Econometrica Supplementary Material

SUPPLEMENT TO "OPTIMAL PRODUCT DESIGN: IMPLICATIONS FOR COMPETITION AND GROWTH UNDER DECLINING SEARCH FRICTIONS"

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APPENDIX A: PROOF OF LEMMA 1

I START BY CHARACTERIZING THE SURPLUS FUNCTION $s_t(x)$, which maps the breadth x of a retailer's variety into the surplus offered by the retailer to its customers. First, $s_t(x)$ is strictly decreasing in x for all $x \in [x_t^*, x_t^* \exp(\eta)]$. To see why this is the case, consider two retailers with varieties x_0 and x_1, with $x_0 < x_1$. Let s_0 denote the optimal surplus offered by the retailer with x_0 and let s_1 denote the optimal surplus offered by the retailer with x_1. Since the retailer with x_0 prefers s_0 to s_1 and the retailer with x_1 prefers s_1 to s_0, it follows that

$$e^{-\lambda_t X_t(1-F_t(s_0))}\big(x_0^{-\alpha} - s_0\big) \geq e^{-\lambda_t X_t(1-F_t(s_1))}\big(x_0^{-\alpha} - s_1\big), \tag{A.1}$$

$$e^{-\lambda_t X_t(1-F_t(s_1))}\big(x_1^{-\alpha} - s_1\big) \geq e^{-\lambda_t X_t(1-F_t(s_0))}\big(x_1^{-\alpha} - s_0\big). \tag{A.2}$$

Combining (A.1) and (A.2) yields

$$e^{-\lambda_t X_t(1-F_t(s_0))}\big(x_0^{-\alpha} - x_1^{-\alpha}\big) \geq e^{-\lambda_t X_t(1-F_t(s_1))}\big(x_0^{-\alpha} - x_1^{-\alpha}\big), \tag{A.3}$$

which implies that $s_0 \geq s_1$. That is, $s_t(x)$ is nondecreasing in x. If $s_0 = s_1$, any retailers carrying a variety with breadth $x \in [x_0, x_1]$ would offer the surplus s_0, and hence, there would be a mass point in the surplus distribution $F_t(s)$ at s_0. This, however, cannot be an equilibrium, since a mass point in $F_t(s)$ at s_0 implies that a retailer could attain a strictly higher profit by offering $s_0 + \epsilon$ rather than by offering s_0, for some arbitrarily small but positive ϵ.

Second, $s_t(x)$ is such that $s_t(x_t^* \exp(\eta)) = 0$. To see why this is the case, suppose that the lowest surplus offered by retailers is some $s_0 > 0$. A retailer offering s_0 only sells to those b_0 buyers who are not in contact with any other retailer carrying a variety that they like. A retailer offering s_0 enjoys a profit of $x^{-\alpha} - s_0$ per unit sold. If the retailer were to offer a surplus of 0, it would still sell only to those b_0 buyers who are not in contact with any other retailer carrying a variety that they like. However, the retailer would enjoy a profit of $x^{-\alpha} > x^{-\alpha} - s_0$ per unit sold. Therefore, the lowest surplus offered by retailers must be equal to 0. Since retailers carrying a broader variety offer a lower surplus, it follows that $s_t(x_t^* \exp(\eta)) = 0$.

To complete the characterization of $s_t(x)$, I use the optimality condition of the retailer's pricing problem and the properties of the surplus distribution. The optimality condition of the retailer's problem is

$$1 = \lambda_t X_t F'\big(s_t(x)\big)\big(x^{-\alpha} - s_t(x)\big). \tag{A.4}$$

The left-hand side of (A.4) is the retailer's marginal cost of offering more surplus to its customers, and it is equal to the retailer's volume. The right-hand side of (A.4) is the

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retailer's marginal benefit of offering more surplus to its customers, and it is equal to the retailer's increase in volume multiplied by its per-unit profit.

The surplus distribution is such that

$$F_t\big(s_t(x)\big) = \left[\int_x^{x_t^* e^\eta} \frac{1}{\log x_t^* e^\eta - \log x_t^*}\, dz\right] \frac{1}{X_t} = \frac{x_t^* e^\eta - x}{\eta X_t}, \tag{A.5}$$

where the expression in (A.5) is obtained from (2.6) and the fact that $s_t(x)$ is strictly decreasing in x. Differentiating (A.5) with respect to x yields

$$F_t'\big(s_t(x)\big) s_t'(x) = -\frac{1}{\eta X_t}. \tag{A.6}$$

Combining (A.4) and (A.6) gives a differential equation for the surplus function

$$s_t'(x) = -\frac{\lambda}{\eta}\big(x^{-\alpha} - s_t(x)\big). \tag{A.7}$$

The unique solution to the differential equation (A.7) that satisfies the boundary condition $s_t(x_t^* \exp(\eta)) = 0$ is

$$s_t(x_0) = \frac{\lambda}{\eta} \int_{x_0}^{x_t^* e^\eta} x^{-\alpha} e^{-\frac{\lambda}{\eta}(x - x_0)}\, dx. \tag{A.8}$$

The expression in (A.8) describes the surplus function $s_t(x)$ for $x \in [x_t^*, x_t^* \exp(\eta)]$. For any $x > x_t^* \exp(\eta)$, the retailer carries the broadest variety in the market. It is easy to check that such a retailer finds it optimal to offer a surplus of 0. For $x < x_t^*$, a retailer carries the most specialized variety in the market. It is easy to check that such a retailer finds it optimal to offer a surplus of $s_t(x_t^*)$.

I can now compute the maximized profit $R_t(x)$ for a retailer carrying a variety with breadth x, which is

$$R_t(x) = \begin{cases} b\lambda_t x\big(x^{-\alpha} - s_t(x_t^*)\big) & \text{for } x < x_t^*, \\ b\lambda_t x e^{-\frac{\lambda_t}{\eta}(x - x_t^*)}\big(x^{-\alpha} - s_t(x)\big) & \text{for } x \in \big[x_t^*, x_t^* e^\eta\big], \\ b\lambda_t x e^{-\frac{\lambda_t}{\eta}(x_t^* e^\eta - x_t^*)} x^{-\alpha} & \text{for } x > x_t^* e^\eta. \end{cases} \tag{A.9}$$

For $x \in [x_t^*, x_t^* \exp(\eta)]$, the expression for $R_t(x)$ is obtained using the fact that $F_t(s_t(x))$ is given by (A.6) and X_t is given by (2.3). For $x > x_t^* \exp(\eta)$, the expression for $R_t(x)$ is obtained by noting that the retailer offers to its buyers a surplus of 0, which is the lowest in the market. For $x < x_t^*$, the expression for $R_t(x)$ is obtained by noting that the retailer offers a surplus of $s_t(x_t^*)$, which is the highest in the market.

APPENDIX B: PROOF OF LEMMA 2

Equation (2.13) implies that a firm's marginal benefit from designing a more specialized variety of the product is equal to the marginal cost from designing a more specialized variety when the firm chooses x_t^* and all other firms choose x_t^*. If, in addition, the firm's marginal cost is lower than the marginal benefit for all $x_t > x_t^*$ and the firm's marginal cost exceeds the marginal benefit for all $x_t < x_t^*$, then equation (2.13) also implies that x_t^* maximizes the firm's profit given that all other firms choose x_t^*.

Let $\mu(x_t)$ denote the derivative with respect to $-x_t$ of the first term on the right-hand side of (2.11). Let $\nu(x_t)$ denotes the derivative with respect to $-x_t$ of the second term on the right-hand side of (2.11). That is, let $\mu(x_t)$ denote the firm's marginal benefit from designing a more specialized variety of the product and let $\nu(x_t)$ denote the firm's marginal cost from designing a more specialized variety of the product. Using (2.10), it is easy to show that $\mu(x_t)$ is such that

$$\mu(x_t) \geq \mu(x_t^*) - bm\frac{\lambda_t}{\eta}\left(x_t^{*-\alpha} - x_t^{-\alpha}\right), \quad \text{for } x_t > x_t^*, \tag{B.1}$$

$$\mu(x_t) \leq \mu(x_t^*) + bm\frac{\lambda_t}{\eta}\left(x_t^{-\alpha} - x_t^{*-\alpha}\right), \quad \text{for } x_t < x_t^*. \tag{B.2}$$

The breadth x_t^* maximizes the firm's profit (2.11) as long as the firm's marginal cost $\nu(x_t)$ is smaller than the lower bound on the marginal benefit on right-hand side of (B.1) for $x_t > x_t^*$, and the marginal cost $\nu(x_t)$ is greater the upper bound on the marginal benefit on the right-hand side of (B.2) for $x_t < x_t^*$. There are many cost functions q such that $\nu(x_t)$ has these properties. For example, a cost function q such that

$$-q'\left(x_t/x_{t-1}^*\right) = -q'\left(x_t^*/x_{t-1}^*\right) + q_0\left[\left(x_t/x_{t-1}^*\right)^{-\beta} - \left(x_t^*/x_{t-1}^*\right)^{-\beta}\right], \tag{B.3}$$

where β and q_0 are parameters such that

$$\beta > \alpha, \quad \text{and} \quad q_0 > \frac{\lambda_t x_t^*}{\eta}\frac{x_{t-1}^{*-\alpha}}{w_t}. \tag{B.4}$$

APPENDIX C: PROPERTIES OF THE FUNCTION Ψ

The function $\Psi(\phi)$ is defined as

$$\Psi(\phi) = \frac{\phi}{\eta}\left[1 - \frac{\phi}{\eta}\int_1^{e^\eta} z^{-\alpha} e^{-\frac{\phi}{\eta}(z-1)}\, dz - e^{-\frac{\phi}{\eta}(e^\eta - 1)} e^{-(\alpha-1)\eta}\right]. \tag{C.1}$$

I am going to establish some properties of $\Psi(\phi)$. In particular, I am going to establish that $\Psi'(\phi) > 0$, $\Psi(0)$ is equal to 0, $\Psi'(0) = [1 - e^{-(\alpha-1)\eta}]/\eta$, and $\Psi(\infty) = \alpha$.

The derivative of $\Psi(\phi)$ with respect to ϕ is

$$\begin{aligned}\Psi'(\phi) = {} & \frac{1}{\eta}\left[1 - \frac{\phi}{\eta}\int_1^{e^\eta} z^{-\alpha} e^{-\frac{\phi}{\eta}(z-1)}\, dz - e^{-\frac{\phi}{\eta}(e^\eta-1)} e^{-(\alpha-1)\eta}\right] \\ & + \frac{\phi}{\eta}\left[-\frac{1}{\eta}\int_1^{e^\eta} z^{-\alpha} e^{-\frac{\phi}{\eta}(z-1)}\, dz + \frac{\phi}{\eta^2}\int_1^{e^\eta} z^{-\alpha}(z-1) e^{-\frac{\phi}{\eta}(z-1)}\, dz\right] \\ & + \frac{\phi}{\eta}\left[\frac{1}{\eta}(e^\eta - 1) e^{-\frac{\phi}{\eta}(e^\eta-1)} e^{-(\alpha-1)\eta}\right].\end{aligned} \tag{C.2}$$

After collecting terms, I can rewrite (C.2) as

$$\begin{aligned}\Psi'(\phi) = {} & \frac{1}{\eta}\left[1 - \frac{\phi}{\eta}\int_1^{e^\eta} z^{-\alpha}\left(2 - \frac{\phi}{\eta}(z-1)\right) e^{-\frac{\phi}{\eta}(z-1)}\, dz\right] \\ & - \frac{1}{\eta} e^{-\frac{\phi}{\eta}(e^\eta-1)} e^{-(\alpha-1)\eta}\left(1 - \frac{\phi}{\eta}(e^\eta - 1)\right).\end{aligned} \tag{C.3}$$

Using the fact that η is small, and hence, $\exp(\eta)$ is close to 1, I can approximate $z^{-\alpha}$ with $1-\alpha(z-1)$ inside the integral of (C.3). That is,

$$\begin{aligned}
&\int_1^{e^\eta} z^{-\alpha}\left(2-\frac{\phi}{\eta}(z-1)\right)e^{-\frac{\phi}{\eta}(z-1)}\,dz \\
&\quad\approx \int_1^{e^\eta}\left[1-\alpha(z-1)\right]\left(2-\frac{\phi}{\eta}(z-1)\right)e^{-\frac{\phi}{\eta}(z-1)}\,dz \\
&\quad= \int_1^{e^\eta}\left(2-\frac{\phi}{\eta}(z-1)\right)e^{-\frac{\phi}{\eta}(z-1)}\,dz-\alpha\int_1^{e^\eta}(z-1)\left(2-\frac{\phi}{\eta}(z-1)\right)e^{-\frac{\phi}{\eta}(z-1)}\,dz. \qquad \text{(C.4)}
\end{aligned}$$

The solution of the first integral in the third line of (C.4) is

$$\int_1^{e^\eta}\left(2-\frac{\phi}{\eta}(z-1)\right)e^{-\frac{\phi}{\eta}(z-1)}\,dz=\frac{1}{\phi/\eta}\left[1-e^{-\frac{\phi}{\eta}(e^\eta-1)}\left(1-\frac{\phi}{\eta}\left(e^\eta-1\right)\right)\right]. \qquad \text{(C.5)}$$

The solution of the second integral in the third line of (C.4) is

$$-\alpha\int_1^{e^\eta}(z-1)\left(2-\frac{\phi}{\eta}(z-1)\right)e^{-\frac{\phi}{\eta}(z-1)}\,dz=-\alpha e^{-\frac{\phi}{\eta}(e^\eta-1)}\left(e^\eta-1\right)^2. \qquad \text{(C.6)}$$

Substituting (C.5) and (C.6) into (C.4) yields

$$\begin{aligned}
\Psi'(\phi) &\approx \frac{1}{\eta}\left[e^{-\frac{\phi}{\eta}(e^\eta-1)}\left(1-\frac{\phi}{\eta}\left(e^\eta-1\right)\right)\left(1-e^{-(\alpha-1)\eta}\right)+\alpha\frac{\phi}{\eta}e^{-\frac{\phi}{\eta}(e^\eta-1)}\left(e^\eta-1\right)^2\right] \\
&= \frac{1}{\eta}e^{-\frac{\phi}{\eta}(e^\eta-1)}\left[\left(1-\frac{\phi}{\eta}\left(e^\eta-1\right)\right)\left(1-e^{-(\alpha-1)\eta}\right)+\alpha\frac{\phi}{\eta}\left(e^\eta-1\right)^2\right] \\
&= \frac{1}{\eta}e^{-\frac{\phi}{\eta}(e^\eta-1)}\left[1-e^{-(\alpha-1)\eta}+\frac{\phi}{\eta}\left(e^\eta-1\right)\left(\alpha e^\eta+e^{-(\alpha-1)\eta}-\alpha-1\right)\right], \qquad \text{(C.7)}
\end{aligned}$$

where the last line in (C.7) is strictly positive because $\alpha e^\eta + e^{-(\alpha-1)\eta} > \alpha + 1$. Hence, $\Psi'(\phi) > 0$.

For $\phi \to 0$, $\Psi'(\phi)$ takes the value

$$\Psi'(0)=\frac{1}{\eta}\left[1-e^{-(\alpha-1)\eta}\right]. \qquad \text{(C.8)}$$

For $\phi \to \infty$, $\Psi(\phi)$ is such that

$$\Psi(\infty)=\lim_{\phi\to\infty}\frac{1-\frac{\phi}{\eta}\int_1^{e^\eta} z^{-\alpha}e^{-\frac{\phi}{\eta}(z-1)}\,dz-e^{-\frac{\phi}{\eta}(e^\eta-1)}e^{-(\alpha-1)\eta}}{\eta/\phi}. \qquad \text{(C.9)}$$

Both the numerator and the denominator converge to 0. Applying de l'Hopital's rule yields

$$\begin{aligned}\Psi(\infty) &= \lim_{\phi\to\infty}\frac{\phi^2}{\eta^2}\left[\int_1^{e^\eta} z^{-\alpha}\left(1-\frac{\phi}{\eta}(z-1)\right)e^{-\frac{\phi}{\eta}(z-1)}\,dz - e^{-\frac{\phi}{\eta}(e^\eta-1)}e^{-(\alpha-1)\eta}\right]\\ &\approx \lim_{\phi\to\infty}\left(\frac{\phi}{\eta}\right)^2\left[e^{-\frac{\phi}{\eta}(e^\eta-1)}\left(e^\eta-1\right)\left(1-e^{-(\alpha-1)\eta}-\alpha\left(e^\eta-1\right)\right)\right]\\ &\quad+\lim_{\phi\to\infty}\alpha\left[1-e^{-\frac{\phi}{\eta}(e^\eta-1)}\left(1+\frac{\phi}{\eta}\left(e^\eta-1\right)\right)\right]\\ &= \alpha. \end{aligned} \tag{C.10}$$

APPENDIX D: PROPERTIES OF THE FUNCTION Γ

I now want to establish some properties of the function $\Gamma(\phi)$, which is defined as

$$\Gamma(\phi)=\frac{\phi}{\eta}\int_1^{e^\eta} z^{-\alpha}e^{-\frac{\phi}{\eta}(z-1)}\,dz. \tag{D.1}$$

Note that I can write (D.1) as

$$\begin{aligned}\Gamma\left(\lambda x^*\right) &= \frac{1}{x^{*-\alpha}}\frac{\lambda x^*}{\eta}\int_1^{e^\eta}\left(zx^*\right)^{-\alpha}e^{-\frac{\lambda}{\eta}(zx^*-x^*)}\,dz\\ &= \frac{1}{x^{*-\alpha}}\frac{\lambda}{\eta}\int_{x^*}^{x^*e^\eta} x^{-\alpha}e^{-\frac{\lambda}{\eta}(x-x^*)}\,dx, \end{aligned}\tag{D.2}$$

where the first line in (D.2) is obtained by defining x^* as λ/ϕ, and the second line in (D.2) is obtained by changing the variable of integration from z to $x = zx^*$.

Multiplying the left- and the right-hand side of (D.2) by $x^{*-\alpha}$ yields

$$\Gamma\left(\lambda x^*\right)x^{*-\alpha}=\frac{\lambda}{\eta}\int_{x^*}^{x^*e^\eta} x^{-\alpha}e^{-\frac{\lambda}{\eta}(x-x^*)}\,dx. \tag{D.3}$$

Differentiating the left- and the right-hand side of (D.3) with respect to x^* yields

$$\begin{aligned}&\Gamma'\left(\lambda x^*\right)x^{*-\alpha}\lambda-\alpha\Gamma\left(\lambda x^*\right)x^{*-\alpha-1}\\ &\quad=\frac{\lambda}{\eta}\left[\left(e^\eta x^*\right)^{-\alpha}e^{-\frac{\lambda}{\eta}(e^\eta x^*-x^*)}e^\eta - x^{*-\alpha}+\frac{\lambda}{\eta}\int_{x^*}^{x^*e^\eta}x^{-\alpha}e^{-\frac{\lambda}{\eta}(x-x^*)}\,dx\right]\\ &\quad=-x^{*-\alpha}\frac{\lambda}{\eta}\left[1-\frac{\lambda}{\eta}\int_{x^*}^{x^*e^\eta}\left(\frac{x}{x^*}\right)^{-\alpha}e^{-\frac{\lambda x^*}{\eta}(\frac{x}{x^*}-1)}\,dx-e^{-\frac{\lambda x^*}{\eta}(e^\eta-1)}e^{-\eta(\alpha-1)}\right]. \end{aligned}\tag{D.4}$$

Multiplying both sides of (D.4) by $x^*/x^{*-\alpha}$ yields

$$\begin{aligned}&\Gamma'\left(\lambda x^*\right)\lambda x^*-\alpha\Gamma\left(\lambda x^*\right)\\ &\quad=-\frac{\lambda x^*}{\eta}\left[1-\frac{\lambda}{\eta}\int_{x^*}^{x^*e^\eta}\left(\frac{x}{x^*}\right)^{-\alpha}e^{-\frac{\lambda x^*}{\eta}(\frac{x}{x^*}-1)}\,dx-e^{-\frac{\lambda x^*}{\eta}(e^\eta-1)}e^{-\eta(\alpha-1)}\right]. \end{aligned}\tag{D.5}$$

Using the fact that $\lambda x^* = \phi^*$, I can rewrite (D.5) as

$$\Gamma'(\phi)\phi - \alpha\Gamma(\phi) = -\frac{\phi}{\eta}\left[1 - \frac{\phi}{\eta}\int_1^{e^\eta} z^{-\alpha} e^{-\frac{\phi}{\eta}(z-1)}\,dz - e^{-\frac{\phi}{\eta}(e^\eta - 1)} e^{-\eta(\alpha-1)}\right]. \tag{D.6}$$

Since the right-hand side of (D.6) is equal to $-\Psi(\phi)$, it follows that

$$\Psi(\phi) = \alpha\Gamma(\phi) - \Gamma'(\phi)\phi. \tag{D.7}$$

Dividing both sides of (D.7) by $\Gamma(\phi)$, yields

$$\frac{\Psi(\phi)}{\Gamma(\phi)} = \frac{\alpha\Gamma(\phi) - \Gamma'(\phi)\phi}{\Gamma(\phi)} = \alpha - \frac{\Gamma'(\phi)\phi}{\Gamma(\phi)}. \tag{D.8}$$

Let $\epsilon(\phi)$ denote the elasticity of $\Gamma(\phi)$ with respect to ϕ. That is,

$$\begin{aligned}\epsilon(\phi) = \frac{\Gamma'(\phi)\phi}{\Gamma(\phi)} &= \frac{\displaystyle\int_1^{e^\eta} z^{-\alpha}\left(1 - \frac{\phi}{\eta}(z-1)\right) e^{-\frac{\phi}{\eta}(z-1)}\,dz}{\displaystyle\int_1^{e^\eta} z^{-\alpha} e^{-\frac{\phi}{\eta}(z-1)}\,dz} \\ &= 1 - \frac{\displaystyle\frac{\phi}{\eta}\int_1^{e^\eta} z^{-\alpha}(z-1) e^{-\frac{\phi}{\eta}(z-1)}\,dz}{\displaystyle\int_1^{e^\eta} z^{-\alpha} e^{-\frac{\phi}{\eta}(z-1)}\,dz}.\end{aligned} \tag{D.9}$$

Let $n(\phi)$ the numerator of the fraction in the second line of (D.9), that is,

$$\begin{aligned}n(\phi) &= \frac{\phi}{\eta}\int_1^{e^\eta} z^{-\alpha}(z-1) e^{-\frac{\phi}{\eta}(z-1)}\,dz \\ &\approx \frac{\phi}{\eta}\int_1^{e^\eta} \big(1 - \alpha(z-1)\big)(z-1) e^{-\frac{\phi}{\eta}(z-1)}\,dz \\ &= \frac{1}{(\phi/\eta)^2}\left[e^{-\frac{\phi}{\eta}(e^\eta - 1)}\left(\left(2\alpha - \frac{\phi}{\eta}\right)\left(1 + \frac{\phi}{\eta}(e^\eta - 1)\right)\right.\right. \\ &\qquad \left.\left. + \alpha\left(\frac{\phi}{\eta}\right)^2 \left(e^\eta - 1\right)^2\right) + \frac{\phi}{\eta} - 2\alpha\right],\end{aligned} \tag{D.10}$$

where the second line is obtained by approximating $z^{-\alpha}$ with $1 - \alpha(z-1)$. Let $d(\phi)$ the denominator of the fraction in the second line of (D.9), that is,

$$\begin{aligned}d(\phi) &= \int_1^{e^\eta} z^{-\alpha} e^{-\frac{\phi}{\eta}(z-1)}\,dz \\ &\approx \int_1^{e^\eta} \big(1 - \alpha(z-1)\big) e^{-\frac{\phi}{\eta}(z-1)}\,dz \\ &= \frac{1}{(\phi/\eta)^2}\left[e^{-\frac{\phi}{\eta}(e^\eta - 1)}\left(\alpha - \frac{\phi}{\eta} + \alpha\frac{\phi}{\eta}(e^\eta - 1)\right) + \frac{\phi}{\eta} - \alpha\right],\end{aligned} \tag{D.11}$$

where the second line is obtained by approximating $z^{-\alpha}$ with $1-\alpha(z-1)$. From the above expressions, it follows that $\lim_{\phi\to\infty} n(\phi)/d(\phi) = 1$. From the above expressions and de l'Hopital's rule, it follows that $\lim_{\phi\to 0} n(\phi)/d(\phi) = 0$. Hence, $\epsilon(0) = 1$ and $\epsilon(\infty) = 0$ and $\Psi(\phi)/\Gamma(\phi)$ is equal to $\alpha - 1$ for $\phi = 0$ and equal to α for $\phi = \infty$.

The derivative of $\epsilon(\phi)$ with respect to ϕ has the opposite sign as

$$\begin{aligned}\tilde{\epsilon}'(\phi) = {} & \left[\int_1^{e^\eta} z^{-\alpha}(z-1)\left(1-\frac{\phi}{\eta}(z-1)\right)e^{-\frac{\phi}{\eta}(z-1)}\,dz\right]\left[\int_1^{e^\eta} z^{-\alpha}e^{-\frac{\phi}{\eta}(z-1)}\,dz\right] \\ & + \frac{\phi}{\eta}\left[\int_1^{e^\eta} z^{-\alpha}(z-1)e^{-\frac{\phi}{\eta}(z-1)}\,dz\right]\left[\int_1^{e^\eta} z^{-\alpha}(z-1)e^{-\frac{\phi}{\eta}(z-1)}\,dz\right]. \end{aligned} \tag{D.12}$$

A linear approximation of $z^{-\alpha}(z-1)(1-(z-1)\phi/\eta)$, $z^{-\alpha}$ and $z^{-\alpha}(z-1)$ around $z = 1$ yields

$$\begin{aligned}\tilde{\epsilon}'(\phi) = {} & \left[\int_1^{e^\eta} (z-1)e^{-\frac{\phi}{\eta}(z-1)}\,dz\right]\left[\int_1^{e^\eta} \big(1-\alpha(z-1)\big)e^{-\frac{\phi}{\eta}(z-1)}\,dz\right] \\ & + \frac{\phi}{\eta}\left[\int_1^{e^\eta} (z-1)e^{-\frac{\phi}{\eta}(z-1)}\,dz\right]\left[\int_1^{e^\eta} (z-1)e^{-\frac{\phi}{\eta}(z-1)}\,dz\right]. \end{aligned} \tag{D.13}$$

Since $\tilde{\epsilon}'(\phi) > 0$, it follows that the derivative of $\epsilon(\phi)$ with respect to ϕ is strictly negative. In turn, this implies that the ratio $\Psi(\phi)/\Gamma(\phi)$ is strictly increasing in ϕ.

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