Econometrica Supplementary Material

SUPPLEMENT TO "THE ASYMPTOTIC VARIANCE OF SEMI-PARAMETRIC ESTIMATORS WITH GENERATED REGRESSORS"

(*Econometrica*, Vol. 81, No. 1, January 2013, 315–340)

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S1. INTRODUCTION

WE PREPARED THIS NOTE in response to the issues raised by an anonymous referee. Lemma 1 in our paper shows how to compute the derivative

$$\frac{\partial}{\partial\alpha}\mathbb{E}\big[t\big(\varphi(x,\alpha_*)\big)\mathbb{E}\big[y|\varphi(x,\alpha)\big]\big]\bigg|_{\alpha=\alpha_*}.$$

In this regard, the referee noted that this raises the issue of whether the derivative

$$\frac{\partial}{\partial\alpha}\mathbb{E}\big[y|\varphi(x,\alpha)\big]$$

exists, because changing α changes the distribution of $\varphi(x,\alpha)$ and its associated sub-σ-field. The referee also implicitly raised the issue of whether the derivative is mean squared continuous in α.

This note will address these issues. In particular, we provide sufficient conditions for the existence and mean squared continuity of the derivative. The main technical tool we use is the theory of Radon measures (see, e.g., Tjur (1980)) defined on manifolds. The main point we want to note here is that Radon measures have similar properties as probability measures. In particular, dominated convergence and other results that allow us to interchange limits and integrals hold for Radon measures.

Organization

In Section S2, we note that the conditional expectation $\mathbb{E}[y|\varphi(x,\alpha)]$ can be given an explicit representation using Tjur (1980, Proposition 9.12.1). Because Tjur (1980, Proposition 8.1.2) was the basis of Hillier and Armstrong (1999; HA hereafter) and because HA was published in *Econometrica*, we use their notation. In Section S3, we introduce a change of variables that facilitates the discussion on the existence of the derivative $\partial\mathbb{E}[y|\varphi(x,\alpha)=s]/\partial\alpha$. This is done by assuming the existence of a certain diffeomorphism. In Section S4, we show how the existence of the derivative $\partial\mathbb{E}[y|\varphi(x,\alpha)=s]/\partial\alpha$ can be established using the standard argument leading to interchangeability of differentiation and integration. In Section S5, we establish mean square continuity again using the standard argument.

 DOI: 10.3982/ECTA9609

S2. SETUP

For simplicity of notation, we will assume that our objective of interest is $E[y|\varphi(x,\alpha)]$. We will write $E[y|\varphi(x,\alpha)] = E[\mu(x)|\varphi(x,\alpha)]$, where $\mu(x) = E[y|x]$. This representation, based on the law of iterated expectations, eliminates reference to the distribution of y and makes the notation even simpler.

We also take α as scalar. This is no restriction in the nonparametric case since α indexes a parametric submodel. It simplifies the notation in the parametric case.

We now use the results in HA (Section 3) to express this expectation as an integral with respect to the Radon measure on a manifold. They considered a mapping $U : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X} \subset R^n$ and $\mathcal{Y} \subset R^k$ with $n > k$. In their equation (14), they noted that the density of $U(x)$ in s is given by the surface integral

$$f_U(s) = \int_{M(s)} \left|\big(DU(x)\big)\big(DU(x)\big)'\right|^{-1/2} p(x)\big(dM(s)\big),$$

where $M(s) = \{x : U(x) = s\}$ is a manifold of dimension $n-k$, $p(x)$ is the probability density function (p.d.f.) of x, $DU(x) = \frac{\partial U}{\partial x'}(x)$ is the $k \times n$ matrix of partial derivatives, and $|\cdot|$ denotes the determinant of a matrix. The density of $U(x)$ could be obtained by the usual change-of-variables method, be it that there may not exist a function $V(x)$ such that $U(x), V(x)$ is a 1–1 mapping on $\mathcal{X}$. Equation (14) in HA does not require the existence of $V(x)$.

Tjur (1980, Proposition 9.12.1, p. 157) implied that the conditional expectation of $\mu(x)$ given $U(x) = s$ is given by

$$\text{(S1)} \qquad \frac{\int_{M(s)} |(DU(x))(DU(x))'|^{-1/2} \mu(x) p(x)(dM(s))}{\int_{M(s)} |(DU(x))(DU(x))'|^{-1/2} p(x)(dM(s))}.$$

If we apply (S1) to the conditional expectation $E[\mu(x)|\varphi(x,\alpha_*) = s]$, taking $U(x) = \varphi(x,\alpha^*)$, we obtain

$$\text{(S2)} \qquad E\big[\mu(x)|\varphi(x,\alpha_*) = s\big] = \frac{\int_{M(s,\alpha_*)} \left| \left(\frac{\partial \varphi(x,\alpha_*)}{\partial x'}\right)\left(\frac{\partial \varphi(x,\alpha_*)}{\partial x'}\right)' \right|^{-1/2} \mu(x) p(x)(dM(s,\alpha_*))}{\int_{M(s,\alpha_*)} \left| \left(\frac{\partial \varphi(x,\alpha_*)}{\partial x'}\right)\left(\frac{\partial \varphi(x,\alpha_*)}{\partial x'}\right)' \right|^{-1/2} p(x)(dM(s,\alpha_*))},$$

where

$$M(s,\alpha_*) = \big\{x : \varphi(x,\alpha_*) = s\big\}.$$

As usual, this conditional expectation is only defined for values s where the density of $\varphi(x, \alpha_*)$ is strictly positive, that is, we require a condition:

CONDITION S1: $\int_{M(s,\alpha_*)} |(\frac{\partial\varphi(x,\alpha_*)}{\partial x'})(\frac{\partial\varphi(x,\alpha_*)}{\partial x'})'|^{-1/2} p(x)(dM(s, \alpha_*)) > 0$.

S3. CHANGE OF VARIABLE

By (S2) we have, for all α where Condition S1 holds,

$$E\big[\mu(x)|\varphi(x,\alpha)=s\big] = \frac{\int_{M(s,\alpha)} \left|\left(\frac{\partial\varphi(x,\alpha)}{\partial x'}\right)\left(\frac{\partial\varphi(x,\alpha)}{\partial x'}\right)'\right|^{-1/2} \mu(x)p(x)(dM(s,\alpha))}{\int_{M(s,\alpha)} \left|\left(\frac{\partial\varphi(x,\alpha)}{\partial x'}\right)\left(\frac{\partial\varphi(x,\alpha)}{\partial x'}\right)'\right|^{-1/2} p(x)(dM(s,\alpha))}.$$

The derivative $\partial E[\mu(x)|\varphi(x,\alpha)=s]/\partial\alpha$ is obtained by differentiating the expression on the right. Since the surface integral is just the area under the integrand above the surface $M(s,\alpha)$, we can see that the derivative would exist if the manifold $M(s,\alpha)$ changes smoothly as a function of α.

The manifold $M(s,\alpha)$ depends on α, which is inconvenient when differentiating the integral. To avoid this problem, we use the change-of-variables technique.[1] For this, we require a differentiable 1–1 mapping on $\mathcal{X}$, that is, a diffeomorphism on $\mathcal{X}$, that maps the level sets of $\varphi(x,\alpha)$ to those of $\varphi(x,\alpha_*)$

CONDITION S2: There is a diffeomorphism $g(\cdot,\alpha):\mathcal{X}\to\mathcal{X}$ that is indexed by α such that

$$\varphi\big(g(x,\alpha),\alpha\big)=\varphi(x,\alpha_*).$$

EXAMPLE 1: Suppose that $x\in R^2$ and $\varphi(x,\alpha)=x_1+\alpha x_2$ with $\alpha_*=1$. Consider the mapping

$$g(x_1,x_2,\alpha):(x_1,x_2)\mapsto\left(x_1,\frac{1}{\alpha}x_2\right),$$

which has an inverse mapping

$$g^{-1}(x_1,x_2,\alpha):(x_1,x_2)\mapsto(x_1,\alpha x_2),$$

and note that

$$\varphi\big(g(x_1,x_2,\alpha),\alpha\big)=x_1+x_2=\varphi(x,\alpha_*).$$

[1]To keep the analysis relatively simple, we consider the case that the image of $\varphi(x,\alpha)$ does not depend on α. The general case can be studied by representing the surface integral as a repeated line integral after obtaining a smooth parametric representation of the surface.

Also note that

$$\varphi(x,\alpha)=s \quad \Leftrightarrow \quad x_1+\alpha x_2=s \quad \Leftrightarrow \quad \varphi\big((x_1,\alpha x_2),\alpha_*\big)=s.$$

Note that under Condition S2, we have

$$\varphi(x,\alpha)=s$$

if and only if

$$\varphi\big(g^{-1}(x,\alpha),\alpha_*\big)=s.$$

This implies that we have

$$\begin{aligned}E\big[\mu(x)|\varphi(x,\alpha)=s\big]&=E\big[\mu(x)|\varphi\big(g^{-1}(x,\alpha),\alpha_*\big)=s\big]\\&=E\big[\mu\big(g(z,\alpha)\big)|\varphi(z,\alpha_*)=s\big],\end{aligned}$$

where $z=g^{-1}(x,\alpha)$. By a change of variables, we can see that z has a density equal to

$$p\big(g(z,\alpha)\big)\left|\frac{\partial g(z,\alpha)}{\partial z'}\left(\frac{\partial g(z,\alpha)}{\partial z'}\right)\right|.$$

Applying (S1) with $U(z)=\varphi(z,\alpha_*)$, we obtain

$$\begin{aligned}&E\big[\mu(x)|\varphi(x,\alpha)=s\big]\\&\quad=E\big[\mu\big(g(z,\alpha)\big)|\varphi(z,\alpha_*)=s\big]\\&\quad=\int_{M(s,\alpha_*)}\frac{\left|\left(\frac{\partial g(z,\alpha)}{\partial z'}\right)\left(\frac{\partial g(z,\alpha)}{\partial z'}\right)'\right|^{1/2}}{\left|\left(\frac{\partial \varphi(z,\alpha_*)}{\partial z'}\right)\left(\frac{\partial \varphi(z,\alpha_*)}{\partial z'}\right)'\right|^{1/2}}\\&\qquad\times\mu(g(z,\alpha))p(g(z,\alpha))(dM(s,\alpha_*))\\&\qquad\Bigg/\left(\int_{M(s,\alpha_*)}\frac{\left|\left(\frac{\partial g(z;\alpha)}{\partial z'}\right)\left(\frac{\partial g(z,\alpha)}{\partial z'}\right)'\right|^{1/2}}{\left|\left(\frac{\partial \varphi(z,\alpha_*)}{\partial z'}\right)\left(\frac{\partial \varphi(z,\alpha_*)}{\partial z'}\right)'\right|^{1/2}}\right.\\&\qquad\left.\times\, p(g(z,\alpha))(dM(s,\alpha_*))\vphantom{\int}\right).\end{aligned}\tag{S3}$$

The function $g(x,\alpha)$ is found by solving

$$\varphi(y,\alpha)=\varphi(x,\alpha_*)$$

for y. This equation can have multiple solutions, which is not a problem because we do not require uniqueness for the diffeomorphism. We do require that $g(x,\alpha)$ be a 1–1 mapping. We now show that for α sufficiently close to α_*, we can choose $g(x,\alpha)$ to be 1–1. If $g(x,\alpha)$ is differentiable with respect to α (which by its definition is the case if $\varphi(x,\alpha)$ is differentiable with respect to both x and α), then we can write

$$g(x,\alpha)=g(x,\alpha_*)+\int_{\alpha_*}^{\alpha}\frac{\partial g}{\partial\alpha}(x,\tilde{\alpha})\,d\tilde{\alpha}=x+\int_{\alpha_*}^{\alpha}\frac{\partial g}{\partial\alpha}(x,\tilde{\alpha})\,d\tilde{\alpha}. \tag{S4}$$

If we differentiate $\varphi(g(x,\alpha),\alpha)=\varphi(x,\alpha_*)$ with respect to α, we find that for all x,α the derivative $\frac{\partial g}{\partial\alpha}(x,\alpha)$ satisfies

$$\frac{\partial\varphi}{\partial x'}\big(g(x,\alpha),\alpha\big)\frac{\partial g}{\partial\alpha}(x,\alpha)+\frac{\partial\varphi}{\partial\alpha}\big(g(x,\alpha),\alpha\big)=0.$$

For α near α_* and for almost all x, if

$$\frac{\partial\varphi}{\partial x}(x,\alpha)\neq 0,\qquad \left|\frac{\partial\varphi}{\partial\alpha}(x,\alpha)\right|<\infty,$$

then $\frac{\partial g}{\partial\alpha}(x,\alpha)$ is bounded for almost all x if α is sufficiently close to α_*. Note that these sufficient conditions hold if $\varphi(x,\alpha)$ is continuously differentiable with respect to x and α, and

$$\frac{\partial\varphi}{\partial x}(x,\alpha_*)\neq 0,\qquad \left|\frac{\partial\varphi}{\partial\alpha}(x,\alpha_*)\right|<\infty$$

for almost all x.

Boundedness of $\frac{\partial g}{\partial\alpha}(x,\alpha)$ implies that $g(x,\alpha)$ is 1–1 if α is sufficiently close to α_*. For this, we invoke the Gale–Nikaido (1965) theorem on the existence of an inverse function. Given the integral representation of $g(x,\alpha)$ above, we need to consider the derivative with respect to (w.r.t.) x:

$$\int_{\alpha_*}^{\alpha}\frac{\partial g}{\partial\alpha}(x,\alpha)\,d\alpha.$$

In particular, we want the derivative w.r.t. x of the integrand $\frac{\partial g}{\partial\alpha}(x,\alpha)$ to be a bounded function of x,α (which also allows us to interchange differentiation and integration).

The cross-derivative $\frac{\partial^2 g}{\partial x' \partial \alpha}(x, \alpha)$ satisfies

$$\frac{\partial g'}{\partial \alpha}(x, \alpha) \frac{\partial^2 \varphi}{\partial x \, \partial x'} \big(g(x, \alpha), \alpha\big) \frac{\partial g}{\partial x'}(x, \alpha) + \frac{\partial \varphi}{\partial x'} \big(g(x, \alpha), \alpha\big) \frac{\partial^2 g}{\partial \alpha \, \partial x'}(x, \alpha) + \frac{\partial^2 \varphi}{\partial \alpha \, \partial x'} \big(g(x, \alpha), \alpha\big) \frac{\partial g}{\partial x'}(x, \alpha) = 0.$$

The earlier conditions ensured that there is a bounded $\frac{\partial g}{\partial \alpha}(x, \alpha)$. Also differentiating the equation in Condition S2 w.r.t. x, we find

$$\frac{\partial \varphi}{\partial x'} \big(g(x, \alpha), \alpha\big) \frac{\partial g}{\partial x'}(x, \alpha) = \frac{\partial \varphi}{\partial x'}(x, \alpha_*),$$

which has a bounded solution for $\frac{\partial g}{\partial x'}(x, \alpha)$ if, in addition to the earlier conditions, we have

$$\left| \frac{\partial \varphi}{\partial x}(x, \alpha_*) \right| < \infty.$$

We conclude that if, in addition,

$$\left| \frac{\partial^2 \varphi}{\partial x \, \partial x'}(x, \alpha) \right| < \infty, \quad \left| \frac{\partial^2 \varphi}{\partial \alpha \, \partial x'}(x, \alpha) \right| < \infty$$

for all x and α near α_*, then all x and α close to α_*, the cross-derivative $\frac{\partial^2 g}{\partial x' \partial \alpha}(x, \alpha)$ is bounded.

By (S4), we have

$$\frac{\partial g(x, \alpha)}{\partial x'} = I + \int_{\alpha_*}^{\alpha} \frac{\partial^2 g}{\partial x' \, \partial \alpha}(x, \tilde{\alpha}) \, d\tilde{\alpha}$$

and this matrix is positive definite if α is sufficiently close to α_*, so that the function $g(x, \alpha)$ has a positive Jacobian for all $x \in \mathcal{X}$ and hence is 1–1.

The referee gave an example in which the definition of the derivative is problematic. In particular, with ι a vector of 1's,

$$\varphi(x, \alpha) = \big(x' \iota\big) \alpha.$$

If $\alpha_* = 0$, the diffeomorphism must satisfy

$$\big(g(x, \alpha)' 1\big) \alpha = 0$$

for all x and $\alpha \neq 0$. It is obvious that in this case, a diffeomorphism that satisfies Condition S2 does not exist.

In the main paper, we require that for almost all x,

$$\frac{\partial \varphi}{\partial x}(x, \alpha_*) \neq 0,$$

so that the example is explicitly excluded. The case that $\varphi(x, \alpha_*)$ is constant in x (on an interval) is interesting, but we exclude it in the paper. To understand why, we can consider the HIT estimator in the case that the population propensity score is constant. The resulting estimator is potentially irregular and we explicitly do not consider such estimators in the paper.

S4. EXISTENCE OF THE DERIVATIVE

We rewrite (S3) as

$$\text{(S5)} \qquad E\big[\mu(x)|\varphi(x, \alpha) = s\big] = \frac{\int_{M(s,\alpha_*)} \psi(x, \alpha)\mu(g(x, \alpha))p(g(x, \alpha))\xi_{s,\alpha_*}(dx)}{\int_{M(s,\alpha_*)} \psi(x; \alpha)p(g(x, \alpha))\xi_{s,\alpha_*}(dx)},$$

where

$$\psi(x, \alpha) = \left|\left(\frac{\partial g(x, \alpha)}{\partial x'}\right)\left(\frac{\partial g(x, \alpha)}{\partial x'}\right)'\right|^{1/2}.$$

Here

$$\xi_{s,\alpha_*}(dx) = \frac{1}{\left|\left(\frac{\partial \varphi(x, \alpha_*)}{\partial x'}\right)\left(\frac{\partial \varphi(x, \alpha_*)}{\partial x'}\right)'\right|^{1/2}}\big(dM(s, \alpha_*)\big)$$

defines the probability measure on $M(s, \alpha_*)$. With (S5), conditions for differentiability are conditions that ensure that we can interchange differentiation and integration. It is obvious that conditions that are sufficient for interchange in the numerator are also sufficient for interchange in the denominator. Condition S1 above ensures that the denominator is strictly positive.

The usual sufficient condition is that the derivative of the integrand w.r.t. α exists, is continuous, and is bounded (by an integrable function). Therefore, the derivative w.r.t. α of $\psi(x, \alpha)$, of $\mu(g(x, \alpha))$, and of $p(g(x, \alpha))$ should be continuous and bounded (by an integrable function).

First, the derivative of $\psi(x, \alpha)$ w.r.t. α depends on $\frac{\partial^2 g}{\partial x' \partial \alpha}(x, \alpha)$. The conditions that ensure that $g(x, \alpha)$ is 1–1 are sufficient for boundedness of

$\frac{\partial^2 g}{\partial x' \partial \alpha}(x, \alpha)$ for α sufficiently close to α_*. They are that $\varphi(x, \alpha)$ is twice continuously differentiable w.r.t. x and x, α, and that

$$\frac{\partial \varphi}{\partial x}(x, \alpha_*) \neq 0, \quad \left|\frac{\partial \varphi}{\partial x}(x, \alpha_*)\right| < \infty, \quad \left|\frac{\partial \varphi}{\partial \alpha}(x, \alpha_*)\right| < \infty,$$
$$\left|\frac{\partial^2 \varphi}{\partial x \, \partial x'}(x, \alpha_*)\right| < \infty, \quad \left|\frac{\partial^2 \varphi}{\partial \alpha \, \partial x'}(x, \alpha_*)\right| < \infty.$$

Under these conditions, $\frac{\partial g}{\partial x'}(x, \alpha)$ is also bounded for α sufficiently close to α_*.

Therefore, the following condition is sufficient for the existence of the derivative (in addition to Conditions S1 and S2)[2]

CONDITION S3:

(i) For all $x \in \mathcal{X}$ and α sufficiently close to α_*, the function $\varphi(x, \alpha)$ is twice continuously differentiable w.r.t. x and x, α, and

$$\frac{\partial \varphi}{\partial x}(x, \alpha_*) \neq 0, \quad \left|\frac{\partial \varphi}{\partial x}(x, \alpha_*)\right| < \infty, \quad \left|\frac{\partial \varphi}{\partial \alpha}(x, \alpha_*)\right| < \infty,$$
$$\left|\frac{\partial^2 \varphi}{\partial x \, \partial x'}(x, \alpha_*)\right| < \infty, \quad \left|\frac{\partial^2 \varphi}{\partial \alpha \, \partial x'}(x, \alpha_*)\right| < \infty.$$

(ii) For all $x \in \mathcal{X}$, $\mu(x)$ is continuously differentiable with a bounded derivative.

(iii) For all $x \in \mathcal{X}$, $p(x)$ is continuously differentiable with a bounded derivative.

The results in the next section show that the expression for the derivative simplifies in α_*.

S5. MEAN SQUARED CONTINUITY

In this section, we give sufficient conditions for mean squared continuity of the derivative, that is, if

$$\kappa(s, \alpha) = \frac{\partial E[\mu(x) | \varphi(x, \alpha) = s]}{\partial \alpha},$$

then mean squared continuity means that

$$\lim_{\alpha \to \alpha_*} \mathbb{E}\Big[\big(\kappa\big(\varphi(x, \alpha), \alpha\big) - \kappa\big(\varphi(x, \alpha_*), \alpha_*\big)\big)^2\Big] = 0.$$

[2]We can replace "bounded" by "bounded by an integrable function." In that case, a condition that the product of integrable bounds is itself integrable should be added.

As a first step, we derive an expression for the derivative $\kappa(s,\alpha)$. We use the same argument that leads to (S5), but instead of the diffeomorphism $g(x,\alpha)$ that satisfies $\varphi(g(x,\alpha),\alpha)=\varphi(x,\alpha_*)$, we use the diffeomorphism $g(x,\tilde{\alpha},\alpha)$ that satisfies

$$\varphi\big(g(x,\tilde{\alpha},\alpha),\tilde{\alpha}\big)=\varphi(x,\alpha).$$

The former diffeomorphism can be denoted as $g(x,\alpha,\alpha_*)$. If we use $g(x,\tilde{\alpha},\alpha)$ in a change of variables, we obtain for (S5) the expression

$$\text{(S6)}\qquad E\big[\mu(x)|\varphi(x,\tilde{\alpha})=s\big] = \frac{\int_{M(s,\alpha)}\psi(x,\tilde{\alpha},\alpha)\mu(g(x,\tilde{\alpha},\alpha))\frac{p(g(x,\tilde{\alpha},\alpha))}{p(x)}\lambda_{s,\alpha}(dx)}{\int_{M(s,\alpha)}\psi(x,\tilde{\alpha},\alpha)\frac{p(g(x,\tilde{\alpha},\alpha))}{p(x)}\lambda_{s,\alpha}(dx)} \equiv \frac{N(\tilde{\alpha})}{D(\tilde{\alpha})}.$$

This expression simplifies if we observe that the diffeomorphism $g(x,\tilde{\alpha},\alpha)$ satisfies

$$g(x,\alpha,\alpha)=x.$$

This implies that because

$$\psi(x,\tilde{\alpha},\alpha)=\left|\left(\frac{\partial g(x,\tilde{\alpha},\alpha)}{\partial x'}\right)\left(\frac{\partial g(x,\tilde{\alpha},\alpha)}{\partial x'}\right)'\right|^{1/2}=\left\|\frac{\partial g(x,\tilde{\alpha},\alpha)}{\partial x'}\right\|,$$

we have $\psi(x,\alpha)=1$ and by Jacobi's formula of the derivative of the determinant,

$$\left.\frac{\partial\left\|\frac{\partial g(x,\tilde{\alpha},\alpha)}{\partial x'}\right\|}{\partial\tilde{\alpha}}\right|_{\tilde{\alpha}=\alpha}=\operatorname{tr}\left(\frac{\partial^2 g}{\partial\tilde{\alpha}\,\partial x'}(x,\alpha,\alpha)\right).$$

This implies that if the analogs of Conditions S1–S3 hold so that we can interchange differentiation and integration, then

$$\begin{aligned}N'(\alpha)&=\int_{M(s,\alpha)}\operatorname{tr}\left(\frac{\partial^2 g}{\partial\tilde{\alpha}\,\partial x'}(x,\alpha,\alpha)\right)\mu(x)\lambda_{s,\alpha}(dx)\\&\quad+\int_{M(s,\alpha)}\mu'(x)\frac{\partial g}{\partial\tilde{\alpha}}(x,\alpha,\alpha)\lambda_{s,\alpha}(dx)\\&\quad+\int_{M(s,\alpha)}\mu(x)\frac{p'(x)}{p(x)}\frac{\partial g}{\partial\tilde{\alpha}}(x,\alpha,\alpha)\lambda_{s,\alpha}(dx)\end{aligned}$$

and

$$D'(\alpha) = \int_{M(s,\alpha)} \operatorname{tr}\left(\frac{\partial^2 g}{\partial\tilde{\alpha}\,\partial x'}(x,\alpha,\alpha)\right)\lambda_{s,\alpha}(dx) + \int_{M(s,\alpha)} \frac{p'(x)}{p(x)}\frac{\partial g}{\partial\alpha}(x,\alpha,\alpha)\lambda_{s,\alpha}(dx),$$

where

$$\lambda_{s,\alpha}(dx) = \frac{p(x)}{\left|\left(\frac{\partial\varphi(x,\alpha)}{\partial x'}\right)\left(\frac{\partial\varphi(x,\alpha)}{\partial x'}\right)'\right|^{1/2}}\big(dM(s,\alpha)\big).$$

Therefore,

$$\text{(S7)}\qquad \frac{\partial E[\mu(x)|\varphi(x,\tilde{\alpha})=s]}{\partial\tilde{\alpha}}\bigg|_{\tilde{\alpha}=\alpha}$$
$$= \frac{N'(\alpha)}{D(\alpha)} - \frac{N(\alpha)}{D(\alpha)}\frac{D'(\alpha)}{N(\alpha)}$$
$$= \mathbb{E}\bigg[\operatorname{tr}\left(\frac{\partial^2 g}{\partial\tilde{\alpha}\,\partial x'}(x,\alpha,\alpha)\right)\mu(x) + \mu'(x)\frac{\partial g}{\partial\alpha}(x,\alpha)$$
$$+ \mu(x)\frac{p'(x)}{p(x)}\frac{\partial g}{\partial\alpha}(x,\alpha)\bigg|\varphi(x,\alpha)=s\bigg]$$
$$- \mathbb{E}\big[\mu(x)|\varphi(x,\alpha)=s\big]$$
$$\times \mathbb{E}\bigg[\operatorname{tr}\left(\frac{\partial^2 g}{\partial\tilde{\alpha}\,\partial x'}(x,\alpha,\alpha)\right) + \frac{p'(x)}{p(x)}\frac{\partial g}{\partial\alpha}(x,\alpha)\bigg|\varphi(x,\alpha)=s\bigg].$$

By (S7), we have

$$\kappa(s_*,\alpha_*)$$
$$= \mathbb{E}\bigg[\operatorname{tr}\left(\frac{\partial^2 g}{\partial\tilde{\alpha}\,\partial x'}(x,\alpha_*,\alpha_*)\right)\mu(x) + \mu'(x)\frac{\partial g}{\partial\alpha}(x,\alpha_*)$$
$$+ \mu(x)\frac{p'(x)}{p(x)}\frac{\partial g}{\partial\alpha}(x,\alpha_*)\bigg|\varphi(x,\alpha_*)=s_*\bigg]$$
$$- \mathbb{E}\big[\mu(x)|\varphi(x,\alpha_*)=s_*\big]$$
$$\times \mathbb{E}\bigg[\operatorname{tr}\left(\frac{\partial^2 g}{\partial\tilde{\alpha}\,\partial x'}(x,\alpha_*,\alpha_*)\right) + \frac{p'(x)}{p(x)}\frac{\partial g}{\partial\alpha}(x,\alpha_*)\bigg|\varphi(x,\alpha_*)=s_*\bigg].$$

We now want to express (S7) also as a conditional expectation given $\varphi(x, \alpha_*) = s_*$. This involves two changes of variables. The first is the same as in Section S3, that is, we transform to

$$z = g^{-1}(x, \alpha, \alpha_*).$$

Note that

$$g(z, \alpha_*, \alpha_*) = z$$

and that the Jacobian is

$$\psi_1(z, \alpha) = \left\| \frac{\partial g(z, \alpha, \alpha_*)}{\partial z'} \right\|$$

so that

$$\psi_1(z, \alpha_*) = 1.$$

The second change is to

$$v = h(z, s, s_*),$$

where the diffeomorphism $h(z, s, s_*)$ is defined by

$$\varphi(z, \alpha_*) = s \quad \Leftrightarrow \quad \varphi\big(h(z, s, s_*), \alpha_*\big) = s_*.$$

Note that we can choose

$$h(z, s_*, s_*) = z.$$

To show that such a diffeomorphism exists, we consider the following equation that holds for all s in a neighborhood of s_*:

$$\varphi\big(h^{-1}(v, s, s_*), \alpha_*\big) = s. \tag{S8}$$

Differentiating w.r.t. s, we find

$$\frac{\partial \varphi}{\partial x'}\big(h^{-1}(v, s, s_*), \alpha_*\big)\frac{\partial h^{-1}}{\partial s}(v, s, s_*) = 1.$$

If

$$\frac{\partial \varphi}{\partial x'}(x, \alpha_*)$$

is bounded and nonzero, there exists a (generally nonunique) solution with $\frac{\partial h^{-1}}{\partial s}(v, s, s_*)$ bounded. As in Lemma 1, we use this to establish the existence of the diffeomorphism. Equation (S8) shows that if φ is continuously differentiable in x, then for all v such that $\varphi)(v, \alpha_*) = s_*$, we have that

$$\lim_{s \to s_*} h^{-1}(v, s, s_*) = v.$$

The two changes in variables are such that

$$\lim_{\alpha \to \alpha_*} g(z, \alpha, \alpha_*) = z,$$

$$\lim_{\alpha \to \alpha_*} \frac{\partial g}{\partial z'}(z, \alpha, \alpha_*) = I,$$

and

$$\lim_{s \to s_*} h^{-1}(v, s, s_*) = v,$$

$$\lim_{s \to s_*} \frac{\partial h^{-1}}{\partial v'}(v, s, s_*) = I.$$

This follows from the equations that define the homeomorphism and continuous differentiability of φ. The two changes of variables introduce two Jacobians in $\kappa(s, \alpha)$ and a transformation of x to z to v. The Jacobians converge to 1 and the transformations converge to identities if $\alpha \to \alpha_*$ and $s \to s_*$. The same argument that established the existence of the derivative allows for the interchange of limit and integral, and this establishes

$$\lim_{s \to s_*, \alpha \to \alpha_*} \kappa(s, \alpha) = \kappa(s_*, \alpha_*).$$

If the derivatives of φ are bounded, then so is $\kappa(s, \alpha)$. This implies that $\kappa(\varphi(x, \alpha), \alpha)$ and $\kappa(\varphi(x, \alpha_*), \alpha_*)$ are bounded (bounded is much easier than bounded by an integrable function). This establishes dominance, and mean squared continuity follows from dominated convergence.

REFERENCES

GALE, D., AND H. NIKAIDO (1965): "The Jacobian Matrix and Global Univalence of Mappings," *Mathematische Annalen*, 159, 81–93. [5]

HILLIER, G., AND M. ARMSTRONG (1999): "The Density of the Maximum Likelihood Estimator," *Econometrica*, 67, 1459–1470. [1]

TJUR, T. (1980): *Probability Based on Radon Measures*. Wiley Series in Probability and Mathematical Statistics. New York: Wiley. [1,2]

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Manuscript received October, 2010; final revision received May, 2012.