamma(\frac{m}{2})+s\beta^{*}(m-1)\Gamma(\frac{m-1}{2})\pi^{1/2}$$

$$+\frac{1}{2}s^2\beta^{*2}(m-1)(m-2)\Gamma(\frac{m-2}{2})\pi+\frac{1}{6}s^3\beta^{*3}(m-1)(m-2)(m-3)\Gamma(\frac{m-3}{2})\pi^{3/2}+$$

$$...+(s\beta^{*})^{m-1}2^{-\frac{m+1}{2}}\pi^{1/2}\}. \qquad (3.7)$$

We know all invariant tests can be written as a function of the maximal invariant. To find the best test within the class of invariant tests we have derived the density function of the maximal invariant and then we have constructed the LBI test. To prove, our test statistic s is UMPI, according to Lehmann (1986, section 6.4) we have to prove that the function of s, $f(w)$ is a monotonic increasing function of s. So far, we cannot prove theoretically that our LBI test statistic s is UMPI. We have simulated this function using a number of different values of s, β^{*} and $X_2{}'P'PX_2$ and in each case found the function (for these particular values) was monotonic. This is quite different from being able to prove that $f(w)$ is monotonic. We do however conjecture that $f(w)$ is monotonic having been unable to disprove this using simulation methods. We therefore conjecture that the test based on s is UMPI where invariance is with respect to transformations of the form (2.2).

3.4 Construction of the LBI test when $u \sim N(0,\sigma^2\Sigma(\phi))$, where $\Sigma(\phi)$ is known

Consider the following model,

$$y = X_1\beta_1 + X_2\beta_2 + u \qquad (3.8)$$

where X_1 is an $n\times q$ nonstochastic matrix, X_2 is an $n\times p$ nonstochastic matrix and $u \sim N[0, \sigma^2 \Sigma(\phi)]$.

3.4.1 Derivation of Density Function

Let $M_1 = I - X_1(X_1' X_1)^{-1} X_1'$, using the same transformation on (3.8) that we used earlier (see section 2) we get,

$$PM_1 u \sim N[0, \sigma^2 (P\Sigma(\phi)P')]$$

and $$Py \sim N\{PX_2\beta_2, \sigma^2 (P\Sigma(\phi)P')\}.$$

We want to find the distribution of the maximal invariant $w = z/(z'z)^{1/2}$, where $z = Py$ and $z'z = r^2$.

We get,

$$f(z) = (2\pi\sigma^2)^{-m/2} \left|P\Sigma(\phi)P'\right|^{-1/2} \exp\{-\frac{1}{2\sigma^2}(z - PX_2\beta_2)'(P\Sigma(\phi)P')^{-1}(z - PX_2\beta_2)\}. \qquad (3.9)$$

Now, changing z to the *m*-dimensional polar co-ordinates $(r, \theta_1, \theta_2, ..., \theta_{m-1})$ as (2.5), where $0 \le r \le \infty$, $0 \le \theta_k \le \pi$, for $k = 1,2,...,(m-2)$ and $0 \le \theta_{m-1} \le 2\pi$, the density of z becomes,

$$f(r, \theta_1, \theta_2, ..., \theta_{m-1}) = (2\pi\sigma^2)^{-m/2} \left|P\Sigma(\phi)P'\right|^{-1/2} \exp[-\frac{1}{2\sigma^2}\{(rw - PX_2\beta_2)'(P\Sigma(\phi)P')^{-1}(rw - PX_2\beta_2)\}] r^{m-1} \prod_{k=1}^{m-2} \sin\theta_k^{\,m-1-k}$$

$$= (2\pi\sigma^2)^{-m/2} \left|P\Sigma(\phi)P'\right|^{-1/2} \exp[-\frac{1}{2\sigma^2}\{r^2 w'(P\Sigma(\phi)P')^{-1} w - 2rw'(P\Sigma(\phi)P')^{-1} PX_2\beta_2 + \beta_2' X_2' P'(P\Sigma(\phi)P')^{-1} PX_2\beta_2\}] r^{m-1} \prod_{k=1}^{m-2} \sin\theta_k^{\,m-1-k}.$$

The marginal density function of $(\theta_1,\theta_2,...,\theta_{m-1})$ can be obtained by integrating out r,

$$f(\theta_1,\theta_2,...,\theta_{m-1}) = (2\pi\sigma^2)^{-m/2}\left|P\Sigma(\phi)P'\right|^{-1/2}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}$$

$$\int_0^\infty \exp[-\frac{1}{2}\{\frac{r^2}{\sigma^2}w'(P\Sigma(\phi)P')^{-1}w - 2w'\frac{r}{\sigma}(P\Sigma(\phi)P')^{-1}PX_2\frac{\beta_2}{\sigma} +$$

$$\frac{\beta_2{}'}{\sigma}X_2{}'P'(P\Sigma(\phi)P')^{-1}PX_2\frac{\beta_2}{\sigma}\}]r^{m-1}dr$$

$$= (2\pi\sigma^2)^{-m/2}\left|P\Sigma(\phi)P'\right|^{-1/2}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}\int_0^\infty \exp[-\frac{1}{2}w'(P\Sigma(\phi)P')^{-1}w\{(\frac{r}{\sigma})^2 -$$

$$2\frac{r}{\sigma}\frac{w'(P\Sigma(\phi)P')^{-1}PX_2}{w'(P\Sigma(\phi)P')^{-1}w}\frac{\beta_2}{\sigma} + (\frac{\beta_2}{\sigma})'\frac{X_2{}'P'(P\Sigma(\phi)P')^{-1}PX_2}{w'(P\Sigma(\phi)P')^{-1}w}\frac{\beta_2}{\sigma}\}]r^{m-1}dr$$

$$= (2\pi)^{-m/2}\left|P\Sigma(\phi)P'\right|^{-1/2}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}$$

$$\int_0^\infty \exp\{-\frac{a_1(w,\phi)}{2}(\lambda - a_2(w,\phi,\beta^*))^2 + b(w,\phi,\beta^*)\}\lambda^{m-1}d\lambda$$

where

$$a_1(w,\phi) = w'(P\Sigma(\phi)P')^{-1}w\,, \qquad (3.10)$$

$$a_2(w,\phi,\beta^*) = \frac{w'(P\Sigma(\phi)P')^{-1}PX_2\beta^*}{w'(P\Sigma(\phi)P')^{-1}w}\,, \qquad (3.11)$$

$$b(w,\phi,\beta^*) = \frac{1}{2}[\frac{\beta^{*\prime}X_2{}'P'(P\Sigma(\phi)P')^{-1}ww'(P\Sigma(\phi)P')^{-1}PX_2\beta^*}{w'(P\Sigma(\phi)P')^{-1}w} \qquad (3.12)$$

$$-\beta^{*\prime}X_2{}'P'(P\Sigma(\phi)P')^{-1}PX_2\beta^*],$$

$\lambda = \frac{r}{\sigma}$ and $\beta^* = \frac{\beta_2}{\sigma}$.

$$f(\theta_1,\theta_2,...,\theta_{m-1}) = \exp(b(w,\phi,\beta^*))(2\pi)^{-m/2}\left|P\Sigma(\phi)P'\right|^{-1/2}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}$$

$$\int_0^\infty \exp\{-\frac{a_1(w,\phi)}{2}(\lambda - a_2(w,\phi,\beta^*))^2\}\lambda^{m-1}d\lambda$$

$$= \exp(b(w,\phi,\beta^*))(2\pi)^{-m/2}\left|P\Sigma(\phi)P'\right|^{-1/2}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}2^{m/2}a_1{}^{-m/2}(w,\phi)$$

$$\int_0^\infty \exp(-\eta^2)(\eta + \sqrt{a_1(w,\phi)}a_2(w,\phi,\beta^*)/\sqrt{2})^{m-1}d\eta$$

where $\eta^2 = \frac{a_1(w,\phi)}{2}(\lambda - a_2(w,\phi,\beta^*))^2$, $\lambda = \sqrt{(2/a_1(w,\phi))}\eta + a_2(w,\phi,\beta^*)$ and $d\lambda = \sqrt{(2/a_1(w,\phi))}d\eta$.

Now following a similar approach to integrating out r to that used earlier (see section 2) we get,

$$f(\theta_1,\theta_2,...,\theta_{m-1}) = \exp(b(w,\phi,\beta^*))\left|P\Sigma(\phi)P'\right|^{-1/2}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}a_1{}^{-m/2}(w,\phi)\pi^{-m/2}$$

$$[\frac{1}{2}\Gamma(m/2) + (m-1)c(w,\phi,\beta^*)\frac{1}{2}\Gamma(\frac{m-1}{2}) + \frac{(m-1)(m-2)}{2}c^2(w,\phi,\beta^*)\frac{1}{2}\Gamma(\frac{m-2}{2}) +$$

$$...+c^{m-1}(w,\phi,\beta^*)\frac{1}{2}\Gamma(1/2)]$$

where $c(w,\phi,\beta^*) = \frac{\sqrt{a_1(w,\phi)}a_2(w,\phi,\beta^*)}{\sqrt{2}} = 2^{-1/2}\frac{w'(P\Sigma(\phi)P')^{-1}PX_2\beta^*}{\{w'(P\Sigma(\phi)P')^{-1}w\}^{1/2}}$. (3.13)

Therefore the density function of maximal invariant w is,

$$f(w) = \frac{1}{2}\pi^{-m/2}\exp(b(w,\phi,\beta^*))\left|P\Sigma(\phi)P'\right|^{-1/2}a_1{}^{-m/2}(w,\phi)[\Gamma(m/2) +$$

$$+(m-1)c(w,\phi,\beta^*)\Gamma(\frac{m-1}{2}) + \frac{(m-1)(m-2)}{2}c^2(w,\phi,\beta^*)\Gamma(\frac{m-2}{2}) +$$

$$...+c^{m-1}(w,\phi,\beta^*)\Gamma(1/2)]. \qquad (3.14)$$

3.4.2 Construction of LBI test statistic

Taking logs on both sides of (3.14) we get,

$$\log f(w) = -\log 2 - \frac{m}{2}\log\pi - \frac{m}{2}\log a_1(w,\phi) + b(w,\phi,\beta^*) + \log\{\left|P\Sigma(\phi)P'\right|^{-1/2}\} +$$
$$\log\{\Gamma(m/2) + (m-1)c(w,\phi,\beta^*)\Gamma(\frac{m-1}{2}) + \frac{(m-1)(m-2)}{2}c^2(w,\phi,\beta^*)\Gamma(\frac{m-2}{2}) +$$
$$\ldots + c^{m-1}(w,\phi,\beta^*)\Gamma(1/2)\},$$

where $a_1(w,\phi)$, $b(w,\phi,\beta^*)$ and $c(w,\phi,\beta^*)$ are defined by (3.10), (3.12) and (3.13).

$$\frac{\partial \log f(w)}{\partial \beta^*} = [\frac{X_2{}'P'(P\Sigma(\phi)P')^{-1}ww'(P\Sigma(\phi)P')^{-1}PX_2\beta^*}{w'(P\Sigma(\phi)P')^{-1}w} - X_2{}'P'(P\Sigma(\phi)P')^{-1}PX_2\beta^*\}$$
$$+\{\Gamma(m/2) + (m-1)c(w,\phi,\beta^*)\Gamma(\frac{m-1}{2}) + \frac{(m-1)(m-2)}{2}c^2(w,\phi,\beta^*)\Gamma(\frac{m-2}{2}) +$$
$$\ldots + c^{m-1}(w,\phi,\beta^*)\Gamma(1/2)\}^{-1}\{\frac{(m-1)}{\sqrt{2}}\Gamma(\frac{m-1}{2})\sqrt{a_1(w,\phi)}\left(\frac{w'(P\Sigma(\phi)P')^{-1}PX_2}{w'(P\Sigma(\phi)P')^{-1}w}\right)' +$$
$$\frac{(m-1)(m-2)}{2}a_1(w,\phi)\frac{w'(P\Sigma(\phi)P')^{-1}PX_2\beta^*}{w'(P\Sigma(\phi)P')^{-1}w}\left(\frac{w'(P\Sigma(\phi)P')^{-1}PX_2}{w'(P\Sigma(\phi)P')^{-1}w}\right)' +$$
$$\ldots + \Gamma(1/2)(m-1)\left(\frac{w'(P\Sigma(\phi)P')^{-1}PX_2\beta^*}{w'(P\Sigma(\phi)P')^{-1}w}\right)^{m-2}\left(\frac{w'(P\Sigma(\phi)P')^{-1}PX_2}{w'(P\Sigma(\phi)P')^{-1}w}\right)'\}$$

$$\left.\frac{\partial \log f(w)}{\partial \beta *}\right|_{\beta^*=0} = \{\Gamma(m/2)\}^{-1}\frac{a_1^{1/2}(w,\phi)}{\sqrt{2}}(m-1)\Gamma(\frac{m-1}{2})\left(\frac{w'(P\Sigma(\phi)P')^{-1}PX_2}{w'(P\Sigma(\phi)P')^{-1}w}\right)'.$$

Hence the LBI test rejects H_0 for,

$$\{\Gamma(m/2)\}^{-1}\frac{a_1^{1/2}(w,\phi)}{\sqrt{2}}(m-1)\Gamma(\frac{m-1}{2})\left(\frac{w'(P\Sigma(\phi)P')^{-1}PX_2}{w'(P\Sigma(\phi)P')^{-1}w}\right)' \geq c_\alpha$$

or equivalently

$$\left\{w'(P\Sigma(\phi)P')^{-1}w\right\}^{1/2}\left(\frac{w'(P\Sigma(\phi)P')^{-1}PX_2}{w'(P\Sigma(\phi)P')^{-1}w}\right)' \geq d_\alpha$$

or

$$\frac{X_2{}'P'(P\Sigma(\phi)P')^{-1}w}{\left[w'(P\Sigma(\phi)P')^{-1}w\right]^{1/2}} \geq d_\alpha,$$

or

$$\frac{X_2{}' P'(P\Sigma(\phi)P')^{-1} Py / (y'M_1 y)^{1/2}}{\{y'P'(P\Sigma(\phi)P')^{-1} Py\}^{1/2} / (y'M_1 y)^{1/2}} \geq d_\alpha,$$

because $w = Py / (y'M_1 y)^{1/2}$, or

$$\frac{X_2{}' P'(P\Sigma(\phi)P')^{-1} Py}{\{y'P'(P\Sigma(\phi)P')^{-1} Py\}^{1/2}} \geq d_\alpha. \qquad (3.15).$$

Here (see King (1980))

$$\begin{aligned} X_2{}' P'(P\Sigma(\phi)P')^{-1} Py &= X_2{}' \{\Sigma(\phi)^{-1} - \Sigma(\phi)^{-1} X_1 (X_1{}' \Sigma(\phi)^{-1} X_1)^{-1} X_1{}' \Sigma(\phi)^{-1}\} y \\ &= (X_2{}' \Sigma(\phi)^{-1/2}{}')\{\Sigma(\phi)^{-1/2} - \Sigma(\phi)^{-1/2} X_1 (X_1{}' \Sigma(\phi)^{-1} X_1)^{-1} X_1{}' \Sigma(\phi)^{-1/2}\} y \\ &= X_2^{*\prime} M_1^* y^* \end{aligned}$$

where

$$y^* = \Sigma(\phi)^{-1/2} y$$

$$X_1^* = \Sigma(\phi)^{-1/2} X_1$$

$$X_2^* = \Sigma(\phi)^{-1/2} X_2$$

$$M_1^* = I - X_1^* (X_1^{*\prime} X_1^*)^{-1} X_1^{*\prime}.$$

Similarly

$$y'P'(P\Sigma(\phi)P')^{-1} Py = y^{*\prime} M_1^* y^*.$$

Thus the LBI test (3.10) can be written as

$$\frac{X_2^{*\prime} M_1^* y^*}{(y^{*\prime} M_1^* y^*)^{1/2}} \geq d_\alpha$$

where d_α is an appropriate critical value.

Therefore our required LBI test statistic is

$$s^* = \frac{X_2^{*\prime} M_1^* y^*}{(y^{*\prime} M_1^* y^*)^{1/2}} . \tag{3.16}$$

In the case of $p = 1$, we are interested in testing $H_0{:}\beta_2 = 0$ against $H_0{:}\beta_2 > 0$ for the regression equation

$$y^* = X_1^* \beta_1 + X_2^* \beta_2 + v , \tag{3.17}$$

where $v = \Sigma(\phi)^{-1/2} u$, $v \sim N(0, \sigma^2 I)$ and ϕ is known.

Using a similar procedure as the unstared case (3.5), we can show that the test statistic t is a monotonic increasing function of the LBI test s^*. Thus we may conclude that our LBI test is equivalent to the one-sided t test.

4.Tests of a Non-linear Regressor

Let us consider the following non-linear model,

$$y = X_1 \beta_1 + g(X_2, \beta_2) + u \tag{4.1}$$

where X_1 is an $n \times q$ nonstochastic matrix, X_2 is an $n \times p$ nonstochastic matrix and $g(X_2, \beta_2)$ is a non-linear function of β_2 and X_2, for which $g(X_2, \beta_2) = 0$ when $\beta_2 = 0$. We wish to test $H_0{:}\beta_2 = 0$ against $H_a{:}\beta_2 > 0$ in the context of the above non-linear regression model for $p = 1$.

4.1 Derivation of Density Function

Let $M_1 = I - X_1(X_1' X_1)^{-1} X_1'$. Then

$$\begin{aligned} M_1 y &= M_1 X_1 \beta_1 + M_1 g(X_2, \beta_2) + M_1 u \\ &= M_1 g(X_2, \beta_2) + M_1 u . \end{aligned} \tag{4.2}$$

Now multiplying (4.2) by P we get,

$$Py = Pg(X_2,\beta_2) + Pu\ ,$$

where P is an $m \times n$ matrix such that $PP' = I_m$, $P'P = M_1$ and $m = n - q$. Then $PM_1 y = Py$ and $PM_1 = P$.

Thus $\quad Py \sim N(Pg(X_2,\beta_2),\sigma^2 I_m)$. (4.3)

Let $z = Py$. We want to find the distribution of $w = z/(z'z)^{1/2}$. Here $z'z = r^2$.

From (4.3),

$$f(z) = (2\pi\sigma^2)^{-m/2}\exp\{-\frac{1}{2\sigma^2}(z - Pg(X_2,\beta_2))'(z - Pg(X_2,\beta_2))\}. \tag{4.4}$$

Now, changing z to the *m*-dimensional polar co-ordinates $(r,\theta_1,\theta_2,\ldots,\theta_{m-1})$ as in (2.5), the density of z becomes,

$$\begin{aligned} f(r,\theta_1,\theta_2,\ldots,\theta_{m-1}) &= (2\pi\sigma^2)^{-m/2}\exp\{-\frac{1}{2\sigma^2}(rw - Pg(X_2,\beta_2))' \\ &\qquad (rw - Pg(X_2,\beta_2))\} r^{m-1}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k} \\ &= (2\pi\sigma^2)^{-m/2}\exp\{-\frac{1}{2\sigma^2}(r^2 w'w - 2w'rPg(X_2,\beta_2) + \\ &\qquad g'(X_2,\beta_2)P'Pg(X_2,\beta_2)\} r^{m-1}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}. \end{aligned}$$

The density function of $(\theta_1,\theta_2,\ldots,\theta_{m-1})$ can be obtained by integrating out r,

$$\begin{aligned} f(\theta_1,\theta_2,\ldots,\theta_{m-1}) &= (2\pi\sigma^2)^{-m/2}\int_0^\infty \exp\{-\frac{1}{2\sigma^2}(r^2 w'w - 2w'rPg(X_2,\beta_2) + \\ &\qquad g'(X_2,\beta_2)P'Pg(X_2,\beta_2)\}\ r^{m-1}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}dr. \end{aligned}$$

Now using similar transformations to those we have used earlier for the linear case we get,

$$f(\theta_1,\theta_2,...,\theta_{m-1}) = (2\pi)^{-m/2}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}\int_0^\infty \exp\{-\frac{1}{2}(\lambda - a(w,\beta_2))^2 + b(w,\beta_2)\}\lambda^{m-1}d\lambda;$$

$$\text{where } a(w,\beta_2) = w'Pg^*(X_2,\beta_2)\,, \tag{4.5}$$

$$b(w,\beta_2) = \frac{1}{2}\{g^{*\prime}(X_2,\beta_2)P'ww'Pg^*(X_2,\beta_2) - g^{*\prime}(X_2,\beta_2)P'Pg^*(X_2,\beta_2)\}\,, \tag{4.6}$$

$$g^*(X_2,\beta_2) = \frac{g(X_2,\beta_2)}{\sigma}\,, \tag{4.7}$$

$\lambda = \dfrac{r}{\sigma}$ and $w'w = 1$.

$$f(\theta_1,\theta_2,...,\theta_{m-1}) = (2\pi)^{-m/2}\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}\exp(b(w,\beta_2))\int_0^\infty \exp\{-\frac{1}{2}(\lambda - a(w,\beta_2))^2\}\lambda^{m-1}d\lambda$$

$$= c\int_0^\infty \lambda^{m-1}\varphi\{(\lambda - a(w,\beta_2))^2\}d\lambda;$$

in which $c = \exp(b(w,\beta_2))\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}$ and

$$\varphi\{(\lambda - a(w,\beta_2))^2\} = (2\pi)^{-m/2}\exp\{-\frac{1}{2}(\lambda - a(w,\beta_2))^2\}\,.$$

Alternatively

$$f(\theta_1,\theta_2,...,\theta_{m-1}) = c\int_0^\infty (\eta + a(w,\beta_2))^{m-1}\varphi(\eta^2)d\eta\ ,$$

where $\eta = \lambda - a(w,\beta_2)$, $d\lambda = d\eta$ and $\lambda = \eta + a(w,\beta_2)$.

Following a similar procedure as for the linear case (section 2), we get

$$f(\theta_1,\theta_2,...,\theta_{m-1}) = \exp(b(w,\beta_2))\prod_{k=1}^{m-2}\sin\theta_k{}^{m-1-k}\{\frac{1}{2}\Gamma(m/2)\pi^{-m/2} +$$
$$\frac{1}{2}a(w,\beta_2)(m-1)\Gamma(\frac{m-1}{2})\pi^{-\frac{m-1}{2}} + \frac{1}{4}a^2(w,\beta_2)(m-1)(m-2)\Gamma(\frac{m-2}{2})\pi^{-\frac{m-2}{2}} +$$
$$...+\frac{1}{2}a^{m-1}(w,\beta_2)(2\pi)^{-m/2}2^{-3/2}\pi^{1/2}\}.$$

Therefore the density function of the maximal invariant w is,

$$\begin{aligned} f(w) &= \frac{1}{2}\pi^{-m/2}\exp(b(w,\beta_2))\{\Gamma(m/2) + a(w,\beta_2)(m-1)\Gamma(\frac{m-1}{2})\pi^{1/2} \\ &+ \frac{1}{2}a^2(w,\beta_2)(m-1)(m-2)\Gamma(\frac{m-2}{2})\pi + \ldots + a^{m-1}(w,\beta_2)2^{-\frac{m+1}{2}}\pi^{1/2}\} \end{aligned} \tag{4.8}$$

where $a(w,\beta_2)$ and $b(w,\beta_2)$ are defined by (4.5) and (4.6). Using this density function of the maximal invariant statistic, we can construct the LBI test for a non-linear regressor.

4.2. Tests of a Non-Linear Regressor

Taking logs of both sides of (4.8) we get

$$\begin{aligned} \log\{f(w)\} &= -\log 2 - \frac{m}{2}\log\pi + b(w,\beta_2) + \log\{\Gamma(m/2) + a(w,\beta_2)(m-1)\Gamma(\frac{m-1}{2})\pi^{1/2} \\ &+ \frac{1}{2}a^2(w,\beta_2)(m-1)(m-2)\Gamma(\frac{m-2}{2})\pi + \ldots + a^{m-1}(w,\beta_2)2^{-\frac{m+1}{2}}\pi^{1/2}\} \end{aligned}$$

$$\begin{aligned} &= -\log 2 - \frac{m}{2}\log\pi + \frac{1}{2}\{g^{*\prime}(X_2,\beta_2)P'ww'Pg^*(X_2,\beta_2) - g^{*\prime}(X_2,\beta_2)P'Pg^*(X_2,\beta_2)\} + \\ &\log\{\Gamma(m/2) + a(w,\beta_2)(m-1)\Gamma(\frac{m-2}{2})\pi^{1/2} + \frac{1}{2}a^2(w,\beta_2)(m-1)(m-2)\Gamma(\frac{m-2}{2})\pi + \\ &\ldots + a^{m-1}(w,\beta_2)2^{-\frac{m+1}{2}}\pi^{1/2}\}. \end{aligned}$$

Therefore the LBI test of $H_0{:}\beta_2 = 0$ against $H_a{:}\beta_2 > 0$ rejects H_0 for large values of $\left.\frac{\partial \log f(w)}{\partial \beta_2}\right|_{\beta_2=0}$ and results in the critical region

$$\frac{\left.\frac{\partial g^*(X_2,\beta_2)}{\partial\beta_2}\right|_{\beta_2=0} M_1 y}{(y'M_1y)^{1/2}} \geq c_\alpha . \tag{4.9}$$

Thus putting the value of $\left.\frac{\partial g^{*}(X_2,\beta_2)}{\partial\beta_2}\right|_{\beta_2=0}$ in the above function we get the LBI test statistic for testing $H_0{:}\beta_2=0$, where $g^{*}(X_2,\beta_2)$ in (4.7) is a non-linear function of X_2 and β_2.

5. **Concluding Remarks**

In this paper, having derived the density function of the maximal invariant statistic, we have constructed an LBI test in the linear regression model against one-sided alternatives. The principle of invariance is used to eliminate nuisance parameters in multiparameter one-sided testing problems. This allows the construction of the LBI test. The resultant LBI test is found to be equivalent to the one-sided *t*-test. In the case of a non-linear regression, we have derived the density function of the maximal invariant statistic and we have constructed the LBI test statistic for a non-linear regressor.

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