

Supplement to “Euler equation estimation: Children and credit constraints”

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S.1. SOLVING THE FOUR-PERIOD MODEL

I normalize all variables with current income and denote lowercase variables as the normalized ones. Hence, for example, $m_1 = (A_0 + Y_1)/Y_1 = G_1^{-1}a_0 + 1$ since $Y_1 = G_1 Y_0$. In all other periods, income is constant. This normalization facilitates solving the model analytically for *all* possible values of income. The resulting consumption function should be multiplied with current period income to give the unnormalized level of consumption, $C_t^* = Y_t c_t^*$. The consumption functions in periods $t = 1, 2, 3$ are independent of whether children arrive deterministically or probabilistically since it is assumed that children, if present in period $t = 1$, will move with certainty in period $t = 2$. Therefore, I first solve for optimal consumption in periods $t = 1, 2, 3$, which are identical for the deterministic and probabilistic versions, and then turn to the initial period consumption, prior to the potential arrival of children. This analysis is split between the model in which children arrive deterministically and the model in which children arrive probabilistically.

In the terminal period, period 3, all resources are consumed ($c_3^* = m_3$) and the *unconstrained* Euler equation linking period 2 and period 3 consumption is then

$$c_2^{-\rho} = m_3^{-\rho}$$

such that inserting normalized resources, $m_3 = m_2 - c_2 + 1$, and rearranging shows that optimal consumption in period 2 is the minimum of available resources, m_2 , and $\frac{1}{2}(m_2 + 1)$. Since income does not fall between periods 1 and 2, and because negative wealth is not allowed, $m_2 \geq 1$ and optimal consumption is then

$$c_2^*(m_2) = \frac{1}{2}(m_2 + 1). \quad (\text{S.1})$$

In period 1, a child may be present and the *unconstrained* Euler equation is given by

$$c_1^{-\rho} \exp(\theta \mathbf{z}_1) = c_2^{-\rho},$$

such that inserting normalized resources and rearranging yields

$$c_1^*(m_1 | \mathbf{z}_1) = \min \left\{ m_1, \frac{m_1 + 2}{1 + 2 \exp(-\rho^{-1} \theta \mathbf{z}_1)} \right\}, \quad (\text{S.2})$$

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where the constraint is binding if $m_1 \leq \underline{m}_1 \equiv \exp(\rho^{-1}\theta\mathbf{z}_1)$. Note that this is certainly the case if nothing is saved from period 0.

Optimal consumption in period $t = 0$ depends on whether children arrive deterministically or probabilistically in period 1. I first derive optimal consumption in the case where children arrive deterministically and then turn to the probabilistic arrival of children.

S.1.1 Initial period consumption: Deterministic arrival of children

In the first period, the *unconstrained* Euler equation is

$$c_0^{-\rho} = G_1^{-\rho} \exp(\theta\mathbf{z}_1) c_1^{-\rho},$$

since income grows with a factor G_1 from period 0 to period 1. Since consumption in period 1 is potentially constrained, this has to be explicitly taken into account. First, assuming that period 1 consumption is less than available resources ($c_1 < m_1$), inserting the optimal consumption found in (S.2) and tedious rearranging yields optimal consumption in this case:

$$c_0^*(m_0|\mathbf{z}_1)^{\det}|_{c_1 < m_1} = \frac{m_0 + 3G_1}{3 + \exp(\rho^{-1}\theta\mathbf{z}_1)}. \quad (\text{S.3})$$

If, on the other hand, consumption in period 1 is constrained ($c_1 = m_1$), inserting this in the Euler equation and rearranging yields

$$c_0^*(m_0|\mathbf{z}_1)^{\det}|_{c_1 = m_1} = \frac{m_0 + G_1}{1 + \exp(\rho^{-1}\theta\mathbf{z}_1)}. \quad (\text{S.4})$$

To determine which of the consumption functions is relevant, note that equation (S.3) would imply a too high level of consumption in period 0 if ignoring that, at some point, consumption in period 1 would be constrained because “too little” is saved in period 0. Hence,

$$\tilde{c}_0^*(m_0|\mathbf{z}_1)^{\det} = \min \left\{ m_0, \frac{m_0 + G_1}{1 + \exp(\rho^{-1}\theta\mathbf{z}_1)}, \frac{m_0 + 3G_1}{3 + \exp(\rho^{-1}\theta\mathbf{z}_1)} \right\},$$

where the level of period $t = 0$ resources implying that consumption in period 1 is constrained is the level of resources, $\underline{m}_0^1 = \exp(\rho^{-1}\theta\mathbf{z}_1)G_1$, that makes the expression in (S.4) to be less than that in (S.3). Hence, when $m_0 \leq \underline{m}_0^1$, optimal consumption in period $t = 0$ is given by equation (S.4), and when $m_0 > \underline{m}_0^1$, optimal consumption is given by equation (S.3).

When households are initiated with zero wealth ($a_{-1} = 0$), available normalized resources in period 0 is 1, $m_0 = 1$, and only equation (S.4) is relevant since $m_0 = 1 \leq \underline{m}_0^1$ for all values of $\theta \geq 0$ and $G_1 \geq 1$. Therefore, assuming no initial wealth and deterministic arrival of children, optimal consumption in period 0 is given by

$$c_0^*(m_0|\mathbf{z}_1)^{\det} = \min \left\{ m_0, \frac{m_0 + G_1}{1 + \exp(\rho^{-1}\theta\mathbf{z}_1)} \right\}, \quad (\text{S.5})$$

where for $m_0 \leq \underline{m}_0^2 \equiv \exp(-\rho^{-1}\theta z_1)G_1$, the constraint is binding and it is optimal to consume everything. This is very intuitive: If income growth is very high, resources next period are much higher and saving today is less attractive. On the other hand, if children affect marginal utility a lot (θ large), the level of resources should be very low before it is optimal not to save anything for the next period, in which a child arrives.

Note that focusing on the situation in which a child arrives in period 1, if $m_0 \leq \underline{m}_0^2$, next-period resources are $m_1 = G_1^{-1}(m_0 - \frac{m_0 + G_1}{1 + \exp(\rho^{-1}\theta)}) + 1$ and we can check whether this is less than \underline{m}_1 which is the case as long as $\theta \geq 0$ and $G_1 \geq 1$. Hence, if $m_0 = 1 \leq \underline{m}_0$, we know that $m_1 \leq \underline{m}_1$ and $c_1^*(m_1 | z_1 = 1) = m_1$. If a child does not arrive, optimal consumption in all periods would be to consume available resources, since in period $t = 0$, borrowing against future income growth is not allowed. This is used when calculating the OLS and IV estimators below.

S.1.2 Initial period consumption: Probabilistic arrival of children

The analysis, if children arrive probabilistically, is slightly more complicated than the above where children arrive deterministically. The *unconstrained* Euler equation is here given by

$$c_0^{-\rho} = G_1^{-\rho} (p \exp(\theta) c_1 |_{z_1=1} + (1-p) c_1 |_{z_1=0}),$$

such that in the case where period 1 consumption is unconstrained ($c_1 < m_1$), inserting optimal consumption from equation (S.2) and rearranging yields

$$c_0^*(m_0) |_{c_1 < m_1} = \frac{m_0 + 3G_1}{1 + [p(\exp(\rho^{-1}\theta) + 2)^\rho + (1-p)3^\rho]^{1/\rho}}. \quad (\text{S.6})$$

However, if households are potentially credit constrained if a child arrives next period, the model has, in general, no analytical solution because the Euler equation is

$$\begin{aligned} c_0^{-\rho} &= G_1^{-\rho} [c_1^{-\rho} (1-p) + p \exp(\theta) m_1^{-\rho}] \\ &= G_1^{-\rho} \left[\left[\frac{1}{3} (G_1^{-1}(m_0 - c_0) + 3) \right]^{-\rho} (1-p) + p \exp(\theta) [G_1^{-1}(m_0 - c_0) + 1]^{-\rho} \right], \end{aligned}$$

with no general analytical solution for c_0 . To complete arguments, I solve for the optimal consumption numerically using the EGM proposed by Carroll (2006), and then use that solution, denoted $c_0^*(m_0) |_{c_1=m_1}^{\text{num}}$. In turn, optimal period 0 consumption is given by

$$c_0^*(m_0) = \min \left\{ m_0, c_0^*(m_0) |_{c_1=m_1}^{\text{num}}, \frac{m_0 + 3G_1}{1 + [p(\exp(\rho^{-1}\theta) + 2)^\rho + (1-p)3^\rho]^{1/\rho}} \right\}. \quad (\text{S.7})$$

Figure S.1(a) reports the consumption function in the deterministic case for the baseline parameters used herein ($p = 0.5$, $\rho = 2$, and $\theta = 0.5$) and Figure S.1(b) reports the consumption function in the probabilistic case. The numerical solutions to both models are reported to complete the solution and confirm the analytical results.

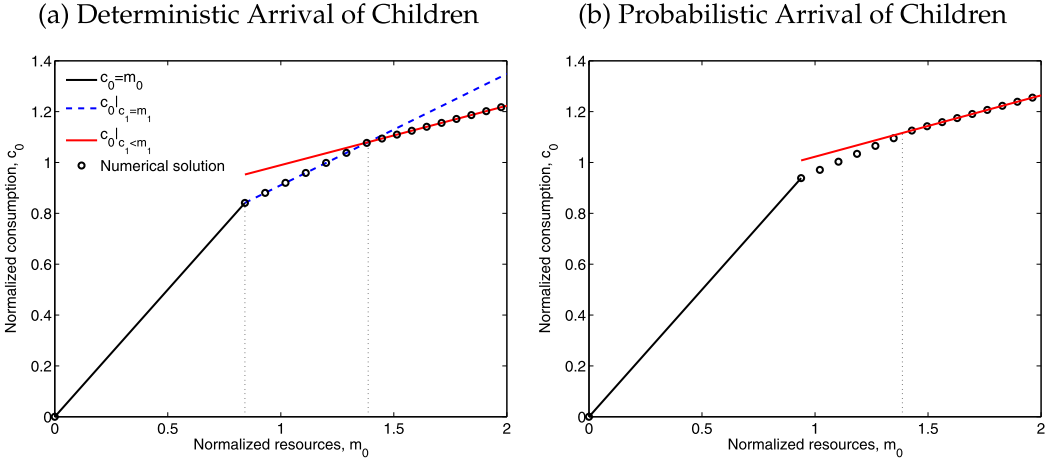


FIGURE S.1. Period 0 optimal consumption functions.

S.1.3 OLS and IV estimates from the four-period model

We have that optimal consumption is given by

$$c_0^*(m_0 | \mathbf{z}_1) = \begin{cases} m_0 & \text{if } m_0 \leq \exp(-\rho^{-1} \theta \mathbf{z}_1) G_1, \\ \frac{m_0 + G_1}{1 + \exp(\rho^{-1} \theta \mathbf{z}_1)} & \text{else,} \end{cases}$$

$$c_1^*(m_1 | \mathbf{z}_1) = \begin{cases} m_1 & \text{if } m_1 \leq \exp(\rho^{-1} \theta \mathbf{z}_1), \\ \frac{m_1 + 2}{1 + 2 \exp(-\rho^{-1} \theta \mathbf{z}_1)} & \text{else,} \end{cases}$$

$$c_2^*(m_2) = \frac{1}{2}(m_2 + 1),$$

$$c_3^*(m_3) = m_3.$$

The OLS estimator is given as

$$\theta_{\text{OLS}}^{\text{young}} = (\Delta \log C_1 |_{\mathbf{z}_1=1} - \Delta \log C_1 |_{\mathbf{z}_1=0}) \rho,$$

$$\theta_{\text{OLS}}^{\text{old}} = -(\Delta \log C_2 |_{\mathbf{z}_1=1} - \Delta \log C_2 |_{\mathbf{z}_1=0}) \rho,$$

while the IV estimator, using the (cohort-)average number of children as an instrument, $Z = p$, is

$$\theta_{\text{IV}}^{\text{young}} = \frac{1}{p} (p \Delta \log C_1 |_{\mathbf{z}_1=1} + (1-p) \Delta \log C_1 |_{\mathbf{z}_1=0}) \rho,$$

$$\theta_{\text{IV}}^{\text{old}} = -\frac{1}{p} (p \Delta \log C_2 |_{\mathbf{z}_1=1} + (1-p) \Delta \log C_2 |_{\mathbf{z}_1=0}) \rho.$$

Insert the optimal consumption for a given set of parameters. Let $m_0 > \exp(-\rho^{-1} \theta \mathbf{z}_1) G_1$ (saves in period 0) and note that $m_0 = 1$, such that this implies that

$\theta > \log(G_1)\rho$. The growth in log consumption is then (using the result that consumption is, then, constrained in period 1)

$$\theta_{\text{OLS}}^{\text{young}} = \begin{cases} \theta - \log(G_1)\rho & \text{if } \theta > \log(G_1)\rho, \\ 0 & \text{if } 0 \leq \theta \leq \log(G_1)\rho \end{cases}$$

$$\leq \theta.$$

Hence, OLS estimates will underestimate the true effect of children on consumption. The IV estimator is

$$\theta_{\text{IV}}^{\text{young}} = \begin{cases} \theta + (1-p)/p \log(G_1)\rho & \text{if } \theta > \log(G_1)\rho, \\ \log G_1 \rho / p & \text{if } 0 \leq \theta \leq \log(G_1)\rho \end{cases}$$

$$\geq \theta$$

such that IV estimation overestimates the effect. However, as θ increases—for a fixed p and G_1 —the overestimation becomes potentially small.

Turning to older households, when children leave, the OLS estimate is

$$\theta_{\text{OLS}}^{\text{old}} = \begin{cases} \rho \log\left(\frac{1+G_1}{G_1}\right) - \rho \log(1 + \exp(-\rho^{-1}\theta)) & \text{if } \theta > \log(G_1)\rho, \\ 0 & \text{if } 0 \leq \theta \leq \log(G_1)\rho \end{cases}$$

$$\leq \theta,$$

such that only if $\theta = 0$ will OLS produce a consistent estimate.¹ Since consumption does not change between periods 1 and 2 if there was no child in period 1, the IV estimator is identical to OLS,

$$\theta_{\text{IV}}^{\text{old}} = \theta_{\text{OLS}}^{\text{old}}.$$

S.2. SOLVING THE LIFE CYCLE MODEL

To reduce the number of state variables, all relations are normalized by permanent income, P_t , and lowercase variables denote normalized quantities (e.g., $c_t = C_t/P_t$). The model is solved recursively by backward induction, starting with the terminal period, T . Within a given period, optimal consumption is found using the endogenous grid method (EGM) of [Carroll \(2006\)](#).

The EGM constructs a grid over end-of-period wealth, a_t , rather than beginning-of-period resources, m_t . Denote this grid of Q points as $\hat{a}_t = (\underline{a}_t, a_t^1, \dots, a_t^{Q-1})$ in which \underline{a}_t is a lower bound on end-of-period wealth that I will discuss in great detail below. The endogenous level of beginning-of-period resources consistent with end-of-period assets, \hat{a}_t , and optimal consumption, c_t^* , is given by $m_t = \hat{a}_t + c_t^*(m_t, \mathbf{z}_t)$.

¹The inequality can be proved by induction and for $G_1 = 1.08$ and $\rho = 2$, the estimate $\theta_{\text{OLS}}^{\text{old}}$ is illustrated in Figure 4.

In the terminal period, independent of the presence of children, households consume all their remaining wealth, $c_T = m_T$. In preceding periods, in which households are retired, consumption across periods satisfies the Euler equation

$$u'(c_t) = \max \left\{ u'(m_t), R\beta \frac{v(\mathbf{z}_{t+1}; \theta)}{v(\mathbf{z}_t; \theta)} u'(c_{t+1}) \right\} \quad \forall t \in [T_r, T],$$

where consumption cannot exceed available resources. When retired, households do not produce new offspring and the age of children (\mathbf{z}_t) evolves deterministically.

The normalized consumption Euler equation in periods prior to retirement is given by

$$u'(c_t) = \max \left\{ u'(m_t + \kappa), R\beta \mathbb{E}_t \left[\frac{v(\mathbf{z}_{t+1}; \theta)}{v(\mathbf{z}_t; \theta)} u'(c_{t+1} G_{t+1} \eta_{t+1}) \right] \right\} \quad \forall t < T_r,$$

such that when $\hat{a}_t > -\kappa$, optimal consumption can be found by inverting the Euler equation

$$c_t^*(m_t, \mathbf{z}_t) = \left(\beta R \mathbb{E}_t \left[\frac{v(\mathbf{z}_{t+1}; \theta)}{v(\mathbf{z}_t; \theta)} (G_{t+1} \eta_{t+1})^{-\rho} \right. \right. \\ \left. \left. \times \check{c}_{t+1}^* \left(\underbrace{(G_{t+1} \eta_{t+1})^{-1} R \hat{a}_t + \varepsilon_{t+1}}_{=m_{t+1}}, \mathbf{z}_{t+1} \right)^{-\rho} \right] \right)^{-1/\rho},$$

where $\check{c}_{t+1}^*(m_{t+1}, \mathbf{z}_{t+1})$ is a linear interpolation function of optimal consumption next period, which is found in the last iteration. Since \hat{a}_t is the constructed grid, it is trivial to determine in which regions the credit constraint is and is not binding. I will discuss this in detail below.

The expectations are over next period arrival of children (\mathbf{z}_{t+1}) and transitory (ε_{t+1}) and permanent income shocks (η_{t+1}). Eight Gauss–Hermite quadrature points are used for each income shock to approximate expectations; $Q = 80$ discrete grid points are used in \hat{a}_t to approximate the consumption function with more mass at lower levels of wealth to approximate accurately the curvature of the consumption function.

The arrival probability of a child next period is a function of the wife's age and number of children today, $\pi_{t+1}(\mathbf{z}_t)$. No more than three children can live inside a household at a given point in time and infants cannot arrive when the household is older than 43. The next period's state is therefore calculated by increasing the age of children by 1 and if the age is 21, the child moves. In principle, there are $22^3 = 10,648$ combinations: three children can be either not present (NC) or aged 0–20. To reduce computation time, children are organized such that child one is the oldest at all times, the second child is the second oldest, and child three is the youngest child. To illustrate, imagine a household that in period t has, say, two children aged 20 and 17, $\mathbf{z}_t = (\text{age}_{1,t} = 20, \text{age}_{2,t} = 17, \text{age}_{3,t} = \text{NC})$; then, in period $t + 1$, only one child will be present, $\mathbf{z}_{t+1} = (\text{age}_{1,t+1} = 18, \text{age}_{2,t+1} = \text{NC}, \text{age}_{3,t+1} = \text{NC})$, given no new offspring arrive. Had new offspring arrived, then $\text{age}_{2,t+1} = 0$.

S.2.1 Explicit credit constraint and self-imposed no-borrowing

Since the EGM works with end-of-period wealth rather than beginning-of-period resources, credit constraints can easily be implemented by adjusting the lowest point in the grid, \underline{a}_t . The potentially binding credit constraint next period is implemented by the rule $c_{t+1}^* = m_{t+1}$ if m_{t+1} is lower than some threshold level, m_{t+1}^* . Including the credit constraint as the lowest point, $\underline{a}_{t+1} = -\kappa$, the lowest level of resources endogenously determined in the last iteration, \underline{m}_{t+1} , is the exact level of resources where households are on the cusp of being credit constrained, that is, $m_{t+1}^* = \underline{m}_{t+1}$. This ensures a very accurate interpolation and requires no additional handling of shadow prices of resources in the constrained Euler equation, denoted λ_{t+1} in Section 2.

Besides the exogenous credit constraint, κ , a “natural” or utility induced self-imposed constraint can be relevant such that the procedure described above should be modified slightly. This is because households want to accumulate enough wealth to buffer against a series of extremely bad income shocks to ensure strictly positive consumption in all periods even in the worst case possible.

PROPOSITION 1. *The lowest possible value of normalized end-of-period wealth consistent with the model, periods prior to retirement, can be calculated as*

$$\underline{a}_t = -\min\{\Omega_t, \kappa\} \quad \forall t \leq T_r - 2,$$

where, denoting the lowest possible values of the transitory and permanent income shock as $\underline{\varepsilon}$, and $\underline{\eta}$, respectively, Ω_t can be found recursively as

$$\Omega_t = \begin{cases} R^{-1}G_{T_r}\underline{\varepsilon}_{T_r}\underline{\eta}_{T_r} & \text{if } t = T_r - 2, \\ R^{-1}(\min\{\Omega_{t+1}, \kappa\} + \underline{\varepsilon}_{t+1})G_{t+1}\underline{\eta}_{t+1} & \text{if } t < T_r - 2. \end{cases}$$

PROOF. To see this, define $\underline{\mathbb{E}}_t[\cdot]$ as the *worst-case* expectation given information in period t and note that in the last period of working life, $T_r - 1$, households must satisfy $A_{T_r-1} \geq 0$. In the second-to-last period during working life, households must then leave a positive amount of resources in the worst case possible,

$$\begin{aligned} \underline{\mathbb{E}}_{T_r-2}[M_{T_r-1}] &> 0, \\ \underline{\mathbb{E}}_{T_r-2}[RA_{T_r-2} + Y_{T_r-1}] &> 0, \\ RA_{T_r-2} + G_{T_r-1}P_{T_r-2}\underline{\varepsilon}_{T_r-1}\underline{\eta}_{T_r-1} &> 0, \\ \Downarrow \\ A_{T_r-2} &> -\underbrace{R^{-1}G_{T_r-1}\underline{\varepsilon}_{T_r-1}\underline{\eta}_{T_r-1}}_{\equiv \Omega_{T_r-2}}P_{T_r-2}. \end{aligned}$$

Combining this with the exogenous credit constraint, κ , end-of-period wealth should satisfy

$$A_{T_r-2} > -\min\{\Omega_{T_r-2}, \kappa\}P_{T_r-2}.$$

In period $T_r - 3$, households must save enough to insure strictly positive consumption next period while satisfying the constraint above, in the worst case possible,

$$\begin{aligned} \mathbb{E}_{T_r-3}[M_{T_r-2}] &> -\min\{\Omega_{T_r-2}, \kappa\} \mathbb{E}_{T_r-3}[P_{T_r-2}], \\ RA_{T_r-3} + G_{T_r-2} P_{T_r-3} \varepsilon_{T_r-2} \underline{\eta}_{T_r-2} &> -\min\{\Omega_{T_r-2}, \kappa\} G_{T_r-2} P_{T_r-3} \underline{\eta}_{T_r-2}, \\ \Leftrightarrow \\ A_{T_r-3} &> -R^{-1} \underbrace{(\min\{\Omega_{T_r-2}, \kappa\} + \varepsilon_{T_r-2})}_{\equiv \Omega_{T_r-3}} G_{T_r-2} \underline{\eta}_{T_r-2} P_{T_r-3}, \end{aligned}$$

such that end-of-period wealth in period $T_r - 3$ should satisfy

$$A_{T_r-3} > -\min\{\Omega_{T_r-3}, \kappa\} P_{T_r-3}.$$

Hence, we can find Ω_t recursively by the formula in Proposition 1 and calculate the lowest value of the grid of normalized end-of-period wealth as $\underline{a}_t = -\min\{\Omega_t, \kappa\}$. \square

REFERENCE

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