Supplementary Material

# Supplement to "The endogenous grid method for discrete-continuous dynamic choice models with (or without) taste shocks": Proof of Theorem 1

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This online appendix provides the calculations for a backward induction proof of Theorem 1, which characterizes the optimal consumption rule for a version of the life-cycle model introduced by Phelps (1962) when it is extended to allow the irreversible retirement choice. This analytical solution is used both to illustrate the DC-EGM algorithm for solving life-cycle problems involving both discrete and continuous choices, and to illustrate its accuracy.

Consider the discrete-continuous (DC) dynamic optimization problem

$$\max_{\{c_t, d_t\}_{t=1}^{T}} \sum_{t=1}^{T} \beta^t \big(\log(c_t) - \delta d_t\big), \tag{S1}$$

involving choice of consumption $c_t$ and when to retire to maximize discounted utility, where $d_t = 0$ denotes retirement, $d_t = 1$ denotes continued work, and $\delta > 0$ is the disutility of work. We assume retirement is absorbing, that is, a retiree cannot return to work.

We solve (S1) subject to a sequence of period-specific borrowing constraints, $c_t \leq M_t$, where $M_t = R(M_{t-1} - c_{t-1}) + y d_{t-1}$ is the consumer's resources at the beginning of period $t$. There is a fixed, nonstochastic gross interest rate $R$ and labor income $y$ for workers. The continuous consumption decision and discrete retirement decision are made at

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the start of each period, whereas interest earnings and labor income are paid at the end of the period. This timing convention is standard in the literature and is the appropriate when we subsequently extend the model to a much wider class of problems where $R$ and $y$ are random variables.

Let $V_t(M)$ and $V_t^R(M)$ be the expected discounted lifetime utility of a worker and a retiree, respectively, in period $t$ of their life. The choice problem of the worker can be expressed recursively through the Bellman equation

$$V_t(M) = \max\big\{v_t(M,0), v_t(M,1)\big\}, \tag{S2}$$

where the *choice-specific value functions* are given as

$$v_t(M,0) = \max_{0\le c\le M}\big\{\log(c) + \beta V_{t+1}\big(R(M-c)\big)\big\}, \tag{S3}$$

$$v_t(M,1) = \max_{0\le c\le M}\big\{\log(c) - \delta + \beta V_{t+1}^R\big(R(M-c)+y\big)\big\}. \tag{S4}$$

The value function for a retiree $V_t^R(M)$ has a closed-form solution given by Phelps (1962, p. 742), so we focus on deriving formulas for $v_t(M,1)$ and finding the optimal consumption rule $c_t(M,1)$ for a worker who has the option to either retire or continue working.

Note that even if $v_t(M,0)$ and $v_t(M,1)$ are concave functions of $M$, the value function $V_t(M)$ is the maximum of these two concave functions and will generally not be globally concave (Clausen and Strub (2013)). Furthermore. $V_t(M)$ will generally have a *kink point* at the value $M = \overline{M}_t$ where the two choice-specific value functions cross: $v_t(\overline{M}_t,1) = v_t(\overline{M}_t,0)$. We refer to these as *primary kinks* since they constitute *optimal retirement thresholds* for the worker, that is, $d_t(M) = 1$ if $M < \overline{M}_t$ and $d_t(M) = 0$ if $M \ge \overline{M}_t$.

The worker is indifferent between retiring and working at the primary kink $\overline{M}_t$, and $V_t(M,1)$ is nondifferentiable at this point. However, the left and right hand derivatives, $V_t^-$ and $V_t^+$, exist and satisfy $V_t^-(\overline{M}_t,1) < V_t^+(\overline{M}_t,1)$. The discontinuity in the derivative of $V_t(M)$ at $\overline{M}_t$ leads to a discontinuity in the optimal consumption function in the *previous* period $t-1$ because the Bellman equation for $V_{t-1}(M)$ depends on $V_t(M)$, so the primary kink in the latter results in a *secondary kink* in $V_{t-1}(M)$. Thus, the primary kinks propagate back in time and manifest themselves in an accumulation of secondary kinks in the value functions in earlier periods, resulting in a increasing number of discontinuities in the consumption functions in earlier periods of the life cycle. The jumps in consumption are caused by the worker's desire to increase saving for an *anticipated retirement* at some point in the future.

THEOREM 1 (Analytical Solution to the Retirement Problem (Revisited)). *Assume that income and disutility of work are time-invariant, and the discount factor $\beta$ and the disutility of work $\delta$ are not too large, that is,*

$$\beta R \le 1 \quad \textit{and} \quad \delta < (1+\beta)\log(1+\beta). \tag{S5}$$

*Then for $\tau \in \{1, \dots, T\}$ the optimal consumption rule in the workers' problem* (S2)–(S4) *is given by*

$$c_{T-\tau}(M,1)=\begin{cases} M & \text{if } M \le y/R\beta, \\ [M+y/R]/(1+\beta) & \text{if } y/R\beta \le M \le \overline{M}^{l_1}_{T-\tau}, \\ \big[M+y\big(1/R+1/R^2\big)\big]/\big(1+\beta+\beta^2\big) & \text{if } \overline{M}^{l_1}_{T-\tau} \le M \le \overline{M}^{l_2}_{T-\tau}, \\ \dots & \dots \\ \left[M+y\left(\sum_{i=1}^{\tau-1}R^{-i}\right)\right]\left(\sum_{i=0}^{\tau-1}\beta^i\right)^{-1} & \text{if } \overline{M}^{l_{\tau-2}}_{T-\tau} \le M \le \overline{M}^{l_{\tau-1}}_{T-\tau}, \\ \left[M+y\left(\sum_{i=1}^{\tau}R^{-i}\right)\right]\left(\sum_{i=0}^{\tau}\beta^i\right)^{-1} & \text{if } \overline{M}^{l_{\tau-1}}_{T-\tau} \le M < \overline{M}^{r_{\tau-1}}_{T-\tau}, \\ \left[M+y\left(\sum_{i=1}^{\tau-1}R^{-i}\right)\right]\left(\sum_{i=0}^{\tau}\beta^i\right)^{-1} & \text{if } \overline{M}^{r_{\tau-1}}_{T-\tau} \le M < \overline{M}^{r_{\tau-2}}_{T-\tau}, \\ \dots & \dots \\ \big[M+y\big(1/R+1/R^2\big)\big]\left(\sum_{i=0}^{\tau}\beta^i\right)^{-1} & \text{if } \overline{M}^{r_2}_{T-\tau} \le M < \overline{M}^{r_1}_{T-\tau}, \\ [M+y/R]\left(\sum_{i=0}^{\tau}\beta^i\right)^{-1} & \text{if } \overline{M}^{r_1}_{T-\tau} \le M < \overline{M}_{T-\tau}, \\ M\left(\sum_{i=0}^{\tau}\beta^i\right)^{-1} & \text{if } M \ge \overline{M}_{T-\tau}. \end{cases} \tag{S6}$$

*The segment boundaries are totally ordered with*

$$y/R\beta < \overline{M}^{l_1}_{T-\tau} < \dots < \overline{M}^{l_{\tau-1}}_{T-\tau} < \overline{M}^{r_{\tau-1}}_{T-\tau} < \dots < \overline{M}^{r_1}_{T-\tau} < \overline{M}_{T-\tau}, \tag{S7}$$

*and the rightmost threshold $\overline{M}_{T-\tau}$, given by*

$$\overline{M}_{T-\tau} = \frac{(y/R)e^{-K}}{1-e^{-K}}, \quad \text{where } K = \delta\left(\sum_{i=0}^{\tau}\beta^i\right)^{-1}, \tag{S8}$$

*defines the smallest level of wealth sufficient to induce the consumer to retire at age $t = T - \tau$.*

Proof. It is straightforward to show using backward induction that the value function for a retiree at age $T-t$ (i.e., $t$ periods before end of life) is a logarithmic function of $M$,[1]

$$V^R_{T-t}(M) = \log(M)\left(\sum_{i=0}^{t}\beta^i\right) + A_t, \tag{S9}$$

[1] See Phelps (1962), Hakansson (1970).

where

$$A_{T-t} = -\log\left(\sum_{i=0}^{t}\beta^i\right)\left(\sum_{i=0}^{t}\beta^i\right) + \beta\big[\log(\beta)+\log(R)\big]\left[\sum_{i=0}^{t-1}\beta^i\left(\sum_{j=0}^{t-1-i}\beta^j\right)\right]. \tag{S10}$$

The optimal consumption rule for a retiree is linear in $M$:

$$c_{T-t}(M,0) = M\left(\sum_{i=0}^{t}\beta^i\right)^{-1}. \tag{S11}$$

Recalling that $v_t(M,1)$ is the discounted utility of a person of age $T-t$ who decides to work (not retire), we can define the optimal retirement threshold at age $t$, $\overline{M}_t$, as the value of $M$ that makes the person indifferent between retiring and not retiring at that age:

$$v_t(\overline{M},0) = v_t(\overline{M},1). \tag{S12}$$

Since we assume $\delta > 0$ (positive disutility from working), it will be optimal for a person of age $t$ to retire if $M \geq \overline{M}_t$ and to work otherwise. We will have a nonconvex kink in the value function for working $v_t(M,1)$ at the point $\overline{M}_t$ since we have

$$V_t(M) = \max\big[v_t(M,0), v_t(M,1)\big]. \tag{S13}$$

As we show below, the two decision-specific value functions are strictly concave and intersect only once at a point $\overline{M}_t$ for which we provide an explicit expression below. We show that $v_t(M,1) > v_t(M,0)$ for $M < \overline{M}_t$ so it is optimal to work in this region, and $v_t(M,1) < v_t(M,0)$ for $M > \overline{M}_t$, so it is optimal to retire in this region.

Let $c_t(M,0)$ be the optimal consumption of a retiree of age $t$. This function is given by formula (S11) above (with trivial reindexing). The optimal consumption of a individual who decides not to retire is

$$c_t(M,1) = \underset{0\leq c\leq M}{\operatorname{argmax}}\big[\log(c) - \delta + \beta V_{t+1}\big(R(M+y_t-c)\big)\big]. \tag{S14}$$

The overall optimal consumption rule is then given by

$$c_t(M) = \begin{cases} c_t(M,1) & \text{if } M < \overline{M}_t, \\ c_t(M,0) & \text{if } M \geq \overline{M}_t. \end{cases} \tag{S15}$$

It is easy to see that due to the nonconvex kink in the value function at $\overline{M}_t$, the optimal consumption function $c_t(M)$ will have a discontinuity at $\overline{M}_t$, and

$$c_t(\overline{M}_t,1) > c_t(\overline{M}_t,0). \tag{S16}$$

This result follows from the condition that

$$V_t'^{-}(\overline{M}_t) < V_t'^{+}(\overline{M}_t). \tag{S17}$$

Since there is a kink at $\overline{M}_t$, the derivative $V_t^{\prime -}(\overline{M}_t)$ must be interpreted as the left hand derivative (derivative from below $\overline{M}_t$); correspondingly, $V_t^{\prime +}(\overline{M}_t)$ is the right hand derivative of $V_t$ at $M = \overline{M}_t$.

We now establish these results by backward induction, starting at period $T-1$, which is the first period where the consumption–retirement decision is nontrivial (it is easy to see that in the final period of life, it is optimal to retire and consume all remaining savings). For notational simplicity, we drop the time subscripts on income, $y = y_t$, since income is constant by assumption. To derive a formula for the retirement threshold $\overline{M}_{T-1}$, consider the $T-1$ optimization problem

$$c_{T-1}(M|d=1) = \underset{0\le c\le M}{\operatorname{argmax}}\big[\log(c) - \delta + \beta \log\big(R(M-c)+y\big)\big]. \tag{S18}$$

The solution to this is given by

$$c_{T-1}(M|d=1) = \begin{cases} M & \text{if } M < y/R\beta, \\ (M+y/R)/(1+\beta) & \text{if } y/R\beta \le M \le \overline{M}_{T-1}. \end{cases} \tag{S19}$$

Note that the worker is liquidity constrained when $M < y/R\beta$ and in this region it is optimal to consume all of her beginning of period savings $M$ and rely on the end-of-period payment of wage earnings $y$ to finance consumption in her last period of life, $T$. The value function for the worker at age $T-1$ is

$$\begin{aligned} &v_{T-1}(M|d=1) \\ &= \begin{cases} \log(M) - \delta + \beta\log(y) \\ \quad \text{if } M < y/R\beta, \\ \log(M+y/R)(1+\beta) - \delta + \beta\big[\log(\beta)+\log(R)\big] - \log(1+\beta)(1+\beta) \\ \quad \text{if } y/R\beta \le M \le \overline{M}_{T-1}, \end{cases} \end{aligned}$$

and the value function for a retiree is given by equation (S9). Equating the values of work and retirement, and solving for the optimal retirement threshold $\overline{M}_{T-1}$ we have

$$\overline{M}_{T-1} = \frac{(y/R)\exp\big\{-\delta/(1+\beta)\big\}}{1-\exp\big\{-\delta/(1+\beta)\big\}}, \tag{S20}$$

provided this is greater than $y/R\beta$ (the threshold below which the consumer is liquidity constrained); otherwise,

$$\overline{M}_{T-1} = \big[y/(R\beta)\big](1+\beta)^{\frac{(1+\beta)}{\beta}} \exp\{-\delta/\beta\}. \tag{S21}$$

However, it is easy to see that assumption $\delta < (1+\beta)\log(1+\beta)$ implies that $\overline{M}_{T-1} > y/R\beta$. It is also easy to see that as the disutility of working $\delta \to \infty$, we have $\overline{M}_{T-1} \to 0$, and as $\delta \to 0$, then $\overline{M}_{T-1} \to \infty$, that is, if there is no disutility of working, the person would never choose to retire.

Note also that at $\overline{M}_{T-1}$ there is a kink in the value function: this is a downward kink (in terms of Clausen and Strub (2013)) as the max of two concave functions $v_{T-1}(M,0)$

and $v_{T-1}(M,1)$, and this kink in the value function results in a *discontinuity* in the optimal consumption function $c_{T-1}(M)$. There is a drop in consumption equal to $(y/R)/(1+\beta)$ at $\overline{M}_{T-1}$, and with two remaining periods in his/her life, a retiree has a "marginal propensity to consume" out of wealth equal to $1/(1+\beta)$ the same as a worker. The discontinuous drop in consumption that occurs at the retirement threshold equals the present value of forgone earnings due to retirement, $y/R$, multiplied by the marginal propensity to consume out of wealth, $1/(1+\beta)$.

To summarize the solution at $T-1$, the optimal retirement threshold is $\overline{M}_{T-1}$ given in equation (S20), the consumption function is given by

$$c_{T-1}(M)=\begin{cases} M & \text{if } M<y/R\beta, \\ (M+y/R)/(1+\beta) & \text{if } y/R\beta\leq M\leq \overline{M}_{T-1}, \\ M/(1+\beta) & \text{if } M>\overline{M}_{T-1}, \end{cases} \tag{S22}$$

and the value function is given by

$$\begin{aligned} &V_{T-1}(M) \\ &=\begin{cases} \log(M)-\delta+\beta\log(y) \\ \quad \text{if } M<y/R\beta, \\ \log(M+y/R)(1+\beta)-\delta+\beta\bigl[\log(\beta)+\log(R)\bigr]-\log(1+\beta)(1+\beta) \\ \quad \text{if } y/R\beta\leq M\leq \overline{M}_{T-1}, \\ \log(M)(1+\beta)+\beta\bigl[\log(\beta)+\log(R)\bigr]-\log(1+\beta)(1+\beta) \\ \quad \text{if } M>\overline{M}_{T-1}. \end{cases} \end{aligned} \tag{S23}$$

Now consider going back one more time period in the backward recursion, to $T-2$. We want to illustrate the possibility of *secondary kinks/discontinuities* in the consumption function for a worker $c_{T-2}(M,1)$ caused by the kinks in $V_{T-1}(M)$. Let $\overline{M}_{T-2}$ denote the *primary kink* due to the retirement threshold at $T-2$ and let $\overline{M}^{r_j}_{T-2}$ denote the secondary kinks, where $j=1,\dots,N_{T-2}$ and $N_{T-t}$ is the number of secondary kinks $t$ periods before the end of life at age $T$.

To see how these secondary kinks arise, consider how the $T-2$ consumption function is determined, as the solution to

$$c_{T-2}(M,1)=\underset{0\leq c\leq M}{\operatorname{argmax}}\bigl[\log(c)-\delta+\beta V_{T-1}\bigl(R(M-c)+y\bigr)\bigr]. \tag{S24}$$

As shown above, $V_{T-1}(M)$ has two kinks: one at $M=y/R\beta$, where the liquidity constraint stops being binding, and the other at $\overline{M}_{T-1}$, where the worker retires. Assume that the initial wealth of the worker at the start of period $T-1$ is low enough so that the worker will be liquidity constrained in period $T-1$. This implies that $R(M-c)+y<y/R\beta$. Then substituting the liquidity-constrained formula for $V_{T-1}(M)$ from (S23) into the period $T-2$ optimization (S24), we find that optimal consumption is given by $c_{t-2}(M,1)=(M+y/R)/(1+\beta)$. However, imposing the liquidity constraint, we must also have $(M+y/R)/(1+\beta)\leq M$, which implies that $M\leq y/R\beta$, and it is easy to verify that for

wealth satisfying this constraint, the worker will be liquidity constrained both in period $T-2$ and in period $T-1$ as well.

However, for wealth above $y/R\beta$, the worker is no longer liquidity constrained in period $T-2$ but our derivation of the worker's consumption in period $T-2$ is still contingent on the assumption that the worker is liquidity constrained in period $T-1$. This will be true provided that the savings and earnings the worker brings to the start of period $T-1$, $R\beta(M+y/R)/(1+\beta)$, is less than $y/R\beta$, which is equivalent to the inequality $M \leq [y/(R\beta)^2](1+\beta-R\beta^2)$. It is not hard to show that when $R=1$, we have $y/\beta < (y/\beta^2)(1+\beta-\beta^2)$ so the interval for which the consumer will consume $(M+y)/(1+\beta)$ is nonempty when $R=1$. For $R>1$, the inequality $y/(R\beta) < [y/(R\beta)^2](1+\beta-R\beta^2)$ is equivalent to $R\beta<1$, so under this assumption this interval will also exist; otherwise, the interval is empty and the consumer goes from consuming $c_{T-2}(M,1)=M$ to consuming an amount we derive below.

In the next region, wealth is sufficiently high in period $T-2$ so the consumer is not liquidity constrained at $T-2$, and the saving and earning will keep the consumer out of the liquidity-constrained region at $T-1$, but the worker's wealth is not high enough to retire at $T-1$. The relevant expression for $V_{T-1}(M)$ in this case is given by the middle expression in equation (S23). This implies an optimal consumption level equal to $c_{T-2}(M,1)=(M+y(1/R+1/R^2))/(1+\beta+\beta^2)$.

For even larger level of wealth there will come a point where the consumer can save enough in period $T-2$ to retire in period $T-1$, that is, savings will exceed the $\overline{M}_{T-1}$ threshold. Thus, there is some wealth level $\overline{M}^r_{T-2}$ at which the the relevant expression for the worker's period $T-1$ value function $V_{T-1}(M)$ is given by the last, retirement, formula in (S23). The optimal consumption in this region is $c_{T-2}(M,1)=(M+y/R)/(1+\beta+\beta^2)$. It is important to carefully check values of $c$ such that savings $M+y-c$ is in the "convex region" of $V_{T-1}(M)$ around the $T-1$ retirement threshold $\overline{M}_{T-1}$. In this region there will be *two local optima* for $c$: one involving the higher consumption $(M+y(1/R+1/R^2))/(1+\beta+\beta^2)$ and the other involving the lower consumption $(M+y/R)/(1+\beta+\beta^2)$ that enables the worker to retire at $T-1$.

These two solutions are reflected in the two possible solutions to the first order condition for optimal consumption given by

$$0=\frac{1}{c}-\begin{cases}(\beta+\beta^2)/\big(M-c+y\big(1/R+1/R^2\big)\big) & \text{if } R(M-c)+y\leq\overline{M}_{T-1},\\ (\beta+\beta^2)/(M-c+y/R) & \text{if } R(M-c)+y>\overline{M}_{T-1}.\end{cases} \tag{S25}$$

For $M<\overline{M}^r_{T-2}$, the global optimum will be $c_{T-2}(M,1)=(M+y(1/R+1/R^2))/(1+\beta+\beta^2)$ and the consumer will be working in both periods $T-2$ and $T-1$. However, for $M>\overline{M}^r_{T-2}$, the consumer will still work at $T-2$ (provided $M<\overline{M}_{T-2}$, the primary kink point at $T-2$, the wealth threshold at which the consumer retires at $T-2$) but will have enough savings to retire at $T-1$. The optimal consumption in this case will be $c_{T-2}(M,1)=(M+y/R)/(1+\beta+\beta^2)$. It is not hard to show that if $M\leq[y/(R\beta)^2]\times(1+\beta-R\beta^2)$, then the quantity $R(M-c_{T-2}(M,1))+y\leq y/R\beta$, that is, the consumer will indeed be in the liquidity-constrained region $M\leq y/R\beta$ at the start of $T-1$ as we assumed would be the case. We also have that $y/R\beta<[y/(R\beta)^2](1+\beta-R\beta^2)$ provided that

$R\beta \leq 1$, which we assumed to be the case. Otherwise this region would be empty and the optimal consumption would be given by $c_{T-2}(M,1) = (M + y(1/R + 1/R^2))/(1 + \beta + \beta^2)$ as derived above. We can check that this consumption function, which is also derived under the assumption that the consumer will not be liquidity constrained at period $T-1$, will result in total savings at $T-1$ that satisfies $R(M-c)+y \geq y/R\beta$ (so the consumer is not liquidity constrained at $T-1$) for wealth at $T-2$ at the lower end of this interval (i.e., at $M = y/R\beta$) provided that $R \leq 1/\beta$.

However, at $M = \overline{M}^r_{T-2}$ the consumer will be indifferent between consuming the larger amount $(M + y(1/R + 1/R^2))/(1 + \beta + \beta^2)$ knowing that he/she will *not* retire at $T-1$ and consuming the lower amount $(M + y/R)/(1 + \beta + \beta^2)$ and knowing that he/she will retire at $T-1$. We find $\overline{M}^r_{T-2}$ as the solution to the equation

$$\begin{aligned}
&\log\bigl(\bigl(M + y\bigl(1/R + 1/R^2\bigr)\bigr)/\bigl(1 + \beta + \beta^2\bigr)\bigr) \\
&\quad + \beta V_{T-1}\bigl(\bigl(y + R\bigl(M - \bigl(M + y\bigl(1/R + 1/R^2\bigr)\bigr)\bigr)/\bigl(1 + \beta + \beta^2\bigr)\bigr)\bigr) \\
&= \log\bigl((M + y/R)/\bigl(1 + \beta + \beta^2\bigr)\bigr) + \beta V_{T-1}\bigl(y + R\bigl(M - (M + y/R)\bigr)/\bigl(1 + \beta + \beta^2\bigr)\bigr).
\end{aligned}$$

Thus, at $M = \overline{M}^r_{T-2}$ the consumer is indifferent between consuming the larger amount $(M + y(1/R + 1/R^2))/(1 + \beta + \beta^2)$ or consuming the smaller amount $(M + y/R)/(1 + \beta + \beta^2)$ that provides the additional savings necessary to enable the consumer to retire at $T-1$.

Now we can express period $T-2$ consumption of the worker as the piecewise linear function

$$c_{T-2}(M,1) = \begin{cases}
M \\
\quad \text{if } M < y/R\beta, \\
(M + y/R)/(1+\beta) \\
\quad \text{if } y/R\beta \leq M \leq \bigl[y/(R\beta)^2\bigr]\bigl(1 + \beta - R\beta^2\bigr), \\
\bigl(M + y\bigl(1/R + 1/R^2\bigr)\bigr)/\bigl(1 + \beta + \beta^2\bigr) \\
\quad \text{if } \bigl[y/(R\beta)^2\bigr]\bigl(1 + \beta - R\beta^2\bigr) \leq M \leq \overline{M}^r_{T-2}, \\
(M + y/R)/\bigl(1 + \beta + \beta^2\bigr) \\
\quad \text{if } \overline{M}^r_{T-2} < M < \overline{M}_{T-2}.
\end{cases} \tag{S26}$$

It is straightforward to verify that $c_{T-2}(M,1)$ has two kinks at $M = [y/(R\beta)^2](1 + \beta - R\beta^2)$ and $M = y/R\beta$ followed by a discontinuity at $M = \overline{M}^r_{T-2}$.

To derive the time $T-2$ retirement threshold $\overline{M}_{T-2}$, we solve for the value of $M$ that makes the consumer indifferent between retiring at $T-2$ and working (but with enough wealth so that the person is above the secondary kink $\overline{M}^r_{T-2}$ where their consumption is given by $c_{T-2}(M,1) = (M + y/R)/(1 + \beta + \beta^2)$),

$$\log(M)\bigl(1 + \beta + \beta^2\bigr) + A_{T-2} = \log(M + y/R)\bigl(1 + \beta + \beta^2\bigr) - \delta + A_{T-2}, \tag{S27}$$

where $A_{T-2}$ is defined in equation (S10) above. Note that the right hand side of (S27) is the value function for a consumer who does not have enough wealth to retire at $T-2$, but since $M > \overline{M}^r_{T-2}$ (the secondary kink point), it follows that the appropriate formula for $V_{T-1}(M)$ will be the one where $M > \overline{M}_{T-1}$ in equation (S23) above. The solution to this equation is $\overline{M}_{T-2}$ given by

$$\overline{M}_{T-2} = \frac{(y/R)e^{-K}}{(1-e^{-K})}, \tag{S28}$$

where $K$ is given by

$$K = \frac{\delta}{(1+\beta+\beta^2)}. \tag{S29}$$

Notice that formulas (S28) and (S20) imply that $\overline{M}_{T-1} < \overline{M}_{T-2}$, that is, the wealth threshold for retirement decreases as one approaches the end of life, $T$.

To summarize the solution we found at $T-2$, the optimal retirement threshold $\overline{M}_{T-2}$ is the solution to equation (S27), and the optimal consumption function is given by

$$c_{T-2}(M) = \begin{cases} M \\ \quad \text{if } M < y/R\beta, \\ (M+y/R)/(1+\beta) \\ \quad \text{if } y/R\beta \le M \le \big[y/(R\beta)^2\big]\big(1+\beta-R\beta^2\big), \\ \big(M+y\big(1/R+1/R^2\big)\big)/\big(1+\beta+\beta^2\big) \\ \quad \text{if } \big[y/(R\beta)^2\big]\big(1+\beta-R\beta^2\big) \le M \le \overline{M}^r_{T-2}, \\ (M+y/R)/\big(1+\beta+\beta^2\big) \\ \quad \text{if } \overline{M}^r_{T-2} < M \le \overline{M}_{T-2}, \\ M/\big(1+\beta+\beta^2\big) \\ \quad \text{if } M > \overline{M}_{T-2}. \end{cases} \tag{S30}$$

The optimal consumption function at $T-2$ has two kinks at $M = y/R\beta$ (the level of wealth at which the consumer is no longer liquidity constrained) and $M = [y/(R\beta)^2] \times (1+\beta-R\beta^2)$, and two discontinuities: one at the secondary kink point $\overline{M}^r_{T-2}$ where consumption drops by $(y/R^2)/(1+\beta+\beta^2)$, and the other at the retirement threshold $\overline{M}_{T-2}$ where consumption drops by $(y/R)/(1+\beta+\beta^2)$. Note that the secondary kink point $\overline{M}^r_{T-2}$ is precisely the amount of wealth where, while the consumer does not yet retire at $T-2$, *they know they will have enough to retire at* $T-1$. Thus, the drop in consumption at this secondary kink point can be regarded as *saving at* $T-2$ *for their anticipated retirement at time* $T-1$.

The value function at $T-2$ can be expressed as

$$V_{T-2}(M) = \begin{cases} \log\big(c_{T-2}(M)\big) - \delta + \beta V_{T-1}\big(R\big(M-c_{T-2}(M)\big)+y\big) & \text{if } M < \overline{M}_{T-2}, \\ \log(M)\big(1+\beta+\beta^2\big) + A_{T-2} & \text{if } M \ge \overline{M}_{T-2}. \end{cases} \tag{S31}$$

Thus, depending on whether the person's wealth at $T-2$ is above or below the secondary kink point $\overline{M}^r_{T-2}$, he/she will know whether he/she will have enough (with their $T-2$ earnings $y$) to retire at $T-1$ or not, and will save/consume accordingly.

Now consider solving the problem at $t = T - 3$, three periods before the end of life. The consumption rule will have three kinks including the level of $M$ where the liquidity constraint no longer binds, and three discontinuities, including the retirement threshold $\overline{M}^{r}_{T-3}$ in period $T - 3$. One additional kink in $c_{T-3}(M)$ is added above the end point $[y/(R\beta)^2](1 + \beta - R\beta^2)$ of the first linear segment of $c_{T-2}(M)$ and reflects the liquidity constraint in period $T - 2$. The additional discontinuity corresponds to the secondary kink point $\overline{M}^{r}_{T-2}$.

Note the pattern here: $c_{T-1}(M)$ has one kink and one discontinuity, $c_{T-2}(M)$ has two kinks and two discontinuities, and $c_{T-3}(M)$ will have three kinks and three discontinuities. The important additional point to notice is that $c_{T-1}$, $c_{T-2}$, and, as we show shortly, $c_{T-3}$, are all *piecewise linear*.

It will be helpful to distinguish the points marking the sequence of connected linear segments of the consumption function due to kinks in the value function arising at the end of the liquidity-constrained region $[0, y/R\beta]$ from those at higher levels of wealth that related to retirement decisions—both current retirement and anticipated future retirements. As we noted there will always be an initial linear segment over the interval $[0, y/R\beta]$ where $c_t(M) = M$ for $M \in [0, y/R\beta]$. Thus there will be a kink in the consumption function at $y/R\beta$ related to the current period liquidity constraint. We have also shown that for $M > \overline{M}_t$ it will be optimal to retire, so there is a *discontinuity* in $c_t(M)$ at $\overline{M}_t$ that relates to the primary kink in the value function and the decision to retire in the current period.

However, at ages $T - t < T - 1$, in addition to these two "current period" kinks/discontinuities, there will be a set of kinks and discontinuities related to the future periods, that is, "future liquidity constraint" kinks $\overline{M}^{l_j}_{T-t}$ and a set of "future retirement threshold" discontinuities $\overline{M}^{r_j}_{T-t}$. These discontinuities correspond to secondary kinks in the same period value function and result from the primary kinks in the value functions of all future periods.

Thus $c_{T-2}(M)$ has one future liquidity constraint kink $\overline{M}^{l_1}_{T-2}$ at $[y/(R\beta)^2] \times (1 + \beta - R\beta^2)$ and one future retirement threshold discontinuity at $\overline{M}^{r_1}_{T-2}$. The former represents the level of saving at which the consumer is not liquidity constrained at age $T - 2$, but will be liquidity constrained at age $T - 1$. The latter is the level of wealth that leads the worker to discontinuously reduce consumption at $T - 2$ so as to have enough savings to retire at $T - 1$.

In period $T - 3$ there will be a total of three discontinuities in $c_{T-3}(M)$. The last discontinuity occurs at the retirement threshold $\overline{M}_{T-3}$, but there will be two additional discontinuities at the secondary kink points in the value function $V_{T-3}$. These are denoted $\overline{M}^{r_1}_{T-3}$ and $\overline{M}^{r_2}_{T-3}$. We have the ordering $\overline{M}_{T-3} > \overline{M}^{r_1}_{T-3} > \overline{M}^{r_2}_{T-3}$. The highest secondary kink point $\overline{M}^{r_1}_{T-3}$ is the level of wealth that leads the consumer to save an amount (including current period wage earnings) of $\overline{M}_{T-2}$, which is the retirement threshold at period $T - 2$. Thus at wealth levels that just exceed $\overline{M}^{r_1}_{T-3}$ the consumer works in period $T - 3$ but discontinuously reduces consumption so as to have enough resources to retire in period $T - 2$. At wealth levels that are just below $\overline{M}^{r_2}_{T-3}$, the consumer works in both periods $T - 3$ and $T - 2$, and retires only in period $T - 1$.

The consumption function $c_{T-3}(M)$ will also have two future liquidity constraint kinks $\overline{M}^{l_1}_{T-3} = [y/(R\beta)^2](1+\beta-R\beta^2)$ and $\overline{M}^{l_2}_{T-3}$ in addition to the current liquidity constraint at $M = y/R\beta$. The first kink will be at the level of saving that is sufficient for the consumer not to be liquidity constrained at age $T-3$ but not enough to avoid being liquidity constrained at age $T-2$. At $\overline{M}^{l_1}_{T-3}$ the consumer switches from consuming according to the second linear segment of $c_{T-3}(M) = (M+y/R)/(1+\beta)$ to consuming on the third linear segment $c_{T-3}(M) = (M+y(1/R+1/R^2))/(1+\beta+\beta^2)$.

At the second future liquidity constraint kink point $\overline{M}^{l_2}_{T-3}$, the worker has sufficient saving to not be liquidity constrained at both ages $T-3$ and $T-2$, but not enough to avoid being liquidity constrained at age $T-1$. At $\overline{M}^{l_2}_{T-3}$ the worker switches from consuming on the third segment of $c_{T-3}(M) = (M+y(1/R+1/R^2))/(1+\beta+\beta^2)$ to the fourth segment, which is the first of the segments created by the retirement threshold kink points $\overline{M}^{r_j}_{T-3}$. Thus for wealth that exceeds $\overline{M}^{l_2}_{T-3}$, consumption switches to $c_{T-3}(M) = (M+y(1/R+1/R^2+1/R^3))/(1+\beta+\beta^2+\beta^3)$. Then for still higher levels of wealth the worker consumes according to the various piecewise linear segments demarcated by the successive future retirement threshold kink points $\overline{M}^{r_j}_{T-3}$, $j = 2, 1$, and finally $\overline{M}_{T-3}$, the retirement threshold at period $T-3$.

Note that the marginal propensity to consume out of wealth is also piecewise linear and monotonically decreasing in $M$. In the liquidity-constrained region the marginal propensity to consume is 1, in the first of the liquidity-constrained consumption segments it is $1/(1+\beta)$, and in the second liquidity-constrained segment it is $1/(1+\beta+\beta^2)$. Then in the remaining retirement related consumption segments, the marginal propensity to consume out of wealth is constant and equal to $1/(1+\beta+\beta^2+\beta^3)$.

In summary, the consumption function at $T-3$ is given by

$$c_{T-3}(M) = \begin{cases} M & \text{if } M < y/R\beta, \\ (M+y/R)/(1+\beta) & \text{if } y/R\beta \le M \le \overline{M}^{l_1}_{T-3}, \\ \big(M+y\big(1/R+1/R^2\big)\big)/\big(1+\beta+\beta^2\big) & \text{if } \overline{M}^{l_1}_{T-3} \le M \le \overline{M}^{l_2}_{T-3}, \\ \big(M+y\big(1/R+1/R^2+1/R^3\big)\big)/\big(1+\beta+\beta^2+\beta^3\big) & \text{if } \overline{M}^{l_2}_{T-3} \le M \le \overline{M}^{r_2}_{T-3}, \\ \big(M+y\big(1/R+1/R^2\big)\big)/\big(1+\beta+\beta^2+\beta^3\big) & \text{if } \overline{M}^{r_2}_{T-3} \le M < \overline{M}^{r_1}_{T-3}, \\ (M+y/R)/\big(1+\beta+\beta^2+\beta^3\big) & \text{if } \overline{M}^{r_1}_{T-3} \le M < \overline{M}_{T-3}, \\ M/\big(1+\beta+\beta^2+\beta^3\big) & \text{if } \overline{M}_{T-3} < M. \end{cases} \tag{S32}$$

The retirement threshold $\overline{M}_{T-3}$ is given by

$$\overline{M}_{T-3} = \frac{(y/R)e^{-K}}{(1-e^{-K})}, \quad \text{where } K = \frac{\delta}{(1+\beta+\beta^2+\beta^3)}. \tag{S33}$$

We solve for the secondary kinks/discontinuities $\{\overline{M}^{l_i}_{T-3}, \overline{M}^{r_j}_{T-3}\}$, $i = 1, 2$ and $j = 1, 2$, in the same way as we did for the period $T-2$: we solve for the level of a wealth that makes the consumer indifferent between consuming the higher level of consumption to the "left" of the kink point (more precisely, the limit of consumption for wealth approaching the kink point from below) and the lower level of consumption to the "right" of the discontinuity (the limit of consumption for wealth approaching the kink point from above).

Finally, the value function is given by

$$V_{T-3}(M) = \begin{cases} \log\bigl(c_{T-3}(M)\bigr) - \delta + \beta V_{T-2}\bigl(R\bigl(M - c_{T-3}(M)\bigr) + y\bigr) & \text{if } M < \overline{M}_{T-3}, \\ \log(M)\bigl(1 + \beta + \beta^2 + \beta^3\bigr) + A_{T-3} & \text{if } M \geq \overline{M}_{T-3}. \end{cases} \tag{S34}$$

Due to the monotonicity of the saving function, the fact that $\overline{M}_{T-2} > \overline{M}^r_{T-2}$ implies that $\overline{M}^{l_1}_{T-3} > M^{l_2}_{T-3} > \overline{M}^{r_1}_{T-3} > \overline{M}^{r_2}_{T-3}$. Similarly, it is not hard to show that $\overline{M}_{T-3} > \overline{M}_{T-2}$.

Having solved for the consumption function explicitly by doing backward induction for three periods, it is easy to see the general pattern. At $t$ periods before the end of life $T$, $t \geq 1$, that is, at period $T - t$, the consumption function $c_{T-t}(M)$ will have a total of $t$ kinks relating to current and future liquidity constraints, namely $y/R\beta$ and $\overline{M}^{l_j}_{T-t}$, $j = 1, \dots, t-1$; $t-1$ discontinuities relating the the future retirement thresholds denoted $\overline{M}^{r_j}_{T-t}$, $j = 1, \dots, t-1$; and one discontinuity at the period $t$ retirement threshold $\overline{M}_{T-t}$. Consequently, $c_{T-t}(M)$ will have $2t + 1$ linear segments. For every period $T - t$, $t \geq 1$, there will be a kink in the consumption function at $M = y/R\beta$ corresponding to the end of the liquidity-constrained region, $[0, y/R\beta]$.

Under the assumptions $R\beta \leq 1$ and $\delta < (1 + \beta)\log(1 + \beta)$, all the kink/discontinuity points define nonempty intervals such that the ordering

$$\begin{aligned} y/R\beta < \overline{M}^{l_1}_{T-t} < \overline{M}^{l_2}_{T-t} < \cdots < \overline{M}^{l_{t-1}}_{T-t} < \overline{M}^{r_{t-1}}_{T-t} < \overline{M}^{r_{t-2}}_{T-t} \\ < \cdots < \overline{M}^{r_2}_{T-t} < \overline{M}^{r_1}_{T-t} < \overline{M}_{T-t} \end{aligned} \tag{S35}$$

holds. The first of the future liquidity constraint kink points is always at the same value of $M$:

$$\overline{M}^{l_1}_{T-t} = \bigl[y/(R\beta)^2\bigr]\bigl(1 + \beta - R\beta^2\bigr) \quad \text{for } t \geq 2. \tag{S36}$$

Period $T - t$ retirement threshold $\overline{M}_{T-t}$ is given by

$$\overline{M}_{T-t} = \frac{(y/R)e^{-K}}{(1 - e^{-K})}, \quad \text{where } K = \delta\left(\sum_{i=0}^{t}\beta^i\right)^{-1}. \tag{S37}$$

The values of the last $t - 2$ future liquidity constraint kink points $\overline{M}^{l_j}_{T-t}$, $j = 2, \dots, t-1$, and the future retirement threshold discontinuity points $\overline{M}^{r_j}_{T-t}$, $j = 1, \dots, t-2$, are determined by the values of wealth that make the consumer indifferent between consuming according to the linear segments of the consumption function on either side of each of these kink points as described above.

The value function $V_{T-t}(M)$ can be expressed recursively in terms of the already defined value function $V_{T-t+1}(M)$ one period ahead,

$$V_{T-t}(M) = \begin{cases} \log\bigl(c_{T-t}(M)\bigr) - \delta + \beta V_{T-t+1}\bigl(R\bigl(M - c_{T-t}(M)\bigr) + y\bigr) & \text{if } M < \overline{M}_{T-t}, \\ \log(M)\left(\sum_{i=0}^{t} \beta^i\right) + A_{T-t} & \text{if } M \geq \overline{M}_{T-t}, \end{cases} \tag{S38}$$

where $A_{T-t}$ was defined in equation (S10) above. It is then straightforward to show with the formal mathematical induction argument the general formula (S6). □

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