Supplement to "Bond risk premia in consumption-based models"

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The Appendix contains the following supplementary material to Creal and Wu (2020): Appendix A discusses the stochastic discount factor, Appendix B lays out the dynamics of the state vector, Appendix C discusses the solution for models with recursive preferences, and Appendix D computes bond prices analytically. Appendix E contains proofs of the propositions in the paper, while Appendix F provides numerical illustrations for the feasible parameter regions. Appendix G provides more details in terms of estimating the time series component of the model, and Appendix H provides details for Sharpe ratios.

Appendix A: Stochastic discount factor

This Appendix provides the derivation for the stochastic discount factor for the agent's problem

$$V_t = \max_{C_t}\big[(1-\beta)Y_t C_t^{1-\eta} + \beta\{E_t[V_{t+1}^{1-\gamma}]\}^{\frac{1-\eta}{1-\gamma}}\big]^{\frac{1}{1-\eta}}, \tag{A.1}$$

$$W_{t+1} = (W_t - C_t)R_{c,t+1}, \tag{A.2}$$

Guess that the solution is $V_t = \phi_t W_t$ for some coefficients ϕ_t, then the agent's problem becomes

$$\phi_t W_t = \max_{C_t}\big[(1-\beta)H_t^{\eta-1}C_t^{1-\eta} + \beta\{E_t[(\phi_{t+1}W_{t+1})^{1-\gamma}]\}^{\frac{1-\eta}{1-\gamma}}\big]^{\frac{1}{1-\eta}}.$$

Substitute in W_{t+1} from the constraint (A.2)

$$\phi_t^{1-\eta} = \max_{C_t}\bigg[(1-\beta)H_t^{\eta-1}\Big(\frac{C_t}{W_t}\Big)^{1-\eta} + \beta\Big(1-\frac{C_t}{W_t}\Big)^{1-\eta}\{E_t[(\phi_{t+1}R_{c,t+1})^{1-\gamma}]\}^{\frac{1-\eta}{1-\gamma}}\bigg]. \tag{A.3}$$

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Take the first-order condition w.r.t. C_t, we get

$$(1-\beta)H_t^{\eta-1}\left(\frac{C_t}{W_t}\right)^{-\eta} = \beta\left(1-\frac{C_t}{W_t}\right)^{-\eta}\left\{E_t\left[(\phi_{t+1}R_{c,t+1})^{1-\gamma}\right]\right\}^{\frac{1-\eta}{1-\gamma}}. \tag{A.4}$$

Use the first-order condition to substitute out the expectation term in (A.3) to solve ϕ_t,

$$\phi_t = (1-\beta)^{\frac{1}{1-\eta}}H_t^{-1}\left(\frac{C_t}{W_t}\right)^{-\frac{\eta}{1-\eta}}.$$

Substitute back to the FOC in (A.4), and and use the budget constraint to get the pricing equation

$$1 = \beta^\vartheta E_t\left[\left(\frac{H_{t+1}^{\eta-1}}{H_t^{\eta-1}}\right)^\vartheta \left(\frac{C_{t+1}}{C_t}\right)^{-\eta\vartheta} R_{c,t+1}^\vartheta\right].$$

Therefore, the pricing kernel is

$$M_{t+1} = \beta^\vartheta \left(\frac{H_{t+1}^{\eta-1}}{H_t^{\eta-1}}\right)^\vartheta \left(\frac{C_{t+1}}{C_t}\right)^{-\eta\vartheta} R_{c,t+1}^{\vartheta-1},$$

and the log SDF is

$$m_{t+1} = \vartheta\ln(\beta) + \vartheta\Delta v_{t+1} - \eta\vartheta\Delta c_{t+1} + (\vartheta-1)r_{c,t+1}, \tag{A.5}$$

APPENDIX B: DYNAMICS OF THE STATE VECTOR

B.1 *General model*

The dynamics of the Gaussian state vector g_t driving Δc_t and π_t are

$$g_{t+1} = \mu_g + \Phi_g g_t + \Phi_{gh}h_t + \Sigma_{gh}\varepsilon_{h,t+1} + \Sigma_{g,t}\varepsilon_{g,t+1}, \quad \varepsilon_{g,t+1} \sim N(0, I),$$

$$\Sigma_{g,t}\Sigma_{g,t}' = \Sigma_{0,g}\Sigma_{0,g}' + \sum_{i=1}^{H}\Sigma_{i,g}\Sigma_{i,g}' h_{it},$$

$$\varepsilon_{h,t+1} = h_{t+1} - E_t[h_{t+1}|h_t],$$

where the volatility dynamics are a noncentral gamma process. They can be written as a Gamma distribution and a Poisson distribution

$$h_{t+1} = \Sigma_h w_{t+1},$$

$$w_{i,t+1} \sim \text{Gamma}(\nu_{h,i} + z_{i,t+1}, 1), \quad i = 1, \ldots, H, \tag{B.1}$$

$$z_{i,t+1} \sim \text{Poisson}(e_i'\Sigma_h^{-1}\Phi_h\Sigma_h w_t), \quad i = 1, \ldots, H. \tag{B.2}$$

This is a discrete-time, multivariate Cox, Ingersoll, and Ross (1985) process. To guarantee positivity and existence of h_t, the process requires $\Sigma_h > 0$, $\Sigma_h^{-1}\Phi_h\Sigma_h > 0$ and the Feller

condition $\nu_{h,i} > 1$ for $i = 1, \dots, H$. The conditional mean and variance of the process are

$$E_t[h_{t+1}|h_t] = \Sigma_h \nu_h + \Phi_h h_t, \tag{B.3}$$

$$V_t[h_{t+1}|h_t] = \Sigma_{h,t} \Sigma'_{h,t}$$

$$= \Sigma_h \operatorname{diag}(\nu_h) \Sigma'_h + \Sigma_h \operatorname{diag}(2\Sigma_h^{-1}\Phi_h h_t) \Sigma'_h, \tag{B.4}$$

where Σ_h is a $H \times H$ matrix of scale parameters, Φ_h is a $H \times H$ matrix of autoregressive parameters, and the intercept is equal to $\Sigma_h \nu_h$. The unconditional mean and variance of h_t are $\bar{\mu}_h = (I_H - \Phi_h)^{-1}\Sigma_h \nu_h$ and $\bar{\Sigma}_h \bar{\Sigma}'_h = (I_H - \Phi_h)^{-1}\Sigma_h \operatorname{diag}(\nu_h)\Sigma'_h(I_H - \Phi_h)^{-1,\prime}$. The unconditional mean of g_t is $\bar{\mu}_g = (I_G - \Phi_g)^{-1}(\mu_g + \Phi_{gh}\bar{\mu}_h)$. The transition density of h_t is

$$p(h_{t+1}|h_t, \nu_h, \Phi_h, \Sigma_h) = |\Sigma_h^{-1}| \prod_{i=1}^{H} (e'_i \Sigma_h^{-1} h_{t+1})^{\frac{\nu_{h,i}-1}{2}} (e'_i \Sigma_h^{-1}\Phi_h h_t)^{-\frac{\nu_{h,i}-1}{2}}$$

$$\times \exp\left(-\sum_{i=1}^{H} e'_i \Sigma_h^{-1} h_{t+1} + e'_i \Sigma_h^{-1}\Phi_h h_t\right)$$

$$\times I_{\nu_{h,i}-1}\left(2\sqrt{(e'_i \Sigma_h^{-1} h_{t+1})(e'_i \Sigma_h^{-1}\Phi_h h_t)}\right), \tag{B.5}$$

where $I_\nu(x)$ is the modified Bessel function. The Laplace transform needed to solve the model with recursive preferences and for pricing assets is

$$\mathbb{E}_t\big[\exp(u'h_{t+1})\big] = \exp\left(\sum_{i=1}^{H} \frac{e'_i \Sigma'_h u}{1 - e'_i \Sigma'_h u} e'_i \Sigma_h^{-1}\Phi_h h_t - \sum_{i=1}^{H} \nu_{h,i} \log(1 - e'_i \Sigma'_h u)\right),$$

which exists only if $e'_i \Sigma'_h u < 1$ for $i = 1, \dots, H$. Further properties of the univariate process are developed by Gouriéroux and Jasiak (2006).

B.2 Long run risk with 2 stochastic volatility factors

In the paper, we include estimation results for a model with 2 stochastic volatility factors:

$$\pi_{t+1} = \bar{\pi}_t + \sqrt{h_{t,\pi}}\varepsilon_{\pi_1,t+1},$$

$$\Delta c_{t+1} = \bar{c}_t + \sqrt{h_{t,c}}\varepsilon_{c_1,t+1},$$

$$\bar{\pi}_{t+1} = \mu_\pi + \phi_\pi \bar{\pi}_t + \phi_{\pi,c}\bar{c}_t + \sigma_\pi \sqrt{h_{t,\pi}}\varepsilon_{\pi_2,t+1},$$

$$\bar{c}_{t+1} = \mu_c + \phi_{c,\pi}\bar{\pi}_t + \phi_c \bar{c}_t + \sigma_{c,\pi}\sqrt{h_{t,\pi}}\varepsilon_{\pi_2,t+1} + \sigma_c \sqrt{h_{t,c}}\varepsilon_{c_2,t+1}.$$

This model maps into the general companion form as follows:

$$g_t = \begin{pmatrix} \pi_t \\ \Delta c_t \\ \bar{\pi}_t \\ \bar{c}_t \end{pmatrix}, \qquad Z_c = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad Z_\pi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mu_g = \begin{pmatrix} 0 \\ 0 \\ \mu_\pi \\ \mu_c \end{pmatrix}, \qquad \bar{\mu}_g = \begin{pmatrix} \bar{\mu}_\pi \\ \bar{\mu}_c \\ \bar{\mu}_\pi \\ \bar{\mu}_c \end{pmatrix},$$

$$\Phi_g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \phi_\pi & \phi_{\pi,c} \\ 0 & 0 & \phi_{c,\pi} & \phi_c \end{pmatrix}, \qquad \Phi_{gh} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Sigma_{gh} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Sigma_{0,g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Sigma_{1,g} = \begin{pmatrix} \frac{1}{\sqrt{1200}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Sigma_{2,g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1200}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Sigma_{3,g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{1200}} & 0 \\ 0 & 0 & \frac{\sigma_{c,\pi}}{\sqrt{1200}} & 0 \end{pmatrix},$$

$$\Sigma_{4,g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{1200}} \end{pmatrix}.$$

We have scaled these matrices by $1/\sqrt{1200}$ during estimation so that the volatility factors h_t are roughly the same magnitude as the Gaussian factors g_t. For the volatility processes, the matrices are

$$\bar{\mu}_h = \begin{pmatrix} \bar{\mu}_{h,\pi} \\ \bar{\mu}_{h,c} \end{pmatrix}, \qquad \nu_h = \begin{pmatrix} \nu_{h,\pi} \\ \nu_{h,c} \end{pmatrix}, \qquad \Phi_h = \begin{pmatrix} \phi_\pi & 0 \\ 0 & \phi_c \end{pmatrix},$$

$$\Sigma_h = \begin{pmatrix} \sigma_{h,\pi} & 0 \\ 0 & \sigma_{h,c} \end{pmatrix}.$$

During estimation, we parameterize the model in terms of the unconditional mean of volatilities $\bar{\mu}_h$.

Gaussian model   For the Gaussian model, we keep everything the same as above except for the scale matrix which is equal to

$$\Sigma_{0,g} = \begin{pmatrix} \sigma_{\pi_1} & 0 & 0 & 0 \\ 0 & \sigma_{c_1} & 0 & 0 \\ 0 & 0 & \sigma_{\pi_2} & 0 \\ 0 & 0 & \sigma_{c,\pi} & \sigma_{c_2} \end{pmatrix},$$

while $\Sigma_{i,g} = 0$ for $i > 0$, and $\bar{\mu}_h, \nu_h = 0$, $\Phi_h = 0$, $\Sigma_h = 0$.

B.3 *Long run risk with* 4 *stochastic volatility factors*

In previous versions of the paper, we estimated models with four volatility factors:

$$\pi_{t+1} = \bar{\pi}_t + \sqrt{h_{t,\pi_1}}\varepsilon_{\pi_1,t+1}, \quad \varepsilon_{\pi_1,t+1} \sim N(0,1),$$

$$\Delta c_{t+1} = \bar{c}_t + \sqrt{h_{t,c_1}}\varepsilon_{c_1,t+1}, \quad \varepsilon_{c_1,t+1} \sim N(0,1),$$

$$\bar{\pi}_{t+1} = \mu_\pi + \phi_\pi \bar{\pi}_t + \phi_{\pi,c}\bar{c}_t + \sqrt{h_{t,\pi_2}}\varepsilon_{\pi_2,t+1}, \quad \varepsilon_{\pi_2,t+1} \sim N(0,1),$$

$$\bar{c}_{t+1} = \mu_c + \phi_{c,\pi}\bar{\pi}_t + \phi_c \bar{c}_t + \sigma_{c,\pi}\sqrt{h_{t,\pi_2}}\varepsilon_{\pi_2,t+1} + \sqrt{h_{t,c_2}}\varepsilon_{c_2,t+1}, \quad \varepsilon_{c_2,t+1} \sim N(0,1).$$

Below, we report estimation results for this model as well. This model maps into the general form as follows:

$$g_t = \begin{pmatrix} \pi_t \\ \Delta c_t \\ \bar{\pi}_t \\ \bar{c}_t \end{pmatrix}, \quad Z_c = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad Z_\pi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mu_g = \begin{pmatrix} 0 \\ 0 \\ \mu_\pi \\ \mu_c \end{pmatrix}, \quad \bar{\mu}_g = \begin{pmatrix} \bar{\mu}_\pi \\ \bar{\mu}_c \\ \bar{\mu}_\pi \\ \bar{\mu}_c \end{pmatrix},$$

$$\Phi_g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \phi_\pi & \phi_{\pi,c} \\ 0 & 0 & \phi_{c,\pi} & \phi_c \end{pmatrix}, \quad \Phi_{gh} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Sigma_{gh} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Sigma_{0,g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Sigma_{1,g} = \begin{pmatrix} \dfrac{1}{\sqrt{12{,}000}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Sigma_{2,g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \dfrac{1}{\sqrt{12{,}000}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Sigma_{3,g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{\sqrt{12{,}000}} & 0 \\ 0 & 0 & \dfrac{\sigma_{c,\pi}}{\sqrt{12{,}000}} & 0 \end{pmatrix},$$

$$\Sigma_{4,g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\sqrt{12{,}000}} \end{pmatrix}.$$

We have scaled these matrices by $1/\sqrt{12{,}000}$ so that the volatility factors h_t are roughly the same magnitude as the Gaussian factors g_t. For the volatility processes, the matrices

are

$$
\bar{\mu}_h = \begin{pmatrix} \bar{\mu}_{h,\pi_1} \\ \bar{\mu}_{h,c_1} \\ \bar{\mu}_{h,\pi_2} \\ \bar{\mu}_{h,c_2} \end{pmatrix}, \qquad \nu_h = \begin{pmatrix} \nu_{h,\pi_1} \\ \nu_{h,c_1} \\ \nu_{h,\pi_2} \\ \nu_{h,c_2} \end{pmatrix}, \qquad \Phi_h = \begin{pmatrix} \phi_{\pi_1} & 0 & 0 & 0 \\ 0 & \phi_{c_1} & 0 & 0 \\ 0 & 0 & \phi_{\pi_2} & 0 \\ 0 & 0 & 0 & \phi_{c_2} \end{pmatrix},
$$

$$
\Sigma_h = \begin{pmatrix} \sigma_{h,\pi_1} & 0 & 0 & 0 \\ 0 & \sigma_{h,c_1} & 0 & 0 \\ 0 & 0 & \sigma_{h,\pi_2} & 0 \\ 0 & 0 & 0 & \sigma_{h,c_2} \end{pmatrix}.
$$

During estimation, we parameterize the model in terms of the unconditional mean of volatilities $\bar{\mu}_h$.

APPENDIX C: RECURSIVE PREFERENCES MODEL SOLUTION

C.1 *Solution for $r_{c,t+1}$*

In order to simplify the expressions, we introduce the following notation:

$$
\begin{aligned}
Z_1 &= (1-\eta)Z_c + \kappa_1 D_g, \\
Z_2 &= -\gamma Z_c + (\vartheta - 1)\kappa_1 D_g, \\
Z_3 &= \Sigma'_{gh}\big((1-\eta)Z_c + \kappa_1 D_g\big) + \kappa_1 D_h \\
&= \Sigma'_{gh} Z_1 + \kappa_1 D_h, \\
Z_4 &= \Sigma'_{gh}\big(-\gamma Z_c + (\vartheta - 1)\kappa_1 D_g\big) + (\vartheta - 1)\kappa_1 D_h \\
&= \Sigma'_{gh} Z_2 + (\vartheta - 1)\kappa_1 D_h,
\end{aligned}
$$

where the vectors Z_c, Z_π are selection vectors and the vectors D_g and D_h are part of the price to consumption ratio $pc_t = D_0 + D'_g g_t + D'_h h_t$.

Step 1: Campbell–Shiller approximation   Let $pc_t = \ln(\frac{P_t}{C_t})$ be the log price to consumption ratio. The return on the consumption asset is

$$
\begin{aligned}
r_{c,t+1} &\equiv \ln\left(\frac{P_{t+1} + C_{t+1}}{P_t}\right) = \ln(C_{t+1}) + \ln\left(\frac{P_{t+1} + C_{t+1}}{C_{t+1}}\right) - \ln(P_t) \\
&= \ln(C_{t+1}) - \ln(C_t) + \ln\left(1 + \frac{P_{t+1}}{C_{t+1}}\right) - \ln(P_t) + \ln(C_t) \\
&= \Delta c_{t+1} - pc_t + \ln\big(1 + \exp(pc_{t+1})\big).
\end{aligned}
$$

Take a first-order Taylor expansion of the function $f(x) = \ln(1 + \exp(x))$ around $\bar{x}$:

$$
\begin{aligned}
r_{c,t+1} &\approx \Delta c_{t+1} - pc_t + \ln\big(1 + \exp(\bar{pc})\big) + \frac{\exp(\bar{pc})}{1 + \exp(\bar{pc})}(pc_{t+1} - \bar{pc}) \\
&= \kappa_0 + \kappa_1 pc_{t+1} - pc_t + \Delta c_{t+1}, \tag{C.1}
\end{aligned}
$$

where $\kappa_0 = \ln(1 + \exp(\bar{p}c)) - \kappa_1 \bar{p}c$ and $\kappa_1 = \frac{\exp(\bar{p}c)}{1+\exp(\bar{p}c)}$.

Step 2: Solve for the price/consumption ratio   The real pricing kernel in (A.5) prices the consumption asset:

$$
\begin{aligned}
1 = E_t\big[\exp(m_{t+1} + r_{c,t+1})\big] &= E_t\big[\exp\big(\vartheta \ln(\beta) + \vartheta \Delta v_{t+1} - \eta\vartheta \Delta c_{t+1} + \vartheta r_{c,t+1}\big)\big] \\
&= \exp\big(\vartheta \ln(\beta) + \vartheta\kappa_0 - \vartheta pc_t\big)E_t\big[\exp\big(\vartheta\Delta v_{t+1} + \vartheta(1-\eta)\Delta c_{t+1} + \vartheta\kappa_1 pc_{t+1}\big)\big], \quad \text{(C.2)}
\end{aligned}
$$

where we have used (C.1). Conjecture a solution for the price to consumption ratio

$$
pc_t = D_0 + D_g' g_t + D_h' h_t \tag{C.3}
$$

for unknown coefficients D_0, D_g and D_h. Substitute the guess and the dynamics of Δc_t and the preference shock into (C.2),

$$
\begin{aligned}
1 &= \exp\big(\vartheta \ln(\beta) + \vartheta\kappa_0 + \vartheta\kappa_1 D_0 - \vartheta pc_t + \vartheta\Lambda_1(g_t)\big) & \text{(C.4)} \\
&\times \exp\big(\vartheta Z_1'\big(\mu_g + \Phi_g g_t + \Phi_{gh}h_t - \Sigma_{gh}(\Sigma_h \nu_h + \Phi_h h_t)\big)\big) & \text{(C.5)} \\
&\times E_t\big[\exp\big((\vartheta\Lambda_2(g_t) + \vartheta\Sigma_{g,t}' Z_1)'\varepsilon_{g,t+1}\big)\big]E_t\big[\exp\big(\vartheta Z_3' h_{t+1}\big)\big]. & \text{(C.6)}
\end{aligned}
$$

Calculate the expectations using the Laplace transform

$$
\begin{aligned}
0 &= \vartheta \ln(\beta) + \vartheta\kappa_0 + \vartheta\kappa_1 D_0 - \vartheta pc_t + \vartheta\Lambda_1(g_t) \\
&+ \vartheta Z_1'\big(\mu_g + \Phi_g g_t + \Phi_{gh}h_t - \Sigma_{gh}(\Sigma_h \nu_h + \Phi_h h_t)\big) \\
&+ \frac{\vartheta^2}{2}\big(\Lambda_2(g_t) + \Sigma_{g,t}' Z_1\big)'\big(\Lambda_2(g_t) + \Sigma_{g,t}' Z_1\big) \\
&- \sum_{i=1}^{H} \nu_{h,i} \ln\big(1 - e_i'\Sigma_h' \vartheta Z_3\big) + \sum_{i=1}^{H} \frac{e_i'\Sigma_h' \vartheta Z_3}{1 - e_i'\Sigma_h' \vartheta Z_3} e_i'\Sigma_h^{-1}\Phi_h h_t.
\end{aligned}
$$

The solution exists if $e_i'\Sigma_h' \vartheta Z_3 < 1$ for $i = 1, \ldots, H$. Solve for pc_t by plugging in the values of $\Lambda_1(g_t)$ and $\Lambda_2(g_t)$ and cancel terms

$$
\begin{aligned}
pc_t &= \ln(\beta) + \kappa_0 + \kappa_1 D_0 \\
&+ Z_1'\big(\mu_g + \Phi_g g_t + \Phi_{gh}h_t - \Sigma_{gh}(\Sigma_h \nu_h + \Phi_h h_t)\big) \\
&- \eta\vartheta Z_1'(\lambda_0 + \lambda_g g_t) \\
&+ \frac{\vartheta}{2} Z_1'\Sigma_{g,t}\Sigma_{g,t}' Z_1 - \frac{1}{\vartheta}\sum_{i=1}^{H} \nu_{h,i} \log\big(1 - e_i'\Sigma_h' \vartheta Z_3\big) + \sum_{i=1}^{H} \frac{e_i'\Sigma_h' Z_3}{1 - \vartheta e_i'\Sigma_h' Z_3} e_i'\Sigma_h^{-1}\Phi_h h_t.
\end{aligned}
$$

We now solve for the coefficients. Both D_0 and D_g are analytical

$$
\begin{aligned}
D_0 = \frac{1}{(1-\kappa_1)}\bigg[&\ln(\beta) + \kappa_0 + Z_1'(\mu_g - \Sigma_{gh}\Sigma_h \nu_h - \eta\vartheta\lambda_0) \\
&- \frac{1}{\vartheta}\sum_{i=1}^{H} \nu_{h,i} \ln\big(1 - e_i'\Sigma_h' \vartheta Z_3\big) + \frac{\vartheta}{2} Z_1'\Sigma_{0,g}\Sigma_{0,g}' Z_1 \bigg],
\end{aligned}
$$

$$D_g = \left(I_G - \kappa_1(\Phi_g - \eta\vartheta\lambda_g)'\right)^{-1}(\Phi_g - \eta\vartheta\lambda_g)'(1 - \eta)Z_c.$$

A solution for D_g only exists when $(I_G - \kappa_1(\Phi_g - \eta\vartheta\lambda_g))$ is invertible. The vector D_h is the solution to the system of equations

$$D_h = (\Phi_{gh} - \Sigma_{gh}\Phi_h)'Z_1 + \frac{\vartheta}{2}(\iota_H \otimes Z_1)'\tilde{\Sigma}_g\tilde{\Sigma}_g'(I_H \otimes Z_1)$$

$$+ \sum_{i=1}^{H} \frac{e_i'\Sigma_h'Z_3}{1 - \vartheta e_i'\Sigma_h'Z_3}\Phi_h'\Sigma_h^{-1,'}e_i, \tag{C.7}$$

where $\tilde{\Sigma}_g\tilde{\Sigma}_g'$ is a $GH \times GH$ block diagonal matrix with $\Sigma_{i,g}\Sigma_{i,g}'$ along the diagonal. This cannot be solved in closed form in the general case. However, if Σ_h and Φ_h are lower triangular, then it can be calculated in closed form recursively for $i = 1, \ldots, H$. We discuss the analytical solution of this equation in more detail in Appendix C.2.

Step 3: Solve for the fixed point   During estimation, we determine the value of $\bar{p}c$ and the log-linearization constants κ_0 and κ_1 as a function of the model parameters by solving the fixed-point problem (averaging of (C.3)),

$$0 = \bar{p}c - D_0(\bar{p}c) - D_g(\bar{p}c)'\bar{\mu}_g - D_h(\bar{p}c)'\bar{\mu}_h,$$

where the coefficients D_0, D_g and D_h are functions of $\bar{p}c$ through κ_0 and κ_1. The parameters $\bar{\mu}_g$ and $\bar{\mu}_h$ are the unconditional means of g_t and h_t.

Step 4: Substitute the solution into the SDF   If the fixed-point problem has a solution, then the return on the consumption asset is

$$r_{c,t+1} \approx \kappa_0 + \kappa_1\left(D_0 + D_g'g_{t+1} + D_h'h_{t+1}\right) - \left(D_0 + D_g'g_t + D_h'h_t\right) + \Delta c_{t+1}$$

by substituting (C.3) into (C.1). We can now write the log-SDF as a function of the r.v.'s $\varepsilon_{g,t+1}$ and h_{t+1} by substituting this, consumption growth Δc_t, and the preference shock into (A.5)

$$m_{t+1} = \vartheta\ln(\beta) + (\vartheta - 1)\left(\kappa_0 - (1 - \kappa_1)D_0\right) - (\vartheta - 1)D_g'g_t$$

$$- (\vartheta - 1)D_h'h_t + \vartheta\Lambda_1(g_t) + Z_2'\left(\mu_g + \Phi_gg_t + \Phi_{gh}h_t - \Sigma_{gh}(\Sigma_h\nu_h + \Phi_hh_t)\right)$$

$$+ \left(\vartheta\Lambda_2(g_t) + \Sigma_{g,t}'Z_2\right)'\varepsilon_{g,t+1} + Z_4'h_{t+1}.$$

C.2 Analytical solution of D_h

The $H \times 1$ vector of loadings D_h are a system of H equations in H unknowns in (C.7). They can be solved analytically when both Φ_h and Σ_h are lower triangular by recursively solving one equation after another. We will consider the simpler case when they are both diagonal. Under this assumption, each equation is independent of one another and they simplify to

$$D_{h,i} = \bar{D}_i + \frac{(\bar{Z}_{3,i} + \kappa_1 D_{h,i})\Phi_{h,i}}{1 - \vartheta\Sigma_{h,i}(\bar{Z}_{3,i} + \kappa_1 D_{h,i})}, \quad i = 1, \ldots, H, \tag{C.8}$$

where $\Phi_{h,i}$ and $\Sigma_{h,i}$ are the ith diagonal elements and $\bar{D}_i$, $\bar{Z}_{3,i}$ are the ith elements of the following quantities:

$$\bar{D} = (\Phi_{gh} - \Sigma_{gh}\Phi_h)'Z_1 + \frac{\vartheta}{2}(\iota_H \otimes Z_1)'\tilde{\Sigma}_g\tilde{\Sigma}_g'(I_H \otimes Z_1),$$

$$\bar{Z}_3 = \Sigma_{gh}'Z_1.$$

Each loading (C.8) for $i = 1, \ldots, H$ is a quadratic equation

$$0 = \kappa_1\vartheta\Sigma_{h,i}D_{h,i}^2 + D_{h,i}(\kappa_1\Phi_{h,i} - \kappa_1\vartheta\Sigma_{h,i}\bar{D}_i - 1 + \vartheta\Sigma_{h,i}\bar{Z}_{3,i})$$
$$+ \bar{D}_i(1 - \vartheta\Sigma_{h,i}\bar{Z}_{3,i}) + \bar{Z}_{3,i}\Phi_{h,i}. \tag{C.9}$$

The solutions are

$$D_{h,i} = \pm\big(\big((\kappa_1\Phi_{h,i} - \kappa_1\vartheta\Sigma_{h,i}\bar{D}_i - 1 + \vartheta\Sigma_{h,i}\bar{Z}_{3,i})^2$$
$$- 4\kappa_1\vartheta\Sigma_{h,i}[\bar{D}_i(1 - \vartheta\Sigma_{h,i}\bar{Z}_{3,i}) + \bar{Z}_{3,i}\Phi_{h,i}]\big)^{1/2}\big)$$
$$/(2\kappa_1\vartheta\Sigma_{h,i})$$
$$- \frac{(\kappa_1\Phi_{h,i} - \kappa_1\vartheta\Sigma_{h,i}\bar{D}_i - 1 + \vartheta\Sigma_{h,i}\bar{Z}_{3,i})}{2\kappa_1\vartheta\Sigma_{h,i}}. \tag{C.10}$$

A real solution exists as long as the discriminant is greater than or equal to zero. If the discriminant is greater than zero, there are two solutions. Only one solution leads to a sensible value. This is the value with a negative sign; see also Campbell, Giglio, Polk, and Turley (2018) for the ICAPM model.

Appendix D: Bond prices

Define

$$Z_5 = Z_4 - \Sigma_{gh}'Z_\pi$$

in addition to Z_1–Z_4 defined in Appendix C.1.

D.1 *Real bonds*

We will guess and verify that the solution for zero coupon bonds is

$$P_t^{(n)} = \exp(\bar{a}_n + \bar{b}_{n,g}'g_t + \bar{b}_{n,h}'h_t)$$

for some unknown coefficients $\bar{a}_n$ and $\bar{b}_{n,g}$ and $\bar{b}_{n,h}$.

For a maturity $n = 1$, the payoff is guaranteed to be $P_{t+1}^{(0)} = 1$ in the next period, in which case $P_t^{(1)} = E_t[M_{t+1}]$. Using standard techniques for affine bond pricing in discrete-time (see Creal and Wu (2015)), we find that at maturity $n = 1$ the bond loadings

are

$$\bar{a}_1 = \ln(\beta) - \eta Z_c'(\mu_g - \Sigma_{gh}\Sigma_h \nu_h - \eta \vartheta \lambda_0)$$

$$- \sum_{i=1}^{H} \nu_{h,i} \log(1 - e_i'\Sigma_h' Z_4) + \frac{(\vartheta - 1)}{\vartheta} \sum_{i=1}^{H} \nu_{h,i} \log(1 - e_i'\Sigma_h' \vartheta Z_3)$$

$$- \frac{(\vartheta - 1)\vartheta}{2} Z_1' \Sigma_{0,g} \Sigma_{0,g}' Z_1 + \frac{1}{2} Z_2' \Sigma_{0,g} \Sigma_{0,g}' Z_2,$$

$$\bar{b}_{1,g} = -(\Phi_g - \eta \vartheta \lambda_g)' \eta Z_c,$$

$$\bar{b}_{1,h} = -(\Phi_{gh} - \Sigma_{gh}\Phi_h)' \eta Z_c,$$

$$+ \left(\sum_{i=1}^{H} \frac{e_i'\Sigma_h' Z_4}{1 - e_i'\Sigma_h' Z_4} e_i'\Sigma_h^{-1}\Phi_h \right)' - (\vartheta - 1)\left(\sum_{i=1}^{H} \frac{e_i'\Sigma_h' Z_3}{1 - \vartheta e_i'\Sigma_h' Z_3} e_i'\Sigma_h^{-1}\Phi_h \right)'$$

$$+ \frac{1}{2}(I_H \otimes Z_2)' \tilde{\Sigma}_g \tilde{\Sigma}_g' (\iota_H \otimes Z_2) - \frac{(\vartheta - 1)\vartheta}{2}(I_H \otimes Z_1)' \tilde{\Sigma}_g \tilde{\Sigma}_g' (\iota_H \otimes Z_1),$$

where bond prices only exist if $e_i'\Sigma_h' Z_4 < 1$ for $i = 1, \dots, H$. At maturity n, we use the fact that $P_t^{(n)} = E_t[\exp(m_{t+1})P_{t+1}^{(n-1)}]$. The bond loadings are

$$\bar{a}_n = \bar{a}_{n-1} + \bar{a}_1 + \sum_{i=1}^{H} \nu_{h,i} \log\left(\frac{1 - e_i'\Sigma_h' Z_4}{1 - e_i'\Sigma_h'(\Sigma_{gh}' \bar{b}_{n-1,g} + \bar{b}_{n-1,h} + Z_4)} \right)$$

$$+ (\mu_g - \Sigma_{gh}\Sigma_h \nu_h - \eta \vartheta \lambda_0)' \bar{b}_{n-1,g} + \frac{1}{2} \bar{b}_{n-1,g}' \Sigma_{0,g} \Sigma_{0,g}' \bar{b}_{n-1,g} + \bar{b}_{n-1,g}' \Sigma_{0,g} \Sigma_{0,g}' Z_2,$$

$$\bar{b}_{n,g} = (\Phi_g - \eta \vartheta \lambda_g)' \bar{b}_{n-1,g} + \bar{b}_{1,g},$$

$$\bar{b}_{n,h} = (\Phi_{gh} - \Sigma_{gh}\Phi_h)' \bar{b}_{n-1,g} + \bar{b}_{1,h}$$

$$+ \left(\sum_{i=1}^{H} \left(\frac{e_i'\Sigma_h'(\Sigma_{gh}' \bar{b}_{n-1,g} + \bar{b}_{n-1,h} + Z_4)}{1 - e_i'\Sigma_h'(\Sigma_{gh}' \bar{b}_{n-1,g} + \bar{b}_{n-1,h} + Z_4)} - \frac{e_i'\Sigma_h' Z_4}{1 - e_i'\Sigma_h' Z_4} \right) e_i'\Sigma_h^{-1}\Phi_h \right)'$$

$$+ \frac{1}{2}(I_H \otimes \bar{b}_{n-1,g})' \tilde{\Sigma}_g \tilde{\Sigma}_g' (\iota_H \otimes \bar{b}_{n-1,g}) + (I_H \otimes Z_2)' \tilde{\Sigma}_g \tilde{\Sigma}_g' (\iota_H \otimes \bar{b}_{n-1,g}).$$

Real yields are $y_t^{(n)} = a_n + b_{n,g}' g_t + b_{n,h}' h_t$ with $a_n = -\frac{1}{n}\bar{a}_n$, $b_{n,g} = -\frac{1}{n}\bar{b}_{n,g}$ and $b_{n,h} = -\frac{1}{n}\bar{b}_{n,h}$.

D.2 *Nominal bonds*

Similar to the solution for the real bond, we guess and then verify. The solution for zero coupon nominal bonds is $P_t^{\$,(n)} = \exp(\bar{a}_n^\$ + \bar{b}_{n,g}^{\$,'} g_t + \bar{b}_{n,h}^{\$,'} h_t)$ for some unknown coefficients $\bar{a}_n^\$$ and $\bar{b}_{n,g}^\$$ and $\bar{b}_{n,h}^\$$. For maturity $n = 1$, the payoff is guaranteed to be $P_{t+1}^{\$,(0)} = 1$

in the next period, in which case $P_t^{\$,(1)} = \mathrm{E}_t[M_{t+1}^\$]$. The solutions are

$$\bar{a}_1^\$ = \ln(\beta) - (\eta Z_c + Z_\pi)'(\mu_g - \Sigma_{gh}\Sigma_h\nu_h - \eta\vartheta\lambda_0)$$

$$+ \frac{(\vartheta-1)}{\vartheta}\sum_{i=1}^{H}\nu_{h,i}\log(1 - e_i'\Sigma_h'\vartheta Z_3) - \sum_{i=1}^{H}\nu_{h,i}\log(1 - e_i'\Sigma_h'Z_5)$$

$$- \frac{(\vartheta-1)\vartheta}{2}Z_1'\Sigma_{0,g}\Sigma_{0,g}'Z_1 + \frac{1}{2}Z_2'\Sigma_{0,g}\Sigma_{0,g}'Z_2 + \frac{1}{2}Z_\pi'\Sigma_{0,g}\Sigma_{0,g}'Z_\pi - Z_2'\Sigma_{0,g}\Sigma_{0,g}'Z_\pi,$$

$$\bar{b}_{1,g}^\$ = -(\Phi_g - \eta\vartheta\lambda_g)'(\eta Z_c + Z_\pi),$$

$$\bar{b}_{1,h}^\$ = -(\Phi_{gh} - \Sigma_{gh}\Phi_h)'(\eta Z_c + Z_\pi)$$

$$- (\vartheta-1)\left(\sum_{i=1}^{H}\frac{e_i'\Sigma_h'Z_3}{1 - \vartheta e_i'\Sigma_h'Z_3}e_i'\Sigma_h^{-1}\Phi_h\right)' + \left(\sum_{i=1}^{H}\frac{e_i'\Sigma_h'Z_5}{1 - e_i'\Sigma_h'Z_5}e_i'\Sigma_h^{-1}\Phi_h\right)'$$

$$+ \frac{1}{2}(I_H \otimes Z_\pi)'\tilde{\Sigma}_g\tilde{\Sigma}_g'(\iota_H \otimes Z_\pi) - (I_H \otimes Z_2)'\tilde{\Sigma}_g\tilde{\Sigma}_g'(\iota_H \otimes Z_\pi)$$

$$+ \frac{1}{2}(I_H \otimes Z_2)'\tilde{\Sigma}_g\tilde{\Sigma}_g'(\iota_H \otimes Z_2) - \frac{(\vartheta-1)\vartheta}{2}(I_H \otimes Z_1)'\tilde{\Sigma}_g\tilde{\Sigma}_g'(\iota_H \otimes Z_1),$$

where bond prices only exist if $e_i'\Sigma_h'Z_5 < 1$ for $i = 1, \ldots, H$. At longer maturities n, we use the fact that $P_t^{\$,(n)} = \mathrm{E}_t[\exp(m_{t+1}^\$)P_{t+1}^{\$,(n-1)}]$. The bond loadings are

$$\bar{a}_n^\$ = \bar{a}_{n-1}^\$ + \bar{a}_1^\$ + (\mu_g - \Sigma_{gh}\Sigma_h\nu_h - \eta\vartheta\lambda_0)'\bar{b}_{n-1,g}^\$$$

$$+ \sum_{i=1}^{H}\nu_{h,i}\log\left(\frac{1 - e_i'\Sigma_h'Z_5}{1 - e_i'\Sigma_h'(\Sigma_{gh}'\bar{b}_{n-1,g}^\$ + \bar{b}_{n-1,h}^\$ + Z_5)}\right)$$

$$+ \frac{1}{2}\bar{b}_{n-1,g}^{\$,'}\Sigma_{0,g}\Sigma_{0,g}'\bar{b}_{n-1,g}^\$ + \bar{b}_{n-1,g}^\$\Sigma_{0,g}\Sigma_{0,g}'(Z_2 - Z_\pi),$$

$$\bar{b}_{n,g}^\$ = (\Phi_g - \eta\vartheta\lambda_g)'\bar{b}_{n-1,g}^\$ + \bar{b}_{1,g}^\$,$$

$$\bar{b}_{n,h}^\$ = (\Phi_{gh} - \Sigma_{gh}\Phi_h)'\bar{b}_{n-1,g}^\$ + \bar{b}_{1,h}^\$$$

$$+ \left(\sum_{i=1}^{H}\left(\frac{e_i'\Sigma_h'(\Sigma_{gh}'\bar{b}_{n-1,g}^\$ + \bar{b}_{n-1,h}^\$ + Z_5)}{1 - e_i'\Sigma_h'(\Sigma_{gh}'\bar{b}_{n-1,g}^\$ + \bar{b}_{n-1,h}^\$ + Z_5)} - \frac{e_i'\Sigma_h'Z_5}{1 - e_i'\Sigma_h'Z_5}\right)e_i'\Sigma_h^{-1}\Phi_h\right)'$$

$$+ \frac{1}{2}(I_H \otimes \bar{b}_{n-1,g}^\$)'\tilde{\Sigma}_g\tilde{\Sigma}_g'(\iota_H \otimes \bar{b}_{n-1,g}^\$) + (I_H \otimes \bar{b}_{n-1,g}^\$)'\tilde{\Sigma}_g\tilde{\Sigma}_g'(\iota_H \otimes (Z_2 - Z_\pi)).$$

Nominal yields are $y_t^{\$,(n)} = a_n^\$ + b_{n,g}^{\$,'}g_t + b_{n,h}^{\$,'}h_t$ with $a_n^\$ = -\frac{1}{n}\bar{a}_n^\$$, $b_{n,g}^\$ = -\frac{1}{n}\bar{b}_{n,g}^\$$ and $b_{n,h}^\$ = -\frac{1}{n}\bar{b}_{n,h}^\$$. The nominal short term interest rate is

$$r_t^\$ = y_t^{\$,(1)} = a_1^\$ + b_{1,g}^{\$,'}g_t + b_{1,h}^{\$,'}h_t. \tag{D.1}$$

APPENDIX E: PROOF OF PROPOSITIONS

E.1 *General case*

Define the fixed-point problem

$$\kappa_1 = \frac{\exp(\bar{p}c)}{1 + \exp(\bar{p}c)},$$

$$\kappa_0 = \ln\big(1 + \exp(\bar{p}c)\big) - \kappa_1 \bar{p}c,$$

$$D_g' = (1 - \eta)Z_c'(\Phi_g - \vartheta\eta\lambda_g)\big(I - \kappa_1(\Phi_g - \vartheta\eta\lambda_g)\big)^{-1},$$

$$Z_1 = (1 - \eta)Z_c + \kappa_1 D_g,$$

$$Z_3 = \Sigma_{gh}'\big((1 - \eta)Z_c + \kappa_1 D_g\big) + \kappa_1 D_h,$$

$$D_h = (\Phi_{gh} - \Sigma_{gh}\Phi_h)'Z_1 + \frac{\vartheta}{2}(\iota_H \otimes Z_1)'\tilde{\Sigma}_g \tilde{\Sigma}_g'(I_H \otimes Z_1)$$

$$+ \sum_{i=1}^{H} \frac{e_i'\Sigma_h' Z_3}{1 - \vartheta e_i'\Sigma_h' Z_3}\Phi_h'\Sigma_h^{-1,'}e_i,$$

$$(1 - \kappa_1)D_0 = \ln(\beta) + \kappa_0 + Z_1'(\mu_g - \Sigma_{gh}\Sigma_h\nu_h - \eta\vartheta\lambda_0)$$

$$- \frac{1}{\vartheta}\sum_{i=1}^{H} \nu_{h,i}\ln\big(1 - e_i'\Sigma_h'\vartheta Z_3\big) + \frac{\vartheta}{2}Z_1'\Sigma_{0,g}\Sigma_{0,g}'Z_1,$$

$$f(\bar{p}c) = D_0 + D_g'\bar{\mu}_g + D_h'\bar{\mu}_h$$

which is solved if $\bar{p}c = f(\bar{p}c)$.

Assumptions The vector of coefficients D_h is a solution to the system of nonlinear equations in (C.7). The system of equations does not necessarily have a real solution for a given parameter vector θ.

In the special case when Σ_h and Φ_h are diagonal, each loading (C.8) reduces to a quadratic equation given by (C.9) that can be solved separately for each element i. The solutions are in (C.10). The fixed-point problem only has a solution when $D_{h,i}$ is real. The coefficient $D_{h,i}$ is real if and only if the parameters satisfy

$$(\kappa_1\Phi_{h,i} - \kappa_1\vartheta\Sigma_{h,i}\bar{D}_i - 1 + \vartheta\Sigma_{h,i}\bar{Z}_{3,i})^2 - 4\kappa_1\vartheta\Sigma_{h,i}\big[\bar{D}_i(1 - \vartheta\Sigma_{h,i}\bar{Z}_{3,i}) + \bar{Z}_{3,i}\Phi_{h,i}\big] \geq 0$$

for $i = 1, \ldots, H$.

In order to solve for the price to consumption ratio pc_t, the conditional expectation in (C.6) must exist. This condition is

$$\vartheta e_i'\Sigma_h'\big[\Sigma_{gh}'\big((1 - \eta)Z_c + \kappa_1 D_g\big) + \kappa_1 D_h\big] < 1, \quad i = 1, \ldots, H. \tag{E.1}$$

This defines another set of restrictions across the parameters θ of the model.

Proof of Proposition 1   First, derive the limiting property for $\bar{p}c \to -\infty$: $\lim_{\bar{p}c \to -\infty} \kappa_1 = 0$ and $\lim_{\bar{p}c \to -\infty} \kappa_0 = 0$. In this case, both D_0 and D_h are finite due to $\vartheta e_i' \Sigma_h' Z_3 < 1$ in Assumption 1. Therefore, $\tilde{p}c$ is finite, so $\lim_{\bar{p}c \to -\infty}(\bar{p}c - \tilde{p}c) \to -\infty$.

Next, derive the limiting property for $\bar{p}c \to \infty$: $\lim_{\bar{p}c \to \infty} \kappa_1 = 1$ and $\lim_{\bar{p}c \to \infty} \kappa_0 = 0$. This implies D_g is finite as long as the eigenvalue of $(\Phi_g - \vartheta \eta \lambda_g)$ for consumption growth is smaller than 1. D_h is finite due to $\vartheta e_i' \Sigma_h' Z_3 < 1$. And $\lim_{\bar{p}c \to \infty}(1 - \kappa_1)D_0 = \lim_{\bar{p}c \to \infty} \ln(\beta) + Z_1'(\mu_g - \Sigma_{gh} \Sigma_h \nu_h - \eta \vartheta \lambda_0) - \frac{1}{\vartheta} \sum_{i=1}^{H} \nu_{h,i} \ln(1 - e_i' \Sigma_h' \vartheta Z_3) + \frac{\vartheta}{2} Z_1' \Sigma_{0,g} \Sigma_{0,g}' Z_1$. The right hand side is finite due to $\vartheta e_i' \Sigma_h' Z_3 < 1$. Therefore, $\lim_{\bar{p}c \to \infty} \kappa_1 = 1$ leads to an infinite D_0. The condition $\lim_{\bar{p}c \to \infty} D_0 \to -\infty$ implies $\lim_{\bar{p}c \to \infty}(\bar{p}c - \tilde{p}c) \to \infty$, which together $\lim_{\bar{p}c \to -\infty}(\bar{p}c - \tilde{p}c) \to -\infty$ guarantees there exists a solution for the fixed-point problem.

With $\kappa_1 < 1$, the condition $\lim_{\bar{p}c \to \infty} D_0 \to -\infty$ is equivalent to

$$
\beta < \lim_{\bar{p}c \to \infty} \exp\left[-\left(Z_1'(\mu_g - \Sigma_{gh} \Sigma_h \nu_h - \eta \vartheta \lambda_0) \right.\right.
$$
$$
\left.\left. - \frac{1}{\vartheta} \sum_{i=1}^{H} \nu_{h,i} \ln(1 - e_i' \Sigma_h' \vartheta Z_3) + \frac{\vartheta}{2} Z_1' \Sigma_{0,g} \Sigma_{0,g}' Z_1 \right) \right].
$$

Therefore, the boundary condition is

$$
\bar{\beta} = \exp\left[-\left(Z_1^{\infty'}(\mu_g - \Sigma_{gh} \Sigma_h \nu_h - \eta \vartheta \lambda_0) \right.\right.
$$
$$
\left.\left. - \frac{1}{\vartheta} \sum_{i=1}^{H} \nu_{h,i} \ln(1 - e_i' \Sigma_h' \vartheta Z_3^{\infty}) + \frac{\vartheta}{2} Z_1^{\infty'} \Sigma_{0,g} \Sigma_{0,g}' Z_1^{\infty} \right) \right],
$$

where

$$
Z_1^{\infty} = (1 - \eta) Z_c + D_g^{\infty},
$$
$$
D_g^{\infty'} = (1 - \eta) Z_c'(\Phi_g - \vartheta \eta \lambda_g)(I - (\Phi_g - \vartheta \eta \lambda_g))^{-1},
$$
$$
Z_3^{\infty} = \bar{Z}_3^{\infty} + D_h^{\infty},
$$
$$
\bar{Z}_3^{\infty} = \Sigma_{gh}' Z_1^{\infty},
$$
$$
D_{h,i}^{\infty} = -\frac{1}{2}\left(\frac{\Phi_{h,i} - 1}{\vartheta \Sigma_{h,i}} - \bar{D}_i^{\infty} + \bar{Z}_{3,i}^{\infty} \right)
$$
$$
- \sqrt{ \frac{1}{4}\left(\frac{\Phi_{h,i} - 1}{\vartheta \Sigma_{h,i}} - \bar{D}_i^{\infty} + \bar{Z}_{3,i}^{\infty} \right)^2 - \frac{1}{\vartheta \Sigma_{h,i}}\left[\bar{D}_i^{\infty}(1 - \vartheta \Sigma_{h,i} \bar{Z}_{3,i}^{\infty}) + \bar{Z}_{3,i}^{\infty} \Phi_{h,i} \right] },
$$
$$
\bar{D}^{\infty} = (\Phi_{gh} - \Sigma_{gh} \Phi_h)' Z_1^{\infty} + \frac{\vartheta}{2}(\iota_H \otimes Z_1^{\infty})' \tilde{\Sigma}_g \tilde{\Sigma}_g'(I_H \otimes Z_1^{\infty}).
$$

E.2 *Special case with Gaussian dynamics*

The fixed-point problem simplifies to

$$\kappa_1 = \frac{\exp(\bar{p}c)}{1 + \exp(\bar{p}c)},$$

$$\kappa_0 = \ln\big(1 + \exp(\bar{p}c)\big) - \kappa_1 \bar{p}c,$$

$$D'_g = (1 - \eta) Z'_c \Phi_g^{\mathbb{Q}\$} \big(I - \kappa_1 \Phi_g^{\mathbb{Q}\$}\big)^{-1},$$

$$Z_1 = (1 - \eta) Z_c + \kappa_1 D_g,$$

$$D_0(1 - \kappa_1) = \ln(\beta) + \kappa_0 + Z'_1 \mu_g^* + \frac{1}{2} \vartheta Z'_1 \Sigma_{0,g} \Sigma'_{0,g} Z_1,$$

$$\tilde{p}c = D_0 + D'_g \bar{\mu}_g$$

which is solved if $\bar{p}c = \tilde{p}c$.

First, the condition in Proposition 1 becomes

$$\beta < \lim_{\bar{p}c \to \infty} \exp\left(-Z'_1 \mu_g^* - \frac{1}{2} \vartheta Z'_1 \Sigma_{0,g} \Sigma'_{0,g} Z_1\right), \qquad (E.2)$$

and $\bar{\beta}$ simplifies to

$$\bar{\beta} = \exp\left[-\left(Z_1^{\infty\prime} \mu_g^* + \frac{\vartheta}{2} Z_1^{\infty\prime} \Sigma_{0,g} \Sigma'_{0,g} Z_1^{\infty}\right)\right],$$

where

$$Z_1^{\infty} \equiv \lim_{\bar{p}c \to \infty} Z_1(\bar{p}c) = (1 - \eta) Z_c + D_g^{\infty},$$

$$D_g^{\infty\prime} \equiv \lim_{\bar{p}c \to \infty} D_g(\bar{p}c)' = (1 - \eta) Z'_c \Phi_g^{\mathbb{Q}\$} \big(I - \Phi_g^{\mathbb{Q}\$}\big)^{-1}.$$

Proof of Corollary 1

1. The condition (E.2) is guaranteed by $Z_1^{\infty\prime} \mu_g^* \leq 0$ and $\vartheta < 0$ for any $\beta \leq 1$. And $\frac{1-\gamma}{1-\psi} > 0$ is equivalent to $\vartheta < 0$,

2. A stronger condition is

$$\beta \leq 1 < \lim_{\bar{p}c \to \infty} \exp\left(-Z'_1 \mu_g^* - \frac{1}{2} \vartheta Z'_1 \Sigma_{0,g} \Sigma'_{0,g} Z_1\right),$$

which can be simplified to

$$\gamma > 1 + \frac{2 Z'_c (I - \Phi_g^{\mathbb{Q}\$})^{-1} \mu_g^*}{Z'_c (I - \Phi_g^{\mathbb{Q}\$})^{-1} \Sigma_{0,g} \Sigma'_{0,g} (I - \Phi_g^{\mathbb{Q}\$,\prime})^{-1} Z_c}, \qquad \text{if } \psi > 1,$$

$$\gamma < 1 + \frac{2 Z'_c (I - \Phi_g^{\mathbb{Q}\$})^{-1} \mu_g^*}{Z'_c (I - \Phi_g^{\mathbb{Q}\$})^{-1} \Sigma_{0,g} \Sigma'_{0,g} (I - \Phi_g^{\mathbb{Q}\$,\prime})^{-1} Z_c}, \qquad \text{if } \psi < 1,$$

hence $\bar{\gamma}(\theta^{\mathbb{P}}, \theta^{\lambda}) = 1 + \dfrac{2 Z_c'(I - \Phi_g^{\mathbb{Q}\$})^{-1} \mu_g^*}{Z_c'(I - \Phi_g^{\mathbb{Q}\$})^{-1} \Sigma_{0,g} \Sigma_{0,g}' (I - \Phi_g^{\mathbb{Q}\$,'})^{-1} Z_c}$, does not depend on ψ.

3. We have $\dfrac{d\vartheta}{d\gamma} = -\dfrac{1}{1-\eta}$, $\dfrac{dD_g^{\infty\prime}}{d\gamma} = 0$ and $\dfrac{dZ_1^{\infty\prime}}{d\gamma} = \dfrac{dD_g^{\infty\prime}}{d\gamma} = 0$. Hence, the derivative of $\ln \bar{\beta}$ w.r.t. γ is

$$\frac{d \ln \bar{\beta}}{d\gamma} = \frac{1}{2(1-\eta)} Z_1^{\infty\prime} \Sigma_{0,g} \Sigma_{0,g}' Z_1^{\infty}$$

$\frac{d \ln \bar{\beta}}{d\gamma} = \frac{1}{\beta} \frac{d\bar{\beta}}{d\gamma}$ implies that the two derivatives have the same sign. Therefore, for $\psi > 1$, then $\frac{d\bar{\beta}}{d\gamma} > 0$; for $\psi < 1$, then $\frac{d\bar{\beta}}{d\gamma} < 0$.

APPENDIX F: NUMERICAL ILLUSTRATIONS

F.1 *Model with two volatility factors*

Figure S1 provides numerical illustrations of Proposition 1 and Corollary 1.[1] The top row takes a special case without stochastic volatility or preference shock. The upper left panel provides a demonstration for part 2 of Corollary 1, where we set $\beta = 0.9998$. A similar pattern holds for other values of $\beta \leq 1$ as well. Dots indicate the existence of a solution, and stars imply no solution. The dashed lines mark the boundaries $\psi = 1$ and $\gamma = \bar{\gamma} = 146.5$. Consistent with part 2 of Corollary 1, when γ and ψ are both bigger than their corresponding boundaries (upper right quadrant) or both smaller than the boundaries (lower left quadrant), a solution exists. The top right panel illustrates part 3 of Corollary 1, with $\psi = 0.8$. As prescribed by the Corollary, when $\psi < 1$, we see a downward sloping line that separates the parameter space for (β, γ) into feasible and infeasible regions. The larger the value of risk aversion γ gets, the smaller the discount rate β needs to be to remain in a region with a valid solution.

While the top panels brings a visualization for Corollary 1, the bottom panels demonstrate how restrictive the space looks in the benchmark setting. The upper-left and lower-right regions of the bottom left panel remain infeasible as before with similar intuition as Gaussian models. The difference is now the upper-right region becomes infeasible in addition to the earlier regions in order to satisfy Assumption 1. This emphasizes that in stochastic volatility models both the intertemporal elasticity of substitution and risk aversion need to be modest. We find although the lower left region is still feasible, the region is much smaller. For $\psi = 0.97$, γ cannot exceed 4.8. For $\psi = 0.52$, γ cannot exceed 6.9. For comparison, the upper bound for γ in the Gaussian case marked by the line is 146.5.

The implications are two-fold. First, much of the economics literature evaluates a model's success according to whether or not it can produce a small value for the risk aversion parameter γ. We need to interpret this result with caution. As we show, for stochastic volatility models, a small value of γ is required to satisfy the constraints of the model. Second, stochastic volatility models have much smaller feasible regions of

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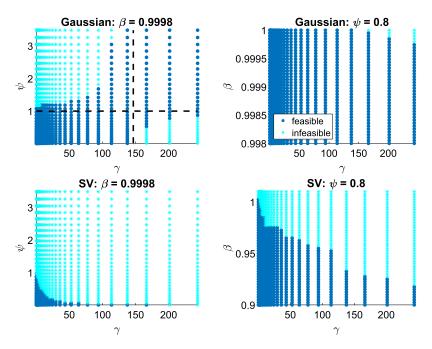[1] The parameters used to make the plots are taken from the estimates in Table S1 of the Appendix.

FIGURE S1. Feasible and infeasible regions of the parameter space. Feasible (dots) and infeasible (stars) regions of the parameters space for the model with 2 stochastic volatility factors. Dashed lines are the theoretical bounds derived in Corollary 1 part 2. The top row is a simplified Gaussian model without stochastic volatility: $\theta^{\mathbb{P}}$ is taken from the estimates of this model, and $\lambda_g = 0$. The bottom row shows our benchmark model with stochastic volatility. Parameters $\theta^{\mathbb{P}}$ are taken from Table S1. θ^{λ} is taken from the global solution of the model. Left: parameter space for (γ, ψ) with $\beta = 0.9998$. Right: Parameter space for (γ, β) with $\psi = 0.8$.

the parameter space, and they are more likely to encounter numerical problems and boundaries.

The bottom right panel is similar to the upper right plot. Again the downward sloping line that divides the regions indicates that with the intertemporal elasticity of substitution less than 1, an agent needs to be less patient as their risk aversion increases. The difference is that the feasible region again is much smaller. For example, for $\gamma = 244$, β can be as big as 0.9996 in the Gaussian model, but it will not be able to exceed 0.93 in the SV setting.

F.2 *Model with four volatility factors*

In previous versions of the paper, we estimated a model with four stochastic variance factors. This model is described in detail in Appendix B.3. We performed the same numerical illustration for this model. The results are reported in Figure S2. The only difference between these figures and the figures from the previous subsection are the values of ψ. This figure has $\psi = 0.7$ instead of $\psi = 0.8$, which reflects the estimated value of ψ between the two models. Otherwise, the graphs are almost identical.
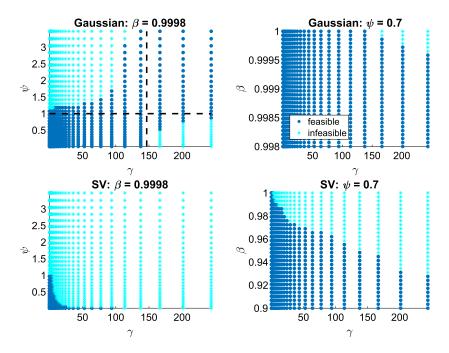
FIGURE S2. Feasible and infeasible regions of the parameter space. Feasible (dots) and infeasible (stars) regions of the parameters space. Dashed lines are the theoretical bounds derived in Corollary 1, part 2. The top row is a simplified model without stochastic volatility: $\theta^{\mathbb{P}}$ is taken from the estimates of this model, and $\lambda_g = 0$. The bottom row shows our benchmark model with stochastic volatility. Parameters $\theta^{\mathbb{P}}$ are taken from Table S1. θ^{λ} is taken from the global solution of the model for the bottom left, and local solution for the bottom right. Left: parameter space for (γ, ψ) with $\beta = 0.9998$. Right: Parameter space for (γ, β) with $\psi = 0.7$.

APPENDIX G: ESTIMATION OF MACROECONOMIC FACTORS

We quantify the distribution of the latent macroeconomic factors related to consumption growth and inflation by Bayesian methods. We use a particle Gibbs sampler which is an MCMC algorithm that uses a particle filter to draw from distributions that are intractable; see Creal and Wu (2017) for an application on interest uncertainty, and Creal (2012) for a survey on particle filtering.

Given a prior distribution $p(\theta^{\mathbb{P}})$ for the $\mathbb{P}$ parameters in the stochastic process

$$\pi_{t+1} = \bar{\pi}_t + \sqrt{h_{t,\pi}}\,\varepsilon_{\pi_1,t+1}, \tag{G.1}$$

$$\Delta c_{t+1} = \bar{c}_t + \sqrt{h_{t,c}}\,\varepsilon_{c_1,t+1}, \tag{G.2}$$

$$\bar{\pi}_{t+1} = \mu_\pi + \phi_\pi \bar{\pi}_t + \phi_{\pi,c}\bar{c}_t + \sigma_\pi\sqrt{h_{t,\pi}}\,\varepsilon_{\pi_2,t+1}, \tag{G.3}$$

$$\bar{c}_{t+1} = \mu_c + \phi_{c,\pi}\bar{\pi}_t + \phi_c \bar{c}_t + \sigma_{c,\pi}\sqrt{h_{t,\pi}}\,\varepsilon_{\pi_2,t+1} + \sigma_c\sqrt{h_{t,c}}\,\varepsilon_{c_2,t+1} \tag{G.4}$$

we sample from the joint posterior distribution

$$p(\theta^{\mathbb{P}}, g_{1:T}, h_{0:T}|m_{1:T}) \propto p(m_{1:T}|g_{1:T}, h_{0:T}, \theta^{\mathbb{P}}) p(g_{1:T}|h_{0:T}, \theta^{\mathbb{P}}) p(h_{0:T}|\theta^{\mathbb{P}}) p(\theta^{\mathbb{P}}), \quad \text{(G.5)}$$

where $m_t = (\Delta c_t, \pi_t)$ and $x_{t:t+k} = (x_t, \dots, x_{t+k})$. Starting with an initial value for the parameters $\theta^{\mathbb{P},(0)}$, the particle Gibbs sampler draws from this distribution by iterating for $j = 1, \dots, M$ between the two full conditional distributions

$$(g_{1:T}, h_{0:T})^{(j)} \sim p(g_{1:T}, h_{0:T}|m_{1:T}, \theta^{\mathbb{P},(j-1)}), \quad \text{(G.6)}$$

$$\theta^{\mathbb{P},(j)} \sim p(\theta^{\mathbb{P}}|m_{1:T}, g_{1:T}^{(j)}, h_{0:T}^{(j)}). \quad \text{(G.7)}$$

This produces a Markov chain whose stationary distribution is the posterior (G.5). The models for consumption growth and inflation above are nonlinear, non-Gaussian state space models. In these models, the full conditional distribution of the latent state variables given the data and model's parameters (G.6) is not easy to sample. The particle Gibbs sampler overcomes this limitation by using a particle filter to jointly sample paths of the state variables $(g_{1:T}, h_{0:T})$ in large blocks. Consequently, it improves the mixing of the MCMC algorithm and the efficiency with which the Markov chain explores the parameter space; see also Creal and Tsay (2015) for a longer discussion.

   Using the particle Gibbs sampler, we estimate the long-run risk model of consumption and inflation in Appendix B.2 (and Appendix B.3). Posterior means and standard deviations for the parameters of the model are in Table S1. The unconditional means of consumption growth and inflation measured in annualized percentage points are $\bar{\mu}_c \times 1200 = 1.68$ and $\bar{\mu}_\pi \times 1200 = 3.96$, respectively. Both expected consumption growth and expected inflation are highly autocorrelated, with posterior mean estimates of $\phi_\pi = 0.984$ and $\phi_c = 0.90$. The estimated eigenvalues of the autocovariance matrix Φ_g are 0.975 and 0.961, respectively, indicating a high degree of persistence in their conditional means.

   In Figure S3, we plot the prior and posterior distributions for key parameters of the model. Relative to the prior, the unconditional means still have considerable uncertainty. This is natural given the high degree of persistence each series has. The posteriors for ϕ_π and $\phi_{c,\pi}$ contract significantly relative to the prior. In the literature, a negative value of $\phi_{c,\pi}$ is important for generating an upward sloping yield curve. While the posterior mean is still slightly negative (see Table S1), it is not statistically significant given the posterior standard deviation is 5 times larger than the point estimate. Figure S3 shows that the posterior for $\phi_{c,\pi}$ is reasonably symmetric around zero. In the paper, we show that our model is able to generate an upward sloping yield curve when $\phi_{c,\pi}$ is either negative or positive.

   Posterior mean estimates (in blue) of the latent state variables together with their 10% and 90% uncertainty bands are plotted in Figure S4. There is considerable variation in the long-run risk factor $\bar{c}_t$ of consumption growth (top left). It shows a noticeable decline during each recession, with the largest decline during the Great Recession. The pattern replicates the long run risk in the literature.

TABLE S1. Estimates of time series parameters for consumption growth and inflation.

Prior					
	μ_π	$\bar{\mu}_\pi$	ϕ_π	$\phi_{\pi,c}$	–
	–	0.0033	0.975	0	
		$(0.708e^{-3})$	(0.10)	(0.05)	
	μ_c	$\bar{\mu}_c$	$\phi_{c,\pi}$	ϕ_c	$\sigma_{c,\pi}$
	–	0.0015	0	0.90	0.00
		$(0.750e^{-3})$	(0.05)	(0.10)	(3.5)
	$\bar{\mu}_{h,\pi}$	$\phi_{h,\pi}$	$\sigma_{h,\pi}$		
	$1.08e^{-5}$	0.975	$7.45e^{-5}$		
	$(6.57e^{-6})$	(0.009)	$(4.123e^{-6})$		
	$\bar{\mu}_{h,c}$	$\phi_{h,c}$	$\sigma_{h,c}$		
	$1.08e^{-5}$	0.975	$7.45e^{-5}$		
	$(6.57e^{-6})$	(0.009)	$(4.123e^{-6})$		
Posterior					
	μ_π	$\bar{\mu}_\pi$	ϕ_π	$\phi_{\pi,c}$	–
	$-0.0940e^{-3}$	0.0029	0.9926	0.0798	
	$(0.9295e^{-4})$	$(0.564e^{-3})$	(0.0158)	(0.0421)	
	μ_c	$\bar{\mu}_c$	$\phi_{c,\pi}$	ϕ_c	$\sigma_{c,\pi}$
	$0.1011e^{-3}$	0.0014	-0.0056	0.9407	-1.516
	$(0.7147e^{-4})$	$(0.268e^{-3})$	(0.0119)	(0.0325)	(0.0177)
	$\bar{\mu}_{h,\pi}$	$\phi_{h,\pi}$	$\sigma_{h,\pi}$		
	$0.0476e^{-4}$	0.9755	$0.0626e^{-6}$		
	$(0.1257e^{-5})$	(0.0070)	$(0.1759e^{-8})$		
	$\bar{\mu}_{h,c}$	$\phi_{h,c}$	$\sigma_{h,c}$		
	$0.1205e^{-4}$	0.9811	$0.1013e^{-6}$		
	$(0.2845e^{-5})$	(0.0058)	$(0.5024e^{-8})$		

Note: Prior (top) and posterior (bottom) mean and standard deviation (in parentheses) of our benchmark two factor volatility model. Consumption growth and inflation are measured in monthly percentage changes. Multiplying the variables by 1200 translates them into annualized percentage points. For example, the unconditional means of consumption growth and inflation measured in annualized percentages are $\bar{\mu}_c \times 1200 = 1.8$ and $\bar{\mu}_\pi \times 1200 = 3.96$, respectively.

G.1 *Four factor volatility model*

In previous versions of the paper, we estimated a model with four volatility factors

$$\pi_{t+1} = \bar{\pi}_t + \sqrt{h_{t,\pi_1}}\varepsilon_{\pi_1,t+1}, \quad \varepsilon_{\pi_1,t+1} \sim N(0,1),$$

$$\Delta c_{t+1} = \bar{c}_t + \sqrt{h_{t,c_1}}\varepsilon_{c_1,t+1}, \quad \varepsilon_{c_1,t+1} \sim N(0,1),$$

$$\bar{\pi}_{t+1} = \mu_\pi + \phi_\pi\bar{\pi}_t + \phi_{\pi,c}\bar{c}_t + \sqrt{h_{t,\pi_2}}\varepsilon_{\pi_2,t+1}, \quad \varepsilon_{\pi_2,t+1} \sim N(0,1),$$

$$\bar{c}_{t+1} = \mu_c + \phi_{c,\pi}\bar{\pi}_t + \phi_c\bar{c}_t + \sigma_{c,\pi}\sqrt{h_{t,\pi_2}}\varepsilon_{\pi_2,t+1} + \sqrt{h_{t,c_2}}\varepsilon_{c_2,t+1}, \quad \varepsilon_{c_2,t+1} \sim N(0,1).$$

This model has 2 volatility factors for similar to those found by Stock and Watson (2007) and Creal (2012). The main conclusions of the paper are the same for this model. In this section of the Appendix, we repeat the same results of previous sections for this model. These results are in Figures S5, S6, S7, and Tables S2 and S3. We do not provide extensive comments on the results because they are largely the same. We do note that the estimates of stochastic volatility from this model of inflation are similar to those found by Stock and Watson (2007) and Creal (2012).
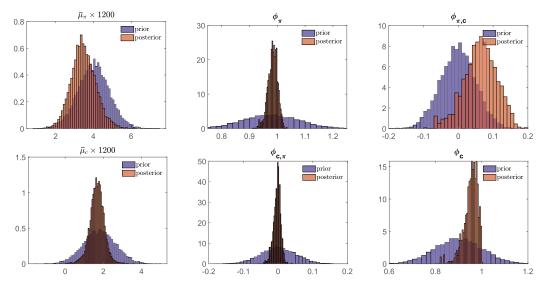
FIGURE S3. Priors and posteriors. Empirical pdf for prior distributions (blue bars) and posterior distributions (red bars). Top left: mean of inflation $\bar{\mu}_\pi \times 1200$; top middle: ϕ_π; top right: $\phi_{\pi,c}$; bottom left: mean of consumption growth $\bar{\mu}_c \times 1200$; bottom middle: $\phi_{c,\pi}$; bottom right: ϕ_c.

APPENDIX H: CONDITIONAL SHARPE RATIOS

The excess return is

$$
\begin{aligned}
rx_{t+1}^{(n),(n-1),\$} = \; & \bar{a}_{n-1}^\$ + \bar{b}_{n-1,g}^{\$,'}[\mu_g + \Phi_g g_t + \Phi_{gh} h_t + \Sigma_{gh}\varepsilon_{h,t+1} + \Sigma_{g,t}\varepsilon_{g,t+1}] \\
& + \bar{b}_{n-1,h}^{\$,'}[\Sigma_h\nu_h + \Phi_h h_t + \varepsilon_{h,t+1}] \\
& - \bar{a}_n^\$ - \bar{b}_{n,g}^{\$,'}g_t - \bar{b}_{n,h}^{\$,'}h_t + \bar{a}_1^\$ + \bar{b}_{1,g}^{\$,'}g_t + \bar{b}_{1,h}^{\$,'}h_t.
\end{aligned}
$$

The expected value or risk premium of the asset is

$$
\begin{aligned}
\mathbb{E}_t\big[rx_{t+1}^{(n),(n-1),\$}\big] = \; & \bar{a}_{n-1}^\$ - \bar{a}_n^\$ + \bar{a}_1^\$ + \bar{b}_{n-1,g}^{\$,'}\mu_g + \bar{b}_{n-1,h}^{\$,'}\Sigma_h\nu_h \\
& + \big(\Phi_g'\bar{b}_{n-1,g}^\$ - \bar{b}_{n,g}^\$ + \bar{b}_{1,g}^\$\big)'g_t \\
& + \big(\Phi_{gh}'\bar{b}_{n-1,g}^\$ + \Phi_h'\bar{b}_{n-1,h}^\$ - \bar{b}_{n,h}^\$ + \bar{b}_{1,h}^\$\big)'h_t.
\end{aligned}
$$

This is only a function of g_t if the preference shock is in the model $\lambda_g \neq 0$. The conditional variance of the return is

$$
\begin{aligned}
\mathbb{V}_t\big[r_{t+1}^{(n),(n-1),\$}\big] = \; & \mathbb{V}_t\big[(\bar{b}_{n-1,g}^{\$,'}\Sigma_{gh} + \bar{b}_{n-1,h}^{\$,'})\varepsilon_{h,t+1}\big] + \mathbb{V}_t\big[\bar{b}_{n-1,g}^{\$,'}\Sigma_{g,t}\varepsilon_{g,t+1}\big] \\
= \; & (\bar{b}_{n-1,g}^{\$,'}\Sigma_{gh} + \bar{b}_{n-1,h}^{\$,'})\Sigma_{h,t}\Sigma_{h,t}'\big(\bar{b}_{n-1,g}^{\$,'}\Sigma_{gh} + \bar{b}_{n-1,h}^{\$,'}\big)' \\
& + \bar{b}_{n-1,g}^{\$,'}\Sigma_{g,t}\Sigma_{g,t}'\bar{b}_{n-1,g}^\$.
\end{aligned}
$$

TABLE S2. Estimates of time series parameters for consumption growth and inflation.

Prior	μ_π	$\bar\mu_\pi$	ϕ_π	$\phi_{\pi,c}$	–
	–	0.0033	0.975	0	
		$(0.708e^{-3})$	(0.10)	(0.05)	
	μ_c	$\bar\mu_c$	$\phi_{c,\pi}$	ϕ_c	$\sigma_{c,\pi}$
	–	0.0015	0	0.90	0.00
		$(0.750e^{-3})$	(0.05)	(0.10)	(3.5)
	$\bar\mu_{h,\pi_2}$	ϕ_{h,π_2}	σ_{h,π_2}	$\bar\mu_{h,\pi_1}$	ϕ_{h,π_1}	σ_{h,π_1}
	$1.97e^{-7}$	0.975	$1.70e^{-11}$	$4.04e^{-6}$	0.975	$6.23e^{-9}$
	$(9.20e^{-7})$	(0.009)	$(2.51e^{-11})$	$(5.08e^{-6})$	(0.009)	$(5.35e^{-10})$
	$\bar\mu_{h,c_2}$	ϕ_{h,c_2}	σ_{h,c_2}	$\bar\mu_{h,c_1}$	ϕ_{h,c_1}	σ_{h,c_1}
	$3.77e^{-8}$	0.975	$1.30e^{-12}$	$1.08e^{-5}$	0.975	$1.02e^{-8}$
	$(8.32e^{-8})$	(0.009)	$(1.73e^{-12})$	$(6.57e^{-6})$	(0.009)	$(1.57e^{-9})$
Posterior	μ_π	$\bar\mu_\pi$	ϕ_π	$\phi_{\pi,c}$	–
	$-0.096e^{-4}$	0.0031	0.978	0.057	
	$0.734e^{-4}$	$(0.497e^{-3})$	(0.014)	(0.033)	
	μ_c	$\bar\mu_c$	$\phi_{c,\pi}$	ϕ_c	$\sigma_{c,\pi}$
	$0.856e^{-4}$	0.0014	-0.002	0.941	-0.394
	$(0.592e^{-4})$	$(0.278e^{-3})$	(0.010)	(0.029)	(0.190)
	$\bar\mu_{h,\pi_2}$	ϕ_{h,π_2}	σ_{h,π_2}	$\bar\mu_{h,\pi_1}$	ϕ_{h,π_1}	σ_{h,π_1}
	$0.269e^{-6}$	0.984	$0.317e^{-9}$	$0.333e^{-5}$	0.989	$0.724e^{-8}$
	$(0.091e^{-6})$	(0.007)	$(0.120e^{-9})$	$(0.077e^{-5})$	(0.004)	$(0.0729e^{-8})$
	$\bar\mu_{h,c_2}$	ϕ_{h,c_2}	σ_{h,c_2}	$\bar\mu_{h,c_1}$	ϕ_{h,c_1}	σ_{h,c_1}
	$0.664e^{-6}$	0.980	$0.340e^{-10}$	$0.891e^{-5}$	0.992	$0.137e^{-7}$
	$(0.372e^{-6})$	(0.009)	$(0.169e^{-10})$	$(0.216e^{-5})$	(0.003)	$(0.225e^{-8})$

Note: Prior (top) and posterior (bottom) mean and standard deviation (in parentheses) of the four factor volatility model. Consumption growth and inflation are measured in monthly percentage changes. Multiplying the variables by 1200 translates them into annualized percentage points. For example, the unconditional means of consumption growth and inflation measured in annualized percentages are $\bar\mu_c \times 1200 = 1.68$ and $\bar\mu_\pi \times 1200 = 3.72$, respectively.

Using the conditional mean and variance expressions, we can calculate the conditional Sharpe ratio of log-returns from

$$s_t^{(n),\$} \equiv \left[\mathbb{E}_t\big(r_{t+1}^{(n),\$}\big) - r_t^\$ + \frac{1}{2}\mathbb{V}_t\big(r_{t+1}^{(n),\$}\big) \right] \Big/ \sqrt{\mathbb{V}_t\big(r_{t+1}^{(n),\$}\big)}$$

which is the same expression as the paper.

APPENDIX I: MCMC AND PARTICLE FILTERS

I.1 *MCMC*

Our MCMC algorithm is the particle Gibbs (PG) sampler. It iterates between two broad steps: (i) drawing the latent state variables $(g_{1:T}, h_{0:T})$ conditional on the model's parameters; and (ii) drawing the model's parameters $\theta^\mathbb{P}$ given the latent state variables. We make heavy use of the fact that the model is a conditionally linear Gaussian state space model.

I.1.1 *Conditionally linear, Gaussian state space form*   Conditional on $h_{0:T}$, the model is a linear, Gaussian state space model. We write the model using the state space form of
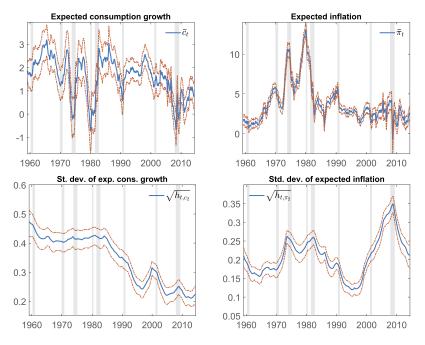
FIGURE S4. Estimated dynamics of consumption growth and inflation for the two factor volatility model. Posterior mean (smoothed) estimates of the factors with their 10% and 90% uncertainty bands for the two factor volatility model. All values are multiplied by 1200. Top left: expected cons. growth $\bar{c}_t$; Top right: expected inflation $\bar{\pi}_t$; Bottom left: standard dev. of expected cons. growth $\sqrt{h_{t,c}}$; Bottom right: standard dev. of expected inflation $\sqrt{h_{t,\pi}}$.

Durbin and Koopman (2012) given by

$$Y_t = Zg_t + d + \eta_t^*, \quad \eta_t^* \sim \mathrm{N}(0, H), \tag{I.1}$$

$$g_{t+1} = Tg_t + c_t + R\varepsilon_{t+1}^*, \quad \varepsilon_{t+1}^* \sim \mathrm{N}(0, Q_t), \tag{I.2}$$
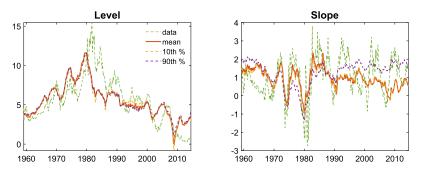

FIGURE S5. Level and slope. Top panel: level defined as average of yields across all maturities. Bottom panel: slope defined as the 5-year minus 3-month yield. Dashed line: data; solid line: mean estimate; dashed-dotted line: 10th percentile; dotted line: 90th percentile. Y-axis: annualized percentage points.
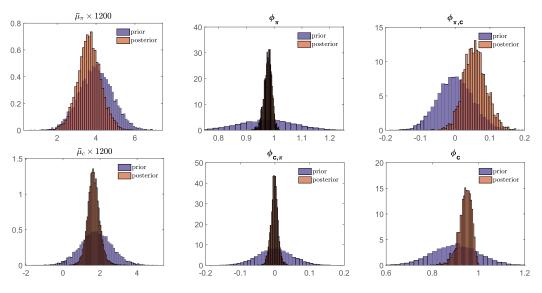
FIGURE S6. Priors and posteriors. Empirical pdf for prior distributions (blue bars) and posterior distributions (red bars). Top left: mean of inflation $\bar{\mu}_\pi \times 1200$; top middle: ϕ_π; top right: $\phi_{\pi,c}$; bottom left: mean of consumption growth $\bar{\mu}_c \times 1200$; bottom middle: $\phi_{c,\pi}$; bottom right: ϕ_c.

where $Y_t = (\Delta c_t \, \pi_t)'$. The models in this paper can be placed in this state space form as

$$Z = \begin{pmatrix} Z_c \\ Z_\pi \end{pmatrix}, \qquad T = \Phi_g, \qquad d = 0_{2 \times 1}, \qquad H = 0_{2 \times 2},$$

$$c_t = \mu_g + \Phi_{gh} h_t + \Sigma_{gh} \varepsilon_{h,t+1} \qquad Q_t = \Sigma_{g,t} \Sigma'_{g,t}.$$

For some models, there are free, estimable parameters in the matrices $(\mu_g, \Phi_{gh}, \Sigma_{gh})$. We can place these in the state vector. This allows any free parameters in $(\mu_g, \Phi_{gh}, \Sigma_{gh})$ to

TABLE S3. Unconditional yield curves.

		3	12	24	36	48	60	level	slope
data		4.94	5.33	5.54	5.72	5.88	5.98	5.57	1.04
SV w/ preference shock		4.91	5.27	5.63	5.85	5.92	5.84	5.57	0.93
	local	4.95	5.20	5.49	5.74	5.95	6.13	5.58	1.18
	10th	4.91	5.25	5.60	5.81	5.88	5.79	5.54	0.88
	90th	4.96	5.15	5.43	5.70	5.96	6.20	5.57	1.24
Gaussian w/ preference shock		5.08	5.25	5.47	5.69	5.89	6.09	5.58	1.01
SV w/o preference shock		5.64	5.63	5.61	5.59	5.57	5.56	5.60	−0.08

Note: Average nominal yields in annualized percentage points across time in the data (first row), our benchmark model with both stochastic volatility and preference shock (second to fifth rows), model without stochastic volatility (sixth row), and model without preference shock (last row) for maturities of 3–60 months. Each column corresponds to one maturity. The last two columns are the average level of yields across all 6 maturities, and the slope is defined as the difference between the 60-month and 3-month yields.
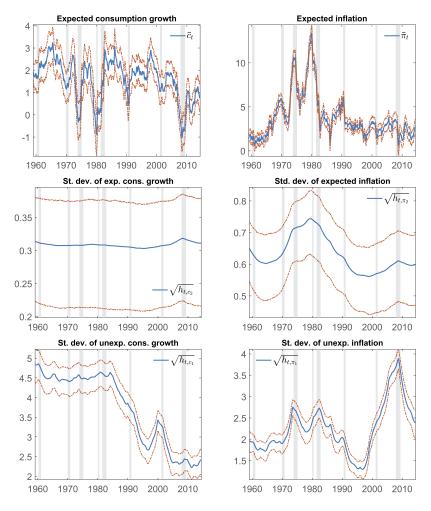
FIGURE S7. Estimated dynamics of consumption and inflation for the four factor volatility model. Posterior mean (smoothed) estimates of the factors with their 10% and 90% uncertainty bands. All values are multiplied by 1200. Top left: expected cons. growth $\bar{c}_t$; Top right: expected inflation $\bar{\pi}_t$; Middle left: standard dev. of expected cons. growth h_{t,c_2}; Middle right: standard dev. of expected inflation h_{t,π_2}; Bottom left: standard dev. of unexpected cons. growth h_{t,c_1}. Bottom right: standard dev. of unexpected inflation h_{t,π_1}.

be drawn jointly with the state variables $g_{1:T}$. It also allows us to marginalize over them when drawing other parameters; see Creal and Wu (2017) for discussion.

I.1.2 *Drawing the state variables*   We draw $(g_{1:T}, h_{0:T})$ from their full conditional distribution in two steps:

$$g_{1:T} \sim p\big(g_{1:T}|Y_{1:T}, h_{0:T}, \theta^{\mathbb{P}}\big),$$

$$h_{0:T} \sim p\big(h_{1:T}|Y_{1:T}, g_{1:T}, \theta^{\mathbb{P}}\big).$$

We draw $g_{1:T}$ conditional on $h_{0:T}$ from the conditionally linear, Gaussian state space model (I.1) and (I.2) using a forward filtering backward sampling algorithm or simulation smoother; see, for example, Durbin and Koopman (2002). Conditional on the draw for $g_{1:T}$, we draw $h_{0:T}$ using a particle Gibbs sampler.

There are two PG samplers developed in the literature. The original PG sampler of Andrieu, Doucet, and Holenstein (2010) with the backward-sampling pass developed by Whiteley (2010); see Creal and Tsay (2015). And, the PG sampler with ancestor sampling (PGAS) of Lindsten, Jordan, and Schön (2014). The former algorithm is simple to implement. We describe its implementation here.

Let J be the total number of particles. In our work, we select $J = 100$. The PG sampler starts with a set of existing particles $h_{0:T}^{(1)}$ that were drawn from the previous iteration.

For $t = 1, \dots, T$, run:

- For $j = 2, \dots, J$, draw from a proposal: $(h_t, h_{t-1})^{(j)} \sim q(h_t, h_{t-1}|g_{t-1:t}, \theta^{\mathbb{P}})$.

- For $j = 1, \dots, J$, calculate the importance weight:

$$w_t^{(j)} \propto \frac{p\big(g_t|g_{t-1}, h_t^{(j)}, h_{t-1}^{(j)}, \theta^{\mathbb{P}}\big) p\big(h_t^{(j)}|h_{t-1}^{(j)}, \theta^{\mathbb{P}}\big)}{q\big(h_t^{(j)}, h_{t-1}^{(j)}|g_{t-1:t}, \theta^{\mathbb{P}}\big)}.$$

- For $j = 1, \dots, J$, normalize the weights: $\hat{w}_t^{(j)} = \frac{w_t^{(j)}}{\sum_{j=1}^{J} w_t^{(j)}}$.

- Conditionally resample the particles $\{h_t^{(j)}\}_{j=1}^{J}$ with probabilities $\{\hat{w}_t^{(j)}\}_{j=1}^{J}$. In this step, the first particle $h_t^{(1)}$ always gets resampled and may be randomly duplicated.

Implementation of the PG sampler is different than a standard particle filter due to the "conditional" resampling algorithm used in the last step. We use the conditional multinomial resampling algorithm from Andrieu, Doucet, and Holenstein (2010).

In the original PG sampler, the particles $\{h_t^{(j)}\}_{j=1}^{J}$ are stored for $t = 1, \dots, T$ and a single trajectory is sampled using the probabilities from the last iteration $\{\hat{w}_T^{(j)}\}_{j=1}^{J}$. An important improvement upon the original PG sampler was introduced by Whiteley (2010), who suggested drawing the path of the state variables from the discrete particle approximation using the backwards sampling algorithm of Godsill, Doucet, and West (2004). On the forwards pass, we store the normalized weights and particles $\{\hat{w}_t^{(m)}, h_{i,t}^{(m)}\}_{m=1}^{M}$ for $t = 1, \dots, T$. We draw a path of the state variables $(h_1^*, \dots, h_T^*)$ from this discrete distribution.

At $t = T$, draw a particle $h_T^* = h_T^{(j)}$ with probability $\hat{w}_T^{(j)}$.

For $t = T - 1, \dots, 0$, run:

- For $j = 1, \dots, J$, calculate the backwards weights: $w_{t|T}^{(j)} \propto \hat{w}_t^{(j)} p(h_{t+1}^*|h_t^{(j)}, \theta)$.

- For $j = 1, \dots, J$, normalize the weights: $\hat{w}_{t|T}^{(j)} = \frac{w_{t|T}^{(j)}}{\sum_{j=1}^{J} w_{t|T}^{(j)}}$.

- Draw a particle $h_t^* = h_t^{(j)}$ with probability $\hat{w}_{t|T}^{(j)}$.

The draw $h_{0:T} = (h_0^*, \ldots, h_T^*)$ is a draw from the full-conditional distribution. In practice, when the dimension H of h_t is high, the number of particles J required for satisfactory performance can be quite large. In this case, we can separate each element of the state vector $h_{i,t}$ for $i = 1, \ldots, H$ and draw them one at a time.

I.1.3 *Drawing the parameters*   We block the parameters into groups that are highly correlated. These groups can be separated into parameters governing the dynamics of g_t and the parameters that enter the dynamics of volatility h_t.

1. *Drawing parameters in $\bar{\mu}_g$, Φ_{gh}, Σ_{gh}:* We place these parameters in the state vector and draw them jointly with the Gaussian state variables.

2. *Drawing parameters in Φ_g, $\Sigma_{0,g}$:* We use the independence Metropolis–Hastings algorithm. Conditional on the volatility state variables $h_{0:T}$, the model is a linear, Gaussian state space model (I.1) and (I.2). We maximize the likelihood using the Kalman filter and calculate the Hessian at the posterior mode. We then draw from a Student's t proposal distribution with mean equal to the posterior mode and covariance matrix equal to the inverse Hessian at the mode.

3. *Drawing parameters of the volatility process $\bar{\mu}_h$, Φ_h, Σ_h:* We use an independence Metropolis–Hastings step. When drawing these parameters, we can marginalize out the Gaussian state variables using the Kalman filter. Conditional on the remaining parameters of the model (which we omit), the target distribution of ν_h, Φ_h, Σ_h can be written as

$$p(\bar{\mu}_h, \Phi_h, \Sigma_h | Y_{1:T}, h_{0:T})$$
$$\propto p(Y_{1:T} | h_{0:T}, \bar{\mu}_h, \Phi_h, \Sigma_h) p(h_{0:T} | \bar{\mu}_h, \Phi_h, \Sigma_h) p(\bar{\mu}_h, \Phi_h, \Sigma_h),$$

where $p(Y_{1:T} | h_{0:T}, \bar{\mu}_h, \Phi_h, \Sigma_h)$ is the likelihood from the Kalman filter, $p(h_{0:T} | \bar{\mu}_h, \Phi_h, \Sigma_h)$ is the transition density of the volatility process (B.5). We maximize this target density and calculate the Hessian at the posterior mode. We then draw from a Student's t-proposal distribution with mean equal to the posterior mode and covariance matrix equal to the inverse Hessian at the mode.

For Gaussian models, we draw the free parameters in $\Sigma_{0,g}$ instead of $\bar{\mu}_h$, Φ_h, Σ_h.

I.2 *Particle filter*

To estimate the structural parameters (β, γ, ψ) and the preference parameters θ^λ, we run cross-sectional regressions on filtered and/or smoothed estimates of the factors. In order to calculate the filtered estimates of the state variables, we use a particle filter. The particle filter we implement is the mixture Kalman filter of Chen and Liu (2000). Let $g_{t|t-1}$ denote the conditional mean and $P_{t|t-1}$ the conditional covariance matrix of the one-step ahead predictive distribution $p(g_t | Y_{1:t-1}, h_{0:t-1}; \theta)$ of a conditionally linear, Gaussian state space model. Similarly, let $g_{t|t}$ denote the conditional mean and $P_{t|t}$ the conditional covariance matrix of the filtering distribution $p(g_t | Y_{1:t}, h_{0:t}; \theta)$. Conditional on the volatilities $h_{0:T}$, these quantities can be calculated by the Kalman filter.

Let J denote the number of particles and let $Y_t = (\pi_t, \Delta c_t)$ be $N \times 1$. The particle filter then proceeds as follows:

At $t = 0$, for $i = 1, \ldots, J$, set $w_0^{(i)} = \frac{1}{J}$ and

- Draw $h_0^{(i)} \sim p(h_0; \theta)$ and calculate $\Sigma_{g,0}^{(i)} \Sigma_{g,0}^{(i),\prime}$.

- Set $g_{1|0}^{(i)} = \bar{\mu}_g + \Phi_{gh} \bar{h}_0^{(i)}$, $P_{1|0}^{(i)} = \Sigma_{g,0}^{(i)} \Sigma_{g,0}^{(i),\prime}$.

- Set $\ell_0 = 0$.

For $t = 1, \ldots, T$ do:

STEP 1: For $i = 1, \ldots, J$:

- Draw from the transition density: $h_{t+1}^{(i)} \sim p(h_{t+1}|h_t^{(i)}; \theta)$ given by

$$z_{j,t+1}^{(i)} \sim \text{Poisson}\big(e_j' \Sigma_h^{-1} \Phi_h h_t^{(i)}\big), \quad j = 1, \ldots, H,$$

$$w_{j,t+1}^{(i)} \sim \text{Gamma}\big(\nu_{h,j} + z_{j,t+1}^{(i)}, 1\big), \quad j = 1, \ldots, H,$$

$$h_{t+1}^{(i)} = \Sigma_h w_{t+1}^{(i)}.$$

- Calculate $c_t^{(i)}$ and $Q_t^{(i)}$ using $h_t^{(i)}$:

$$c_t^{(i)} = \Phi_{gh} h_t^{(i)} + \Sigma_{gh} \varepsilon_{h,t+1}^{(i)},$$

$$Q_t^{(i)} = \Sigma_{g,t}^{(i)} \Sigma_{g,t}^{(i),\prime}.$$

- Run the Kalman filter:

$$v_t^{(i)} = Y_t - Z g_{t|t-1}^{(i)} - d,$$

$$F_t^{(i)} = Z P_{t|t-1}^{(i)} Z' + H,$$

$$K_t^{(i)} = P_{t|t-1}^{(i)} Z' \big(F_t^{(i)}\big)^{-1},$$

$$g_{t|t}^{(i)} = g_{t|t-1}^{(i)} + K_t^{(i)} v_t^{(i)},$$

$$P_{t|t}^{(i)} = P_{t|t-1}^{(i)} - K_t^{(i)} Z P_{t|t-1}^{(i)},$$

$$g_{t+1|t}^{(i)} = T g_{t|t}^{(i)} + c_t^{(i)},$$

$$P_{t+1|t}^{(i)} = T P_{t|t}^{(i)} T' + R Q_t^{(i)} R'.$$

- Calculate the weight:
$\log(w_t^{(i)}) = \log(\hat{w}_{t-1}^{(i)}) - 0.5N \log(2\pi) - 0.5 \log |F_t^{(i)}| - \frac{1}{2} v_t^{(i)\prime} (F_t^{(i)})^{-1} v_t^{(i)}.$

STEP 2: Calculate an estimate of the log-likelihood: $\ell_t = \ell_{t-1} + \log(\sum_{i=1}^{J} w_t^{(i)})$.

STEP 3: For $i = 1, \ldots, J$, calculate the normalized importance weights: $\hat{w}_t^{(i)} = \dfrac{w_t^{(i)}}{\sum_{j=1}^{J} w_t^{(j)}}$.

STEP 4: Calculate the effective sample size $E_t = \frac{1}{\sum_{j=1}^{J} (\hat{w}_t^{(j)})^2}$.

STEP 5: If $E_t < 0.5J$, resample $\{g_{t+1|t}^{(i)}, P_{t+1|t}^{(i)}, h_{t+1}^{(i)}\}_{i=1}^{J}$ with probabilities $\hat{w}_t^{(i)}$ and set $\hat{w}_t^{(i)} = \frac{1}{J}$.

STEP 6: Increment time and return to STEP 1.

Within the particle filter, we use the residual resampling algorithm of Liu and Chen (1998). We set $J = 100,000$.

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