Consistent Implementation

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January 22, 2001

Abstract

In this paper we examine conditions under which a social choice function may be implemented consistently. That is, if agents possess consistent conjectures about each others behaviour, then the resulting equilibrium will be consistent. We examine the question of whether a social choice function can be implemented by equilibria with consistent conjectures. We relate the work to literature on both consistent conjectures and mechanism design.

1 Introduction

Mechanism design or implementation theory is concerned with designing mechanisms or game forms that lead to the successful implementation of a particular social choice function for a particular game theoretic solution concept. The implementation theory literature has examined a number of different solution concepts – dominant strategies, Nash equilibrium, Bayesian perfect equilibria, subgame perfect equilibria and other refinements.

In contrast, the industrial organization literature has been concerned with the extent to which behavioural conjectures, such as Nash behaviour, are consistent. By this we mean consistent in the sense that conjectural variations do in fact equate with the slopes of their respective player's reaction function. Nash equilibria and their refinements are in general not consistent in this sense, except in equilibrium. Thus, for example in oligopoly, Cournot-Nash behaviour is not consistent in this sense [4, 19, 3, 8].

An alternative and more general solution concept, is that of conjectural equilibria. Under conjectural equilibria, some functional form is generally

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assumed for the conjectural variation. A solution for the game is then sought subject to these more general conjectures. If these conjectural variation functions are appropriately chosen, so that the conjectural variation and the slope of the reaction functions are in fact equal, then the conjectural equilibrium will be consistent.

It is an open question whether or not consistency should be viewed as desirable. It is therefore important that implementation theory take into consideration whether or not the equilibria used to implement a particular social choice function does in fact involve consistent behaviour. This leads to the question which we examine in this paper. Is consistent implementation possible? We approach this problem more generally, as a problem of conjectural implementation, and then use a consistency property to establish the existence of a conjectural equilibrium that is also consistent. To do this we show that consistency implies the existence of a fixed point for a communication process that has been suitably augmented by a learning rule for behavioural conjectures.

In this paper we examine a situation of incomplete information, rather than complete information. We do not however follow the approach taken by the Bayesian implementation literature [15, 16, 17, 18, 7, 2]. The reason for this is that conjectural equilibria are not the same as Bayesian (perfect) Nash equilibria. Unlike Bayesian Nash equilibria, conjectural equilibria do not have an interpretation in terms of subjective probabilities. Conjectural variations have more in common with decision heuristics. In addition, as already stated, conjectural equilibria do not usually coincide with Nash equilibria.

2 Conjectural Variations and Conjectural Implementation

Conjectural variations are concerned with guesses that agents make about the likely behaviour of other agents. Formally, *conjectural variations* refer to the assumed or conjectured slope of an agent j's reaction function with respect to other agents' strategies. The slope of the actual reaction function of an agent participating in a game may differ from the slope conjectured by other agents. If the conjectured slope of the reaction function differs from the actual slope of the reaction function, conjectures are said to be *inconsistent*. If the conjectured slope is the same as the actual slope of the reaction function, then conjectures are said to be *consistent*.

In this section, we are interested in establishing the basic framework in order to design rules whereby social institutions can make collective choices.

In particular, we develop a framework whereby a social choice rule (organization) can be implemented by a mechanism (game form) in which the equilibria are consistent conjectural equilibria. We consider an economy/society with n agents, indexed by the agent set $N = \{1, ..., n\}$. Each agent $i \in N$ is characterized by a triple $e_i := (p_i, T_i, \omega_i)$ is consisting of agent i's preference profile p_i, i's set of conjectural variations T_i and i's endowment ω_i.The set of all possible characterizations of agent i is given by E_i and the set of all environments is give by

$$E = E_1 \times E_2 \times ...E_n. \tag{1}$$

We let M_i define the ith individuals message space and let each $m_i \in M_i$ represent a message or signal announced by the ith agent. The message space of the society is given by $M = M_1 \times M_2 \times \ldots M_n$. Note that $m \in M$ is a vector of messages announced by agents. At time s agents announce their messages according to the following update rule parametrized by the environment $e \in E$:

$$m(s) = f(m(s-1), e).$$

The message correspondence of the society $\mu : E \to M$ is defined as

$$\mu(e) := \{m \in M | f(m, e) - m = 0\}.$$

The outcome function of the game is the mapping $g : M \to C$ where C is the set of feasible actions for the economy. The definition of a mechanism can now be given

Definition 2.1 *(Mechanism) A* mechanism *defined on a family of environments E is a triple $\mathcal{M} := (M, \mu, g)$, where M is the message space, μ an equilibrium correspondence and g an outcome function.*

A social choice function $f : E \to C$ is a rule that assigns a collective choice $f(e_1, ..., e_n) \in C$ to an environment $e \in E$. Implementation theory associates a particular social choice rule with a mechanism. More formal definition is given as follows.

Definition 2.2 *(Implementation) A mechanism $\mathcal{M} = (M, \mu, g)$ implements a social choice function $f(.)$ if there is an equilibrium $\mu_1^*(.), ..., \mu_n^*(.)$ such that $g(\mu_1^*(.), ..., \mu_n^*(.)) = f(e_1, ..., e_n)$*

Under conjectural variations, agents announce strategies publicly, but also update their type according to a conjectural learning rule $\tau_i : M \times T_i \to T_i$. Each agent i's $(i = 1, ..., n)$ messages and conjectural variations are therefore updated according to the following system of coupled difference equations:

$$\begin{aligned} m_i(s+1) &= f^i(m_1(s), ..., m_i(s), ...m_n(s), t_i^1(s), \ldots, t_i^n(s)) \\ \vec{t}_i(s+1) &= \tau(m_1(s), ..., m_i(s), ...m_n(s), t_i^1(s), \ldots, t_i^n(s)) \end{aligned}$$

with initial state $m_1(0), ..., m_i(0), ...m_n(0), t_i^1(0), \ldots, t_i^n(0))$, where the ith agent's conjectural variation is represented by the vector $\vec{t}(s) := (t_i^1(s), \ldots, t_i^n(s)) \in T_i$ at time s. The response function of agent i can therefore be represented as a mapping $f^i : M \times T_i \to M_i$. Note that although each agent's announced strategies are public information, their conjectures about the behaviour of others are held privately and not revealed.

In order for conjectures to be consistent they must be equal to the slope of the reaction function at each $s \in \mathbb{R}_+$.We now provide a definition for the consistency of conjectures based on the convergence of a sequence of messages and conjectural variations. We will show in the next section, that if the sequence converges then a consistent conjectural equilibrium exists.

Definition 2.3 *(Consistent Conjectural Sequence) For each agent $i \in N$, at time s we let $m_i^{'}(s)$ denote the Jacobian of the message $m_i(s)$ with respect $m_j(s)$ for each $j \in N$. If the communication process generates a sequence of pairs $\left(m_i^{'}(s), \vec{t}_i(s)\right)$such that $t_i(s) \to m_i^{'}(s)$ as $s \to \infty$ for every agent $i \in N$, then we call such a sequence a* consistent conjectural sequence.

To show that implementation in consistent conjectures is possible, we need to first show that a conjectural equilibrium exists. The following theorem establishes the existence of a conjectural equilibria by utilizing the consistency property provided by the definition of a consistent conjectural sequence.

Theorem 2.1 *(Existence of a Consistent Conjectural Equilibrium) A consistent conjectural equilibrium exists, if the product space $X = M \times T$ is a convex compact subset of R^N, and if the augmented communication process $\tilde{f}$ is a continuous map from X into itself.*

Proof We need to show that a consistent conjectural sequence converges. First, consider the case of linear reaction functions. In this case, regardless of the chosen strategy of any given agent, the conjectural variation concerning the slope of that agent's reaction function will already have obtained a fixed point. Hence, to show existence of a consistent conjectural equilibrium one need only show existence of an equilibrium in behavioural strategies.

This will not be the case for nonlinear reaction functions. In the nonlinear case, we simply need to show existence of a fixed point for the system

$$\begin{aligned} m_i(s+1) &= f^i(m_1(s), ..., m_i(s), ...m_n(s), t_i^1(s), \ldots, t_i^n(s)) \\ \vec{t}_i(s+1) &= \tau(m_1(s), ..., m_i(s), ...m_n(s), t_i^1(s), \ldots, t_i^n(s)) \end{aligned}$$

for each $i \in N$ and for all s. We therefore, need to show that the system of $2n$ equations possesses a fixed point. Writing the above system more compactly, with $\tilde{f} : M \times T \rightarrow M \times T$, and taking $x \in M \times T$

$$x(s+1) = \tilde{f}(x(s)).$$

Because the strategy space M is a convex compact set, we need only show that T is a convex compact set. T consists of the space of derivatives of the reaction function. If these derivatives exist and are continuously differentiable, then continuity of $\tilde{f}$ is assured. Provided that the reaction functions have slopes lying between $[t_{\min}, t_{\max}]$ with $t_{\min} > -\infty$ and $t_{\max} < \infty$, then T will be a convex compact subset of R^n. Hence X will be convex and compact. Because X is a convex compact subset of R^n the mapping possesses a fixed point by Brouwer's fixed point theorem.QED.

3 Implementation in Consistent Conjectures

Now that we have established existence of a consistent conjectural equilibrium, we need to establish that a social choice function does in fact exist, and that it may be implemented by the outcome a conjectural variations game form. To do this we introduce the notion of an integer game. This is a standard approach used in Nash implementation theory to establish the conditions under which a social choice function may be implemented [9, 10]. Consistent implementation differs from Nash implementation in the nature of the equilibrium, but is close to Nash implementation in that behavioural conjectures play a role in both Nash and consistent implementation.

Here we present a modified version of the integer game found in Osborne and Rubinstein [14]. We define the following environment $\mathcal{E} :=< N, C, P, \mathcal{G} >$ for the integer game, where N is the set of agents, C is the space of outcomes, P the set of all preference profiles $\succeq$ defined to be partial orderings on C, and $\mathcal{G}$ is the set of possible game forms or mechanisms. We wish to construct a mechanism or game form $G =< N, (A_i)_{i \in N}, g >$ on this environment, where N is the set of agents, A_i is the ith agent's action space and $g : A \rightarrow C$ is the outcome function for G that maps over the action space $A := \times_{i\in N} A_i$. This game form G is defined as an integer game, where the action space of each agent $i \in N$ is defined by $A_i := (p_i, m_i, \vec{t_i}, k_i)$ consisting of an announced preference profile $p_i :=\succeq$, a message and a vector of conjectural variations, and an integer k_i.

The proof proceeds analogous to standard implementation proofs based on integer games. As player types are conjectural variations we will assume that they are private information, and are unknown to other players. The question can be posed as to how the outcome function can be evaluated by an integer, as in the final stage of this game it is necessary to compare types in order to determine the outcome. Conjectural variations are communicated to the social planner in a process similar to a sealed bid auction. By doing this an integer game can still be employed to evaluate the outcome function, but at the same time preserve the confidential nature of conjectural variations.

Formally consistent implementation will therefore require some concept of privacy preservation. Privacy preservation relies on the message correspondence being a *coordinate correspondence*, so we first define what is meant by a coordinate correspondence before introducing the formal definition of *privacy preservation.*

Definition 3.1 *(Coordinate correspondence) Given a family of environments $E = E_1 \times E_2 \times ...E_n$, where each E_i is the family of characteristics of agent i. The correspondence $\mu : E \to M$ is called a* coordinate correspondence *if $\forall i \in N, \exists \mu_i : E_i \to M$ such that $\mu(e) = \cap \mu_{i\in N}(e_i) \, \forall e \in E$.*

Definition 3.2 *(Privacy Preservation) If the correspondence $\mu(e)$ associated with some mechanism $\mathcal{M}$ is a coordinate correspondence then $\mathcal{M}$ is* privacy preserving.

In addition a consistent implementable social choice function f should be monotonic. To explain, the property of monotonicity in social choice functions, the following definition is provided.

Definition 3.3 *(Monotonicity) A social choice rule $f : P \to C$ is said to be* monotonic *if there exists a $c \in C$ such that $c \in f(\succeq)$ and $c \notin f\left(\succeq'\right)$ implies there is an agent $i \in N$ and $b \in C$ such that $c \succeq_i b$ and $b \succ_i' c$.*

The reader will note that monotonicity is also a necessary condition for the social choice functions being Nash implementable. However social choice functions also require no veto power as well as monotonicity as a sufficient condition to be Nash implementable. We now provide the following definition for the no veto property.

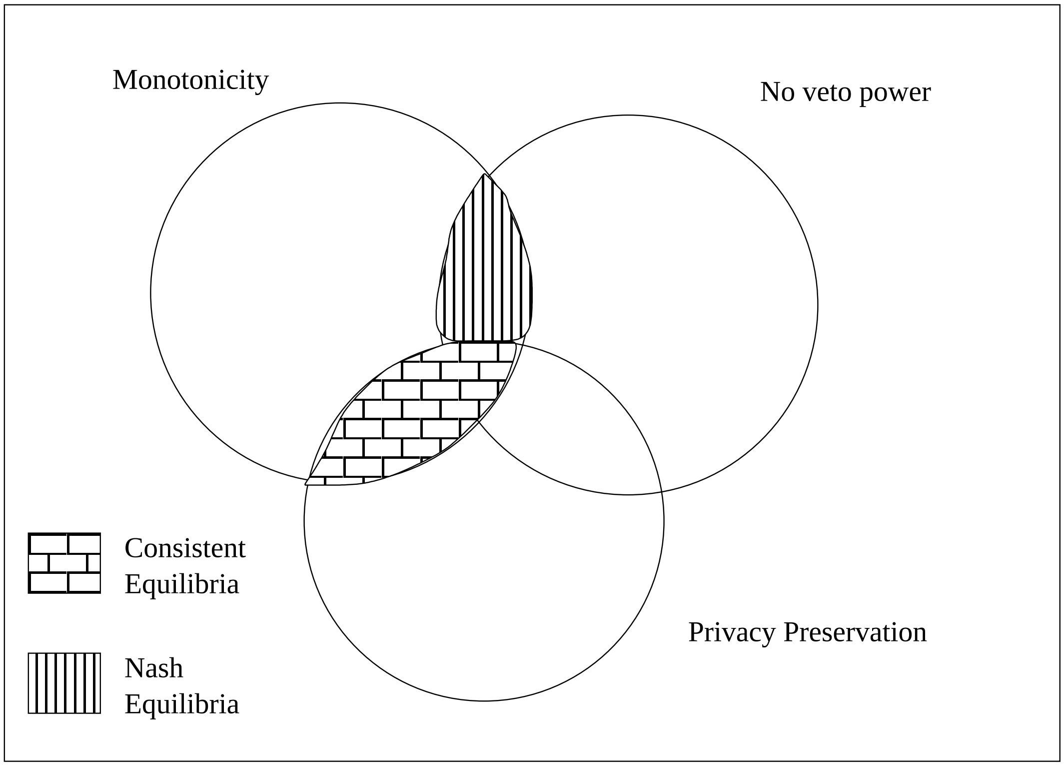

Figure 1: the relationship between monotonicity, no veto power and privacy preservation

Definition 3.4 *(No Veto Power) A social choice rule $f : P \to C$ possesses the* no veto power *property if $c \in f(\succeq)$ iff for at least $|N| - 1$ agents $c \succeq_i y$, for any $y \in C$.*

The relationship between consistent implementation and Nash implementation is illustrated by figure 1.

It can be seen that in general privacy preservation is a condition that is not necessarily required for Nash implementation. The reason for this is that Nash implementation is conducted under the assumption of complete information.

Theorem 3.1 *(Implementation) A mechanism (game form) $\mathcal{M} = (M, \mu, g)$ consistently implements a social choice function $f(.)$ if there is a consistent equilibrium $\mu_1^*(.), ..., \mu_n^*(.)$ such that $g(\mu_1^*(.), ..., \mu_n^*(.)) \in f(\succeq)$.*

Proof Recall that a consistent conjectural equilibrium is a pair (m^*, t^*), in the integer game for agent $i \in N$ we replace this with the augmented strategy quadruple $a_i = (p^*, m*, t^*, k^*)$. The outcome of the game is $c^* = g(m^*)$. We need to show that $c^* \in f(\succeq)$. Note that $c^* = (m^*, t^*)$. Define an outcome function as follows: if $\exists\, j \in N$ and $\exists\, (\succeq, c, k) = (p_i, c_i, k_i)$, with $c \in f(\succeq)$ $\forall i \in N \setminus \{j\}$, then

i)

$$g((p_i, c_i, k_i)) = \left\{ \begin{array}{l} c_j, \text{ if } c \succeq_j c_j \\ c, \text{ if } c \prec_j c_j \end{array} \right\}$$

ii) Otherwise $g((p_i, c_i, k_i)) = c_k$ where k is an integer such that $t_k > t_j$ $\forall j \in N$. $t \in T$.

Let $c \in f(\succeq)$, then an equilibrium $a_i = (p, c, 0)$ will be a consistent conjectural equilibrium of the integer game. Any deviation (p', c', k') will have the consequence that $c \prec_j c_j$. Consider a consistent conjectural equilibrium a_i^* of the game $< G, \succeq >$. We first show that $c^* \in f(\succeq)$. We consider three cases

Case 1 Consider an equilibrium $a_i^* = (p^{'}, c^*, k^{'}) \forall i \in N$ such that $c^* \in f(e)$. If this were not so then $c^* \notin f(e)$ but if this were the case monotonicity of f implies there is an agent for whom deviation is profitable, i.e. $\exists b \in C$ such that $c^* \succeq_i^{'} b$ and $b \succ_i c^*$ But then player i would choose b rather than c^*.

Case 2 Consider an $a_i = (\succeq^{'}, c^*, k^{'}), \forall i \in N$ and $c^* \notin f(\succeq^{'})$. If $\exists b \in C$ such that $b \succ_i c^*$ then it pays agent i to deviate to $(\succeq^{'}, b, k^{''})$ where $k^{''} > k^{'}$ (by property ii) of the outcome function. This would make the outcome b preferable to the consistent equilibrium c^*. Hence c^* is the preferred outcome of all agents except agent i. If $f(\succeq)$ has the no veto power property then $c^* \in f(\succeq)$. But what if $f(\succeq)$ has veto power. If this is the case, then any attempt by agent i to veto the outcome c^* will result $\mu(e) \notin \cap \mu_{j \in N}(e_j)$. Hence, the message correspondence will not be a coordinate correspondence and consequently $\mathcal{M}$ will not possess the *privacy preservation* property. Thus $c^* \in f(\succeq)$ but only if agents publicly announce their conjectures. If conjectural variations are private information, then we exclude this situation from the beginning. If $\mathcal{M}$ is *privacy preserving* then no agent can veto an outcome and we can drop the requirement of no veto power.

Case 3 Suppose that $\exists i, j$ such that $a_i^* \neq a_j^*$. We show either that for at least $|N| - 1$ agents c^* is the most preferred outcome or alternatively, that under privacy preservation c^* is the most preferred outcome. Because privacy preservation subsumes no veto power, as shown in Case 2, it suffices to show that under privacy preservation $c^* \in f(\succeq)$. This can now be shown. First assume $c^* \notin f(\succeq)$ then $\exists h$ who can deviate by choosing $(\succeq', b, k'')$ for some $k'' > k_l \; \forall l \neq h$. However such a player h is then exercising a right of veto over the outcome $c^* \in f(\succeq)$. The alternative equilibrium $b \succ_h c^*$ resulting from h's exercise of veto power cannot be a coordinate correspondence because $\mu(e) \notin \cap \mu_{j \in N}(e_j)$. As a consequence of this $\mu_h(e_h) \cap \mu_j(e_j) = \emptyset, \forall j \in N \setminus \{h\}$. Thus $c^* \in f(\succeq)$ so long as privacy preservation holds.

QED.

Remark 3.1 *An example of a game form that consistently implements some social choice function without requiring no veto power, is a Stackelberg game. In this game form the leader can effectively veto certain outcomes. Because conjectures remain private, players cannot update their conjectures and are effectively pre-programmed as either leader or follower. If leadership were endogenous, one may conjecture that no veto power would be required for consistent implementation.*

4 Conclusion

In this paper, we show that a mechanism exists which implements a social choice function in consistent conjectural equilibria. The sufficient conditions for the existence of a consistently implementable social choice function is that it is monotonic and the mechanism or game form possesses a privacy preservation property. We show this by first modifying the communication process of Hurwicz [6] and Mount and Reiter [12, 13] to provide a more general definition of consistent equilibrium. We then provide a new version of the integer game that incorporates aspects of sealed bid auctions in which agents communicate conjectural variations to the social planner, but conceal this information from other players.

Possible extensions of the work concern questions of informational viable. It would be interesting to know whether or not there are mechanisms that possess social choice functions that are

i) consistently implementable, but not informationally viable; or

ii) informationally viable, but not consistently implementable.

With regard to the latter, we note that some non-Walrasian mechanisms are known to be informationally viable, for example the model of the firm presented by Milgrom and Roberts [11]. Further work also needs to be done on the relationship between Bayesian and consistent implementation, as both relate to the problem of designing mechanisms for environments with incomplete information. The results presented in this paper also have implication for the design of economic policy in imperfectly competitive markets. It is an open question whether or not consistency can be viewed as a desirable quality in markets.

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