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SUPPLEMENT TO "STRATEGIC LEARNING AND THE TOPOLOGY OF SOCIAL NETWORKS": EXAMPLES

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IN THIS SUPPLEMENT, we give two examples showing that the assumptions of bounded out-degree and L-connectedness are crucial. Our approach in constructing equilibria will be to prescribe the initial moves of the agents and then extend this to an equilibrium strategy profile.

Define the set of times and histories agents have to respond to as $\mathscr{H} = \{(i, t, a) : i \in V, t \in \mathbb{N}_0, a \in [0, 1] \times \{0, 1\}^{|N(i)|\cdot t}\}$. The set $[0, 1] \times \{0, 1\}^{|N(i)|\cdot t}$ is interpreted as the pair of the private belief of i and the history of actions observed by agent i up to time t. If $a \in [0, 1] \times \{0, 1\}^{|N(i)|\cdot t}$, then for $0 \leq t' \leq t$, we let $a_{t'} \in [0, 1] \times \{0, 1\}^{|N(i)|\cdot t'}$ denote the history restricted to times up to t'. We say that a subset $\mathcal{H} \subseteq \mathscr{H}$ is *history-closed* if, for every $(i, t, a) \in \mathcal{H}$, we have that for all $0 \leq t' \leq t$ that $(i, t', a_{t'}) \in \mathcal{H}$.

For a strategy profile $\bar{Q}$, denote the optimal expected utility for i under any response as $u_i^\star(\bar{Q}) = \sup_{\bar{R}} u_i(\bar{R})$, where the supremum is over strategy profiles $\bar{R}$ such that $R^j = Q^j$ for all $j \neq i$ in V.

DEFINITION S.1: On a history-closed subset $\mathcal{H} \in \mathscr{H}$, a *forced response* $q_{\mathcal{H}}$ is a map $q_{\mathcal{H}} : \mathcal{H} \to \{0, 1\}$ denoting a set of actions we force the agents to make. A strategy profile $\bar{Q}$ is $q_{\mathcal{H}}$*-forced* if, for every $(i, t, a) \in \mathcal{H}$, agent i at time t has seen history a from her neighbors and then she selects action $q_{\mathcal{H}}(i, t, a)$. A strategy profile $\bar{Q}$ is a $q_{\mathcal{H}}$ *equilibrium* if it is $q_{\mathcal{H}}$-forced and for every agent $i \in V$, it holds that $u_i(\bar{Q}) \geq u_i(\bar{R})$ for any $q_{\mathcal{H}}$-forced strategy profile $\bar{R}$ such that $R^j = Q^j$ for all $j \neq i$ in V.

The following lemma can be proved by a minor modification of Theorem D.5, so we omit the proof.

LEMMA S.2: *Let $\mathcal{H} \in \mathscr{H}$ be history-closed and let $q_{\mathcal{H}}$ be a forced response. There exists a $q_{\mathcal{H}}$ equilibrium.*

Having constructed $q_{\mathcal{H}}$ equilibria, we then will want to show that they are equilibria. To do that, we appeal to the following lemma.

LEMMA S.3: *Let $\bar{Q}$ be a $q_{\mathcal{H}}$ equilibrium. Suppose that for every agent i, any strategy profile $\bar{R}$ that attains $u_i^\star(\bar{Q})$ has that for all t,*

$$\mathbb{P}\big[\bar{Q}_t^i\big(I_i, A_{[0,t)}^{N(i)}\big) \neq \bar{R}_t^i\big(I_i, A_{[0,t)}^{N(i)}\big), \big(i, t, \big(I_i, A_{[0,t)}^{N(i)}\big)\big) \in \mathcal{H}\big] = 0. \tag{S.1}$$

Then $\bar{Q}$ is an equilibrium.

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PROOF: If $\bar{Q}$ is not an equilibrium, then by compactness there exists a strategy profile for $\bar{R}$ that attains $u_i^\star$ and differs from $\bar{Q}$ only for agent i. By Eq. (S.1), this implies that agent i following $\bar{R}$ must take the same actions almost surely as if they were following $\bar{Q}$ until the end of the forced moves. Hence it is $q_{\mathcal{H}}$-forced and so $\bar{R}$ is a $q_{\mathcal{H}}$ equilibrium. It follows that i cannot increase the expected utility of $\bar{Q}$, which is, therefore, an equilibrium. *Q.E.D.*

To show that every agent follows the forced moves almost surely, we now give a lemma that gives a sufficient condition for an agent to act myopically, according to her posterior distribution. For an equilibrium strategy profile $\bar{Q}$, let $\bar{Q}^\dagger_{i,t,a}$ be the strategy profile where the agents follow $\bar{Q}$ except that if agent i has $a = (I_i, A^{N(i)}_{[0,t)})$, then from time t onward, agent i acts myopically, taking action $B^i_{t'}(G, \bar{Q}^\dagger_{i,t,a})$ for time $t' \geq t$. We denote

$$\begin{aligned} Y_\ell &= Y_\ell(i,t,a) \\ &:= \mathbb{E}\bigl[\bigl|\mathbb{P}\bigl[S=1|\mathcal{F}^i_{t+\ell}\bigl(G, \bar{Q}^\dagger_{i,t,a}\bigr)\bigr] - 1/2\bigr| |\mathcal{F}^i_t, a = \bigl(I_i, A^{N(i)}_{[0,t)}\bigr)\bigr]. \end{aligned}$$

We will show that the following conditions are sufficient for agent i to act myopically. For $\ell \in \{1,2,3\}$, we set $\mathcal{B}_\ell = \{2Y_0 > \frac{\lambda^2(1/2 - Y_{\ell-1})}{1-\lambda}\}$ and we set

$$\mathcal{B}_4 = \left\{2Y_0 > \lambda^2\left(\frac{1}{2} - Y_2\right) + \frac{\lambda^3\left(\frac{1}{2} - Y_3\right)}{1-\lambda}\right\}.$$

Since $\bar{Q}$ and $\bar{Q}^\dagger_{i,t,a}$ are the same up to time $t-1$, we have that $\mathcal{F}^i_t(G, \bar{Q})$ is equal to $\mathcal{F}^i_t(G, \bar{Q}^\dagger_{i,t,a})$. As Y_ℓ is the expectation of a submartingale, it is increasing. Hence, after rearranging we see that $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{B}_3 \subseteq \mathcal{B}_4$.

LEMMA S.4: *Suppose that for strategy profile $\bar{Q}$, agent i has an optimal response such that for any $\bar{R}$ such that $R^j = Q^j$ for all $j \neq i$ in V, then $u_i(\bar{Q}) \geq u_i(\bar{R})$. Then for any t,*

$$\mathbb{P}\bigl[A^i_t(G, \bar{Q}) \neq B^i_t, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4\bigr] = 0,$$

that is, agent i acts myopically at time t under $\bar{Q}$ almost surely on the event $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$.

PROOF: If agent i acts under $\bar{Q}^\dagger_{i,t,a}$, then her expected utility from time t onward given a is

$$\begin{aligned} &u_{i,t,a}\bigl(\bar{Q}^\dagger_{i,t,a}\bigr) \\ &\quad := (1-\lambda)\sum_{t'=t}^{\infty} \lambda^{t'} \mathbb{E}\bigl[\mathbb{P}\bigl[A^i_{t'}\bigl(G, \bar{Q}^\dagger_{i,t}\bigr) = S\bigr]|\mathcal{F}^i_t, a = \bigl(I_i, A^{N(i)}_{[0,t)}\bigr)\bigr] \end{aligned}$$

$$\geq (1-\lambda)\lambda^t\bigg(\frac{1}{2}+Y_0+\lambda\bigg(\frac{1}{2}+Y_1\bigg)$$
$$+\lambda^2\bigg(\frac{1}{2}+Y_2\bigg)+\frac{\lambda^3}{1-\lambda}\bigg(\frac{1}{2}+Y_3\bigg)\bigg)$$

under $\bar{Q}^\dagger_{i,t,a}$. Now assume that the action of agent i at time t under $\bar{Q}$ is not the myopic choice. Then her expected utility is at most

$$u_{i,t,a}(\bar{Q}) \leq (1-\lambda)\lambda^t\bigg(\frac{1}{2}-\bigg|\mathbb{P}\big[S=1|\mathcal{F}^i_t, a=\big(I_i, A^{N(i)}_{[0,t)}\big)\big]-\frac{1}{2}\bigg|$$
$$+\lambda\mathbb{E}\big[\mathbb{P}\big[A^i_{t+1}(G,\bar{Q})=S\big]|\mathcal{F}^i_t, a=\big(I_i, A^{N(i)}_{[0,t)}\big)\big]+\frac{\lambda^2}{1-\lambda}\bigg).$$

We note that at time $t+1$, the information available about S is the same under both strategies since the only difference is the choice of action by agent i at time t; hence, as i takes the optimal action under $\bar{Q}^\dagger$,

$$\frac{1}{2}+Y_1=\mathbb{E}\big[\mathbb{P}\big[A^i_{t+1}\big(G,\bar{Q}^\dagger_{i,t,a}\big)=S\big]|\mathcal{F}^i_t, a=\big(I_i, A^{N(i)}_{[0,t)}\big)\big]$$
$$\geq \mathbb{E}\big[\mathbb{P}\big[A^i_{t+1}(G,\bar{Q})=S\big]|\mathcal{F}^i_t, a=\big(I_i, A^{N(i)}_{[0,t)}\big)\big].$$

Since $\bar{Q}$ is optimal for i, we have that

$$0\geq u_{i,t,a}\big(\bar{Q}^\dagger_{i,t,a}\big)-u_{i,t}(\bar{Q})$$
$$\geq (1-\lambda)\lambda^t\bigg(2Y_0-\lambda^2\bigg(\frac{1}{2}-Y_2\bigg)-\frac{\lambda^3}{1-\lambda}\bigg(\frac{1}{2}-Y_3\bigg)\bigg). \tag{S.2}$$

Condition (S.2) does not hold under $\mathcal{B}_4$, so $\mathbb{P}[A^i_t(G,\bar{Q})\neq B^i_t, \mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3\cup\mathcal{B}_4]=0$. *Q.E.D.*

S.1. THE ROYAL FAMILY

In the main theorem, we require that the graph G not only be strongly connected, but also be L-connected and have bounded out-degrees, which are local conditions. In the following example, the graph is strongly connected and has bounded out-degrees, but is not L-connected. We show that for bounded private beliefs, asymptotic learning does not occur in all equilibria.[1]

Consider the graph in Figure S.1. The vertex set is composed of two groups of agents: a "royal family" clique of R agents who all observe each other, and

[1]We draw on Bala and Goyal's (1998) *royal family* graph.

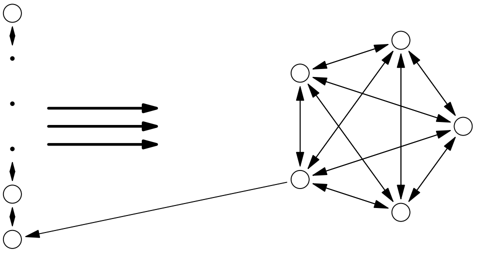

FIGURE S.1.—The royal family. Each member of the public (on the left) observes each royal (on the right) as well as her next door neighbors. The royals observe each other and one royal observes one member of the public.

$n \in \mathbb{N} \cup \{\infty\}$ agents—the "public'—who are connected in an undirected chain and, in addition, can all observe all the agents in the royal family. Finally, a single member of the royal family observes one of the public, so that the graph is strongly connected.

We choose μ_0 and μ_1 so that $\mathbb{P}[Z_0^i \in (1,2) \cup (-2,-1)] = 1$ and set the forced moves so that all agents act myopically at time 1. By Lemma S.2, we can extend this to a forced equilibrium $\bar{Q}$. By Lemma S.3, it is sufficient to show that no agent can achieve his or her optimum without choosing the myopic action in the first round. By our choice of μ_0 and μ_1, we have that

$$\left|\mathbb{P}\big[S=1|\mathcal{F}_0^i\big]-\frac{1}{2}\right| = \frac{e^{|Z_0^i|}}{1+e^{|Z_0^i|}}-\frac{1}{2} \geq \frac{e}{1+e}-\frac{1}{2} \geq \frac{1}{5}.$$

Hence, in the notation of Lemma S.4, we have that $Y_0 \geq \frac{1}{5}$ when $t=0$ for all i and a almost surely. Moreover, after the first round, all agents see the royal family and can combine their information. Since the signals are bounded, it follows that for some $c = c(\mu_0, \mu_1) > 0$, independent of R and n,

$$\mathbb{E}\left[\frac{1}{2}-\left|\mathbb{P}\big[S=1|\mathcal{F}_1^i\big]-\frac{1}{2}\right|\middle|\mathcal{F}_0^i\right] \leq e^{-cR}.$$

Hence, if R is a large constant, then $\mathcal{B}_2$ holds, so by Lemma S.4, if an agent is to attain her maximal expected utility given the actions of the other agents, she must act myopically almost surely at time 0. Thus $\bar{Q}$ is an equilibrium.

Let $\mathcal{J}$ denote the event that all agents in the royal family have a signal favoring state 1. On this event, under $\bar{Q}$, all agents in the royal family choose action 1 at time 0 and this is observed by all the agents, so $\mathcal{J} \in \mathcal{F}_1^i$ for all i. Since agents observe at most one other agent, this signal overwhelms their other information and so

$$\mathbb{P}\big[S=1|\mathcal{F}_1^i, \mathcal{J}\big] \geq 1-e^{-cR}$$

for all $i \in V$. Thus if R is a large constant, $\mathcal{B}_1$ holds for all the agents at time 1, so by Lemma S.4, they all act myopically and choose action 1 at time 1. Since $\mathcal{J} \in \mathcal{F}_1^i$, they also all knew this was what would happen, so they gain no extra information. By iterating this argument, we see that all agents choose 1 in all subsequent rounds. However, $\mathbb{P}[\mathcal{J}, S = 0] \geq e^{-c'R}$, where c' is independent of R and n. Hence as we let n tend to infinity, the probability of learning does not tend to 1, and when n equals infinity, the probability of learning does not equal 1.

S.2. THE MAD KING

More surprising is that there exist *undirected* (i.e., 1-connected) graphs with equilibria where asymptotic learning fails; These graphs have unbounded out-degrees. Note that in the myopic case, learning is achieved on these graphs (Mossel, Sly, and Tamuz (2014)) and so this is an example in which strategic behavior impedes learning.

In this example, we consider a finite graph that includes five classes of agents. There is a king labeled u and a regent labeled v. The court consists of R_C agents and the bureaucracy of R_B agents. The remaining n are the people. Note again that the graph is undirected.

- The king is connected to the regent, the court, and the people.
- The regent is connected to the king and to the bureaucracy.
- The members of the court are each connected only to the king.
- The members of the people are each connected only to the king.
- The members of the bureaucracy are each connected only to the regent.

See Figure S.2.

As in the previous example, we will describe some initial forced equilibrium and then appeal to existence results to extend it to an equilibrium. We suppose

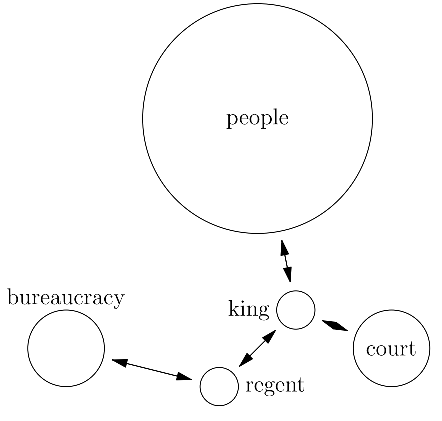

FIGURE S.2.—The mad king.

that μ_0 and μ_1 are such that $\mathbb{P}[Z_0^i \in (1, 1+\varepsilon) \cup (-\sqrt{7}, -\sqrt{7}+\varepsilon)] = 1$, where ε is some very small positive constant, and we will choose R_C, λ, and R_B so that e^{R_C} is much smaller than $\frac{1}{1-\lambda}$, which in turn will be much smaller than R_B:

$$e^{R_C} \ll \frac{1}{1-\lambda} \ll R_B.$$

The equilibrium we describe will involve the people being forced to choose action 0 in rounds 0 and 1, as otherwise the king "punishes" them by withholding his information. As an incentive to comply, he offers them the opinion of his court and, later, of his bureaucracy. While the opinion of the bureaucracy is correct with high probability, it is still bounded, and so even as the size of the public tends to infinity, the probability of learning stays bounded away from 1.

We now describe a series of forced moves for the agents, fixing $\delta > 0$ to be some small constant.

- The regent acts myopically at time 0. If for some state s, $\mathbb{P}[S = s|\mathcal{F}_1^v] \geq 1 - e^{-\delta R_B}$, then the regent chooses states s in round 1 and all future rounds; otherwise his moves are not forced.
- The king acts myopically in rounds 0 and 1 unless one or more of the people choose action 1 in round 0 or 1, in which case he chooses action 1 in all future rounds; otherwise, if s is the action of the regent at time 1, then from time 2, the king takes action s until the regent deviates and chooses another action.
- The members of the bureaucracy act myopically in rounds 0 and 1. If s is the action of the regent at time 1, then from time 2, the members of the bureaucracy take action s until the regent deviates and chooses another action.
- The members of the court act myopically in rounds 0 and 1. At time 2, they copy the action of the king from time 1. If s is the action of the king at time 2, then from time 3, the members of the bureaucracy take action s until the king deviates and chooses another action.
- The people choose action 0 in rounds 1 and 2. At time 2, they copy the action of the king from time 1. If s is the action of the king at time 2, then from time 3, the people take action s until the king deviates and chooses another action.

By Lemma S.2, this can be extended to a forced equilibrium strategy $\bar{Q}$. We will show that this is also an equilibrium strategy in the unrestricted game by establishing Eq. (S.1). In what follows, when we say acts optimally or in an optimal strategy, we mean for an agent with respect to the actions of the other agents under $\bar{Q}$.

First consider the regent. By our choice of μ_0, μ_1, we have that $Y_0 > \frac{1}{5}$. Let $\mathcal{J} = \mathcal{J}_0 \cup \mathcal{J}_1$, where $\mathcal{J}_s$ denotes the event that $\mathbb{P}[S = s|\mathcal{F}_1^v] \geq 1 - e^{-\delta R_B}$. Since the regent views all the myopic actions of the bureaucracy, he knows the correct value of S except with probability exponentially small in R_B, so for $s \in \{0, 1\}$, if $\delta > 0$ is small enough,

$$\mathbb{P}[\mathcal{J}_s|S = s] \geq 1 - e^{-\delta R_B}$$

and hence for large enough R_B, we have that $Y_1 \geq \frac{1}{2} - 2e^{-\delta R_B}$, which implies that $\mathcal{B}_2$ holds at time 1. By Lemma S.4, in any optimal strategy, the regent acts myopically in round 0 and so follows the forced move. On the event $\mathcal{J}_s$, the regent follows s in all future steps. At time 1, condition $\mathcal{B}_1$ holds, so again the regent follows the forced move in any optimal strategy. We next claim that for large enough R_B,

$$\mathbb{P}\big[\mathbb{P}\big[S = s|\mathcal{F}_2^v\big] \geq 1 - e^{-\delta R_B/2}|\mathcal{J}_s\big] = 1. \tag{S.3}$$

Assuming Eq. (S.3) holds, then condition $\mathcal{B}_1$ again holds, so the regent must choose s at time 2 in any optimal strategy. By construction of the forced moves from time 2 onward, the king and bureaucracy simply imitate the regent and so he receives no further information from time 2 onward. Thus again using Lemma S.4, we see that under any optimal strategy, the regent must follow his forced moves.

To establish that the regent follows the forced moves in any optimal strategy, it remains to show that Eq. (S.3) holds. The information available to the regent at time 2 includes the actions of the king and the bureaucracy at times 0 and 1. Consider the actions of the bureaucracy at times 0 and 1. At time 0, they follows their initial signal. At time 1, they also learn the initial action of the regent, who acts myopically. By our assumption on μ_0 and μ_1 that $\mathbb{P}[Z_0^i \in (1, 1 + \varepsilon) \cup (-\sqrt{7}, -\sqrt{7} + \varepsilon)] = 1$, an initial signal toward 0 is much stronger than an initial signal toward 1, since whenever Z is negative, it is at most $-\sqrt{7} + \varepsilon$. For i, a member of the bureaucracy, we have that $Z_1^i \geq 2$ if both i and the regent choose action 1 at time 1. However, if either i or the regent choose action 0 at time 1, then $Z_t^i \leq -\sqrt{7} + \varepsilon + 1 + \varepsilon < -1$. Since the actions of i and the regent at time 0 are known to the regent at time 1, he gains no extra information at time 2 from his observation of i at time 1 since he can correctly predict his action.

The information the regent has available at time 2 is thus his information from time 1 together with the information from observing the king. The information available to the king is a function of his initial signal and that of the regent and the court. Since this is only $R_C + 1$ members and we choose R_B to be much larger than R_C, it is insignificant compared to the information the regent observed from the court at time 0 and hence (S.3) holds. Thus, there is no optimal strategy for the regent that deviates from the forced moves.

As we noted above, the members of the bureaucracy have $|Z_0^i|, |Z_1^i| \geq 1$ almost surely. For $t \geq 1$, let $\mathcal{M}_{s,t}$ denote the event that the regent chose action s for times 1 up to t. As argued above, $\mathcal{J}_s \subset \mathcal{M}_{s,t}$ for all t under $\bar{Q}$. This analysis holds even if a single member of the bureaucracy adopts a different strategy, as we have taken R_B to be large, so this change is insignificant. Given that $\mathcal{M}_{s,t}$ holds, the only additional information available to agent i, a member of the bureaucracy, is his or her original signal and the action at time 1 of the regent. Thus

$$\mathbb{P}\big[S = s|\mathcal{F}_t^i, \mathcal{M}_{s,t}\big] \geq 1 - e^{-\delta R_B/2}.$$

It follows then by Lemma S.4 that acting myopically at times 0 and 1, and then imitating the regent until he changes his action is the sole optimal strategy for a member of the bureaucracy.

Next consider the forced responses of the king. Since under $\bar{Q}$, the people always choose action 0 at times 0 and 1, the rule forcing the king to choose action 1 after seeing a 1 from the people is never invoked. We claim that, provided R_B is taken to be sufficiently large, the king acts myopically at times 0 and 1. At time 0, the posterior probability of $S = 1$ is bounded away from 1/2, so Y_0 is bounded away from 0 while $\frac{1}{2} - Y_2 \leq 2e^{-\delta R_B/2}$, so by Lemma S.4, the king must act myopically. Similarly, at time 1, since our choice of μ_0 and μ_1 to have their log-likelihood ratio concentrated around either 1 or $-\sqrt{7}$, a posterior calculation gives that

$$\begin{aligned}&\big|Z_1^u - \#\big\{i \in N(u) : A_0^i(\bar{Q}) = 1\big\} + \sqrt{7}\#\big\{i \in N(u) : A_0^i = 0\big\}\big|\\ &\quad \leq \varepsilon(2 + R_C)\end{aligned}$$

and thus for some $\varepsilon(R_C) > 0$ sufficiently small, we can find an $\varepsilon'(\varepsilon, R_C) > 0$ such that $Y_0 = |\frac{e^{Z_1^u}}{1+e^{Z_1^u}} - \frac{1}{2}| > \varepsilon'$. Since we again have that $\frac{1}{2} - Y_1 \leq 2e^{-\delta R_B/2}$, taking $R_B = R_B(\varepsilon, R_C)$ to be sufficiently large, $\mathcal{B}_2$ holds and so the king must act myopically. It remains to see that the king should imitate the regent from time 2 onward unless the regent subsequently changes his action in any optimal strategy. This follows from a similar analysis to the case of the members of the bureaucracy, so we omit it.

We next move to an agent i, a member of the court. At time 0, the agent has $Y_0 > \frac{e}{1+e} - \frac{1}{2} > \frac{1}{5}$. Agent i at time 1 views the action of the king, who has in turn viewed the actions of the whole court at time 0, so $\frac{1}{2} - Y_2 \leq e^{-cR_C}$. At time 2, the agent sees the action of the king who has imitated the action of the regent at time 1, so $\frac{1}{2} - Y_3 \leq e^{-\delta R_B/2}$. Hence provided that R_C is sufficiently large and $R_B(R_C, \lambda)$ is sufficiently large, then $\mathcal{B}_4$ holds and i must act myopically at time 0. The information of a member of the court at time 1 is a combination of their initial signal and the action of the king at time 1. Similarly to a member of the bureaucracy, by the choice of μ_0 and μ_1, we have that $|Z_1^i| \geq 1$ and so $Y_0 > \frac{1}{5}$. Also $\frac{1}{2} - Y_2 \leq e^{-\delta R_B/2}$, since this includes the information from the action of the regent at time 1. Thus $\mathcal{B}_3$ holds and i must act myopically at time 1. At time 2, agents i knows the action of the king from round 2, so $Y_0 \geq \frac{1}{2} - e^{-cR_C}$

and $\frac{1}{2} - Y_1 \leq e^{-\delta R_B/2}$, and so $\mathcal{B}_2$ holds and i must act myopically at time 2. Finally, from time 3 onward, agent i knows the action of the regent at time 1. As with the king and the bureaucracy, this will not be changed unless i receives new information, that is, the king changes his action sometime after time 2. Thus any optimal strategy of i follows the forced moves.

This finally leaves the people. Let agent i be one of the people. We first check that it is always better for them to wait and just say 0 in rounds 0 and 1 so as to get more information from the king, their only source. If agent i chooses action 1 at time 0, then the total information he or she receives is a function of the initial signals of i and the king. Thus, since the signals are uniformly bounded, even if the agent knew the signals exactly, we would have that for some $c'(\mu_0, \mu_1)$, the expected utility from such a strategy is at most $1 - e^{-2c'}$. If an agent acts with 0 at time 0 but 1 at time 1, she can potentially receive information from the initial signals of the king, court, and regent as well as her own; still, the optimal expected utility, even using all of this information, is at most $1 - e^{-c'(R_C+3)}$. Consider instead the expected utility following the forced moves. On the event $\mathcal{J}$, agent i will have expected utility at least $\lambda^3(1 - e^{-\delta R_B})$, which is greater than $1 - e^{-c'(R_C+3)}$ provided that λ is sufficiently close to 1 and R_B is sufficiently large. Thus agent i must choose action 0 at times 0 and 1 in any optimal strategy. The analysis of rounds 2 and onward follows similarly to the court and thus any optimal strategy of i follows all the forced moves.

This exhaustively shows that there is no alternative optimal strategy for any of the agents that differs from the forced moves. Thus $\bar{Q}$ is an equilibrium. However, on the event $\mathcal{J}_1$, all the agents' actions converge to 1. However, $\mathbb{P}[\mathcal{J}, S = 0] \geq e^{-c''R_B} > 0$, where c'' is independent of R_C, R_B, λ, and n. Hence, as we let n tend to infinity, the probability of learning does not tend to 1.

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